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FIXED POINT THEOREMS FOR EXPANSION MAPPINGS IN 2 NON-ARCHIMEDEAN MENGER PM-SPACE

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Abstract. The aim of this paper is to generalize the results of Ahmad, Ashraf and Rhoades [1] in the setting of 2 Non Archimedean Menger PM-space introduced by Renu Chugh and Sumitra [2]. In fact, 2 non-Archimedean Menger PM-space (briefly 2 N. A. Menger PM-space) is the generalization of 2-metric space in probabilistic setting, i.e., the case where instead of the distances between two or more points one knows only the probability of a possible value of this distance and distance is represented by a distribution function.

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1. Introduction

Wang, Li. Gao and Iseki [11] presented some interesting work on expansion mappings in metric spaces which correspond to some contractive mappings in [6]. Rhoades [7, 8] and Taniguchi [10] generalized the results of [11] for pairs of mapping. Pant, Dimri and Singh [5] introduced the notion of expansion mappings on PM-spaces. Later, Vasuki [9] also established some results for expansion mappings in Menger spaces.

In this paper, we prove common fixed point theorems for compatible the mappings satisfying expansion type condition in 2 N. A. Menger PM-space.

2. Preliminaries

Definition 2.1. *Let X be any non-empty set and D be the set of all left continuous distribution functions. An ordered pair (X, F) is said to be 2 non-Archimedean probabilistic metric space (briefly 2 N. A. PM-space) if F is a mapping from $X \times X \times X$ into D satisfying the following conditions, where the value of F at $(x, y, z) \in X \times X \times X$ is represented by $F_{x,y,z}$ or $F(x, y, z)$ for each $x, y, z \in X$ and $s, t > 0$ such that*

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(i) $F(x,y,z;t)=1$ *for all* $t>0$ *if and only if at least two of the three points are equal.*

(ii) $F(x,y,z)=F(x,z,y)=F(z,x,y)$

(iii) $F(x,y,z;0)=0$

(iv) *If* $F(x,y,s;t_1)=F(x,s,z;t_2)=F(s,y,z;t_3)=1$, *then* $F(x,y,z;\max\{t_1,t_2,t_3\})=1$.

Definition 2.2. *A t-norm is a function* $\Delta:[0,1]\times[0,1]\times[0,1]\to[0,1]$, *which is associative, commutative, non-decreasing in each coordinate and* $\Delta(a,1,1)=a$ *for each* $a\in[0,1]$.

Definition 2.3. *A 2 N. A. Menger PM-space is an ordered triplet* (X,F,Δ), *where* Δ *is a t-norm and* (X,F) *is a 2 N. A. PM-space satisfying the following condition,*

$$F(x,y,z;\max\{t_1,t_2,t_3\})\geq\Delta\left(F(x,y,s;t_1),F(x,s,z;t_2),F(s,y,z;t_3)\right)$$

for each $x,y,z\in X$, $t_1,t_2,t_3\geq 0$.

Definition 2.4. *Let* (X,F,Δ) *be 2 N. A. Menger PM-space and* Δ *a continuous t-norm, then* (X,F,Δ) *is Hausdorff in the topology induced by the family of neighborhoods,*

$$U_x(\epsilon,\lambda,a_1,a_2,\ldots,a_n);x,a_i\in X,\epsilon>0,i=1,2,\ldots,n\in Z^+,$$

where Z^+ *is the set of all positive integers and*

$$\begin{aligned}U_x(\epsilon,\lambda,a_1,a_2,\ldots,a_n)&=\{y\in X;F(x,y,a_i;\epsilon)>1-\lambda,1\leq i\leq n\}\\&=\bigcap_{i=1}^{n}\{y\in X;F(x,y,a_i;\epsilon)>1-\lambda,1\leq i\leq n\}\end{aligned}$$

Definition 2.5. *A 2 N. A. Menger PM-space* (X,F,Δ) *is said to be of type* (C_g), *if there exists a* $g\in\Omega$ *such that*

$$g(F(x,y,z;t))\leq g(F(x,y,a;t))+g(F(x,a,z;t))+g(F(a,y,z;t))$$

for each $x,y,z\in X$, $t\geq 0$, *where*

$$\begin{aligned}\Omega=&\{g/g:[0,1]\to[0,\infty)\text{ is continuous, strictly decreasing and}\\&g(1)=0\text{ and }g(0)<\infty\}.\end{aligned}$$

Definition 2.6. *A 2 N. A. Menger PM-space* (X,F,Δ) *is said to be of type* (D_g), *if there exists a* $g\in\Omega$ *such that*

$$g(\Delta(t_1,t_2,t_3))\leq g(t_1)+g(t_2)+g(t_3)$$

for each $t_1,t_2,t_3\in[0,1]$.

Remark 1. If 2 N. A. Menger PM-space is of type (D_g), then (X, F, Δ) is of type (C_g).

Definition 2.7. *A sequence $\{x_n\}$ in 2 N. A. Menger PM-space (X, F, Δ) converges to x if and only if for each $\epsilon > 0, \lambda > 0$, there exists $M(\epsilon, \lambda)$ such that*

$$g\left(F\left(x_n, x, a; \epsilon\right)\right) < g(1 - \lambda)$$

for every $n > M$.

Definition 2.8. *A sequence $\{x_n\}$ in 2 N. A. Menger PM-space is Cauchy sequence if and only if for each $\epsilon > 0, \lambda > 0$, there exists an integer $M(\epsilon, \lambda)$ such that*

$$g\left(F\left(x_n, x_{n+p}, a; \epsilon\right)\right) < g(1 - \lambda)$$

for every $n, p \geq M$ and $p \geq 1$.

Definition 2.9. *Two self mappings A, S of a 2 N. A. Menger PM-space are said to be compatible if*

$$\lim_n g\left(F\left(ASx_n, SAx_n, a; t\right)\right) = 0$$

for every $t > 0, a \in X$, where $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = z$ *for some $z \in X$.*

Example 1. Let $X = R$ be the set of real numbers equipped with 2-metric defined as

$$d(x, y, z) = \begin{cases} 0 & \text{if at least two of the three points are equal} \\ 2, & \text{otherwise} \end{cases}$$

Set $F(x, y, z; t) = \dfrac{t}{t + d(x, y, z)}$.
Then, (X, F, Δ) is 2 N. A. Menger PM-space with Δ as continuous t-norm satisfying $\Delta(r, s, t) = \min(r, s, t)$ or $pro(r, s, r)$.

Proof. (i) $F(x, y, z; 0) = \dfrac{0}{0 + d(x, y, z)} = 0$.
(ii) and (iii) are trivial.
For (iv) condition, let $F(x, y, s; t_1) = F(x, s, z; t_2) = F(s, y, z; t_3) = 1$, then we have to show that $F(x, y, z; \max\{t_1, t_2, t_3\}) = 1$.
Now, $F(x, y, s; t_1) = 1$ if and only if $\dfrac{t_1}{t_1 + d(x, y, s)} = d(x, y, s) = 0$.
Similarly, $F(x, s, z; t_2) = 1$ if and only if $d(x, s, z) = 0$ and $F(s, y, z; t_3) = 1$ if and only if $d(s, y, z) = 0$.
Now, $d(x, y, z) \leq d(x, y, s) + d(x, s, z) + d(s, y, z) \leq 0 + 0 + 0 = 0$.
Hence, $F(x, y, z; \max\{t_1, t_2, t_3\}) = \dfrac{\max\{t_1, t_2, t_3\}}{\max\{t_1, t_2, t_3\} + 0} = 1$

Now, let us check the last condition, i.e.,

$$F\left(x,y,z;\max\left\{t_1,t_2,t_3\right\}\right) \geq \Delta\left[F\left(x,y,s;t_1\right),F\left(x,s,z;t_2\right),F\left(s,y,z;t_3\right)\right]$$

Let $\max\left\{t_1,t_2,t_3\right\} = T$, then to prove

$$F\left(x,y,z;T\right) \geq \Delta\left[F\left(x,y,s;t_1\right),F\left(x,s,z;t_2\right),F\left(s,y,z;t_3\right)\right]$$

i.e.,

$$\frac{T}{T+d(x,y,z)} \geq \Delta\left[\frac{t_1}{t_1+d(x,y,s)},\frac{t_2}{t_2+d(x,s,z)},\frac{t_3}{t_3+d(s,y,z)}\right]$$

But d can have two values. i.e., either zero or 2. So, the following cases arise;

CASE 1. When every d on the right is zero while d on left may occur with zero or 2. That is, again two subcases as;

Subcase 1. When d on left is 0. Then,

$$\frac{T}{T+0} \geq \Delta\left[\frac{t_1}{t_1},\frac{t_2}{t_2},\frac{t_3}{t_3}\right]$$

That is, $1 \geq \Delta\left[1,1,1\right] = 1$, which is true.

Subcase 2. When d on the left is 2, which is not possible if every d on the right is zero.

CASE 2. When two d's on the right are with zero and one d as 2, i.e., let $d(x,y,s) = 0$, $d(x,s,z) = 0$ and $d(s,y,z) = 2$, then

$$\frac{T}{T+0} \geq \Delta\left[1,1,\frac{t_3}{t_3+2}\right] \text{ or } \frac{T}{T+2} \geq \Delta\left[1,1,\frac{t_3}{t_3+2}\right]$$

which is again true.

CASE 3. When one d on the right is zero and others are 2, then it is again true.

Hence (X,F,Δ) is a 2 N. A. Menger PM-space. □

Example 2. Let $X = R$ with 2-metric defined as

$$d(x,y,z) = \min\left[|x-y|,|y-z|,|z-x|\right]$$

for all $x,y,z \in X$ and $t > 0$.

Define $F(x,y,z;t) = \dfrac{t}{t+d(x,y,z)}$, with $\Delta(r,s,t) = \min\left(r,s,t\right)$ or $r \cdot s \cdot t$.

Then,

(i) $F(x,y,z;0) = \dfrac{0}{0+d(x,y,z)} = 0.$

(ii) and (iii) are trivial.
(iv) Let $F(x, y, s; t_1) = F(x, s, z; t_2) = F(s, y, z; t_3) = 1$.
Then to prove that $F(x, y, z; \max\{t_1, t_2, t_3\}) = 1$.
Now, $F(x, y, s; t_1) = 1$ if and only if $\dfrac{t_1}{t_1 + d(x, y, s)} = 1$ if and only if $d(x, y, s) = 0$.
Also, $F(x, s, z; t_2) = 1$ if and only if $\dfrac{t_2}{t_2 + d(x, s, z)} = 1$ if and only if $d(x, s, z) = 0$.
Similarly, $F(s, y, z; t_3) = 1$ if and only if $\dfrac{t_3}{t_3 + d(s, y, z)} = 1$ if and only if $d(s, y, z) = 0$.
Now,

$$\begin{aligned} d(x, y, z) &\leq d(x, y, s) + d(x, s, z) + d(s, y, z) \\ &= 0 + 0 + 0 = 0 \\ &= 0. \end{aligned}$$

Let $\max\{t_1, t_2, t_3\} = T$.
So,

$$F(x, y, z; \max\{t_1, t_2, t_3\}) = F(x, y, z; T) = \frac{T}{T + d(x, y, z)} = 1$$

Also, we can check

$$F(x, y, z; \max\{t_1, t_2, t_3\}) \geq \Delta\left[F(x, y, s; t_1), F(x, s, z; t_2), F(s, y, z; t_3)\right]$$

Thus, (X, F, Δ) is a 2 N. A. Menger PM-space.

Lemma 1. If A and S are compatible maps of a 2 N. A. Menger PM-space (X, F, Δ), where Δ is continuous and $\Delta(x, x, x) \geq x$ for all $x \in [0, 1]$ and $Ax_n, Sx_n \to z$ for some $z \in X$, where $\{x_n\}$ is a sequence in X, then $SAx_n = Az$ provided A is continuous.

Proof. Suppose A is continuous and $\{x_n\}$ is a sequence in X, such that $\lim_n Ax_n = \lim_n Sx_n = z$ for some $z \in X$.
So, $ASx_n \to Az$ as $n \to \infty$.
Since A and S are compatible maps so,

$$g(F(ASx_n, Az, a; t)) = \lim_n g(F(SAx_n, ASx_n, a; t)) \to 0 \text{ as } n \to \infty,$$

which implies $SAx_n \to Az$. □

Lemma 2. ([9]). Let $\{y_n\}$ be a sequence in Menger PM-space (X, F, Δ), where Δ is a continuous t-norm satisfying $\Delta(x, x) \geq x$ for all $x \in [0, 1]$. If there exists a positive number $q \in (0, 1)$, such that

$$F(y_n, y_{n+1}; qx) \geq F(y_{n-1}, y_n; x), n = 1, 2, 3, \ldots$$

then $\{y_n\}$ is a Cauchy sequence.

Lemma 3. Let $\{y_n\}$ be a sequence in 2 N. A. Menger PM-space (X, F, Δ), where Δ is a continuous t-norm satisfying $\Delta(x, x, x) \geq x$ for all $x \in [0, 1]$. If there exists a positive number $h \in (0, 1)$, such that

$$g\left(F\left(y_n, y_{n+1}, a; ht\right)\right) \leq g\left(F\left(y_{n-1}, y_n, a; t\right)\right), n = 1, 2, 3, \ldots \tag{1}$$

then $\{y_n\}$ is a Cauchy sequence.

Proof. It follows from (1)

$$\begin{aligned} g\left(F\left(y_n, y_{n+1}, a; \frac{(1-h)\epsilon}{2h}\right)\right) &\leq g\left(F\left(y_{n-1}, y_{n-2}, a; \frac{(1-h)\epsilon}{2h^2}\right)\right) \\ &\vdots \quad \vdots \quad \vdots \\ &\leq g\left(F\left(y_2, y_1, a; \frac{(1-h)\epsilon}{2h^{n-1}}\right)\right). \end{aligned}$$

Since, $0 < h < 1$, for $\epsilon > 0, \lambda > 0$, there exists a positive integer N such that

$$g\left(F\left(y_n, y_{n-1}, a; \frac{(1-h)\epsilon}{2h}\right)\right) \leq g\left(1-\lambda\right), \text{ for every } n \geq N \tag{2}$$

That is,

$$F\left(y_n, y_{n-1}, a; \frac{(1-h)\epsilon}{2h}\right) \geq (1-\lambda), \text{ for every } n \geq N$$

(as g strictly decreasing).
It is sufficient to prove that for any positive integer p,

$$g\left(F\left(y_n, y_{n+p}, a; \epsilon\right)\right) \leq g\left(1-\lambda\right), \text{ for every } n \geq N \tag{3}$$

For $p = 1$, (3) holds.
Suppose that (3) holds for $1 < p \leq k$, then we prove (3) for $p = k + 1$.
For this it suffices to show that

$$F\left(y_n, y_{n+p}, a; \epsilon\right) \leq (1-\lambda), \text{ for every } n \geq N \tag{4}$$

As g is strictly decreasing, so using (1),

$$\begin{aligned} F\left(y_n, y_{n+k+1}, a; \epsilon\right) &\geq F\left(y_{n-1}, y_{n+k}, a; \frac{\epsilon}{h}\right) \\ &\geq \Delta\left[F\left(y_{n-1}, y_{n+k}, y_n; \frac{(1-h)\epsilon}{2h}\right),\right. \\ &\quad \left. F\left(y_{n-1}, y_n, a; \frac{(1-h)\epsilon}{2h}\right), F\left(y_n, y_{n+k}, a; \epsilon\right)\right] \\ &> \Delta\left(1-\lambda, 1-\lambda, 1-\lambda\right) \geq 1-\lambda, n \geq N \end{aligned}$$

Hence (4) holds for $p = k + 1$. Thus (3) is proved (as g is strictly decreasing). Therefore, $\{y_n\}$ is a Cauchy sequence. □

In 2001, Ahmad, Ashraf and Rhoades [1] proved the following result;

Theorem 1. *Let (X, D) be a complete D-metric space. Let S be a surjective self-map on X and T an injective self-map of X satisfying the following condition;*
there exists $q > 1$ such that,

$$D(Sx, Sy, Sz) \geq qD(Tx, Ty, Tz)\,, \text{ for all } x, y, z \in X.$$

If S and T commute each other, then there exists a unique common fixed point of S and T.

3. Main Result

Now, we give the analogue of this theorem for compatible maps in the setting of 2 N. A. Menger PM-space as follows.

Theorem 2. *Let S and T be compatible self-maps of a complete 2 N. A. Menger PM-space (X, F, Δ), where Δ is a continuous t-norm satisfying $\Delta(x, x, x) \geq x$ with the following conditions;*

(i) $g(F(Sx, Sy, a; qt)) \geq g(F(Tx, Ty, a; t))$ *for all* $x, y, a \in X$, $t > 0$ *and* $q > 1$.

(ii) *S is surjective*

(iii) *One of S and T is continuous*

Then S and T have a unique common fixed point.

Proof. Let $x_\circ \in X$, since S is surjective, we can choose a point $x_1 \in X$ such that $Sx_1 = Tx_\circ$. Inductively, we can define a sequence such that

$$y_n = Sx_{n+1} = Tx_n \tag{5}$$

Now,

$$\begin{aligned} g(F(y_n, y_{n+1}, a; qt)) &= g(F(Sx_{n+1}, Sx_{n+2}, a; qt)) \\ &\geq g(F(Tx_{n+1}, Tx_{n+2}, a; t)) \\ &= g(F(y_n, y_{n+1}, a; t)) \end{aligned}$$

By Lemma (3), $\{y_n\}$ is a Cauchy sequence. But X is complete and hence $\{y_n\}$ is convergent. Let it converge to z. i.e., $\lim_n y_n = \lim_n Sx_n = \lim_n Tx_n = z$. Now, we suppose that S is continuous. Since S and T are compatible, so, by Lemma (1) $S^2 x_n$ and $TSx_n \to Sz$ as $n \to \infty$.
Using (i), we get

$$g(F(SSx_n, Sx_n, a; qt)) \geq g(F(TSx_n, Tx_n, a; t))\,.$$

Taking $n \to \infty$, we get

$$g\left(F\left(Sz, z, a; qt\right)\right) \geq g\left(F\left(Sz, z, a; t\right)\right)$$

which implies $Sz = z$.
Again by (i), we have

$$g\left(F\left(Sz, Sx_n, a; qt\right)\right) \geq g\left(F\left(Tz, Tx_n, a; t\right)\right)$$

which implies $Tz = z$.
Thus, $z = Sz = Tz$. i.e., z is a common fixed point of S and T.
Let w be another fixed point of S and T, then (i) gives

$$g\left(F\left(Sz, Sw, a; qt\right)\right) \geq g\left(F\left(Tz, Tw, a; t\right)\right)$$

which implies $z = w$. □

Remark 2. We can remove the continuity of maps from Theorem 1 in the form of following result:

Theorem 3. *Let S and T be compatible self-maps of a complete 2 N. A. Menger PM-space (X, F, Δ), where Δ is a continuous t-norm satisfying $\Delta(x, x, x) \geq x$ with the following conditions;*

(i) $g\left(F\left(Sx, Sy, a; qt\right)\right) \geq g\left(F\left(Tx, Ty, a; t\right)\right)$ *for all* $x, y, a \in X$, $t > 0$ *and* $q > 1$.

(ii) S *is surjective*

(iii) If one of the spaces $S(X)$ *or* $T(X)$ *is complete,*

Then S and T have a unique common fixed point.

Proof. Let $x_\circ \in X$, since S is surjective we can choose a point $x_1 \in X$ such that $Sx_1 = Tx_\circ$. Inductively, we can define a sequence $y_n = Sx_{n+1} = Tx_n$
Now,

$$\begin{aligned} g\left(F\left(y_n, y_{n+1}, a; qt\right)\right) &= g\left(F\left(Sx_{n+1}, Sx_{n+2}, a; qt\right)\right) \\ &\geq g\left(F\left(Tx_{n+1}, Tx_{n+2}, a; t\right)\right) \\ &= g\left(F\left(y_n, y_{n+1}, a; t\right)\right) \end{aligned}$$

By Lemma (3), $\{y_n\}$ is a Cauchy sequence. But X is complete and hence $\{y_n\}$ is convergent. Let it converges to z. i.e., $\lim_n y_n = \lim_n Sx_n = \lim_n Tx_n = z$.
If $S(X)$ is complete, then there exists a point $u \in X$ such that $Su = z$.
From (i), we get

$$g\left(F\left(Su, Sx_n, a; qt\right)\right) \geq g\left(F\left(Tu, Tx_n, a; t\right)\right).$$

Taking $n \to \infty$, we get

$$g\left(F\left(Su, z, a; qt\right)\right) \geq g\left(F\left(Tu, z, a; t\right)\right)$$

which implies $Tu = z$. Therefore, $Su = Tu = z$.
Now, S and T are compatible and $Su = Tu$. Hence $Sz = STu = TSu = Tz$. i.e., $Sz = Tz$.
Now, we claim that z is a fixed point of S and T.
Again, by (i), we have

$$g\left(F\left(Sz, Sx_n, a; qt\right)\right) \geq g\left(F\left(Tz, Tx_n, a; t\right)\right)$$

Taking $n \to \infty$, we get

$$g\left(F\left(Sz, z, a; qt\right)\right) \geq g\left(F\left(Tz, z, a; t\right)\right)$$

or

$$g\left(F\left(Sz, z, a; qt\right)\right) \geq g\left(F\left(Sz, z, a; t\right)\right)$$

Thus, $z = Sz = Tz$. i.e., z is a common fixed point of S and T.
Let w be another fixed point of S and T, then (i) gives

$$g\left(F\left(Sz, Sw, a; qt\right)\right) \geq g\left(F\left(Tz, Tw, a; t\right)\right)$$

which implies $z = w$. Hence the theorem is proved. □

Remark 3. Our results extend, generalize and unify the results of various authors mentioned in the introduction of this note in the framework of 2 N. A. Menger PM-space.

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