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GROWTH AND OSCILLATION THEORIES OF DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper we investigate the complex oscillation and the growth of some differential polynomials generated by the solutions of the differential equation

$$f^{''} + A_1(z) f^{'} + A_0(z) f = F,$$

where $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F are meromorphic functions of finite order.

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1. Introduction and main results

Throughout this paper we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [8, 13]). Moreover, we assume some knowledge of the application of Nevanlinna theory into complex differential equations (see[10]). In addition, we will use $\lambda(f)$ and $\lambda(1/f)$ to denote respectively the exponents of convergence of the zero-sequence and the pole-sequence of a meromorphic function f, $\rho(f)$ to denote the order of growth of f, $\overline{\lambda}(f)$ and $\overline{\lambda}(1/f)$ to denote respectively the exponents of convergence of the sequence of distinct zeros and distinct poles of f. A meromorphic function $\varphi(z)$ is called small function of a meromorphic function $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$, where $T(r, f)$ is the Nevanlinna characteristic function of f.

In order to express the rate of growth of meromorphic solutions of infinite order, we recall the following definition.

Definition 1.1. *[5, 14, 16] Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by*

$$\rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r}. \tag{1.1}$$

To give some estimates of fixed points, we recall the following definitions.

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Definition 1.2. *[5, 12, 15] Let f be a meromorphic function. Then the exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by*

$$\overline{\tau}(f) = \overline{\lambda}(f - z) = \overline{\lim_{r\to+\infty}} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}, \tag{1.2}$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$.

Definition 1.3. *[9, 14] Let f be a meromorphic function. Then the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by*

$$\overline{\lambda}_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}. \tag{1.3}$$

Definition 1.4. *[11, 14] Let f be a meromorphic function. Then the hyper-exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by*

$$\overline{\tau}_2(f) = \overline{\lambda}_2(f - z) = \overline{\lim_{r\to+\infty}} \frac{\log\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}. \tag{1.4}$$

Thus $\overline{\tau}_2(f) = \overline{\lambda}_2(f - z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

Consider the second-order linear differential equation

$$f^{''} + A_1(z) f^{'} + A_0(z) f = F, \tag{1.5}$$

where $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F are meromorphic functions of finite order. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [17]). However, there are a few studies on the fixed points of solutions of differential equations. It was in year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see [5]). In [4], Z. X. Chen and K. H. Shon have studied the differential equation

$$f^{''} + A_1(z) e^{az} f^{'} + A_0(z) e^{bz} f = 0 \tag{1.6}$$

and have obtained the following result:

Theorem A. *[4] Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ be meromorphic functions with $\rho(A_j) < 1$ $(j = 0, 1)$, a, b be complex numbers such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every meromorphic solution $f(z) \not\equiv 0$ of the equation (1.6) has infinite order.*

In the same paper, Z. X. Chen and K. H. Shon have investigated the fixed points of solutions, their 1st and 2nd derivatives and the differential polynomials and have obtained the following theorem.

Theorem B. *[4] Let $A_j(z)$ $(j=0,1)$, a,b,c satisfy the additional hypotheses of Theorem A. Let d_0,d_1,d_2 be complex constants that are not all equal to zero. If $f(z)\not\equiv 0$ is any meromorphic solution of equation* (1.6)*, then:*

(i) $f, f^{'}, f^{''}$ all have infinitely many fixed points and satisfy

$$\overline{\tau}(f)=\overline{\tau}\left(f^{'}\right)=\overline{\tau}\left(f^{''}\right)=\infty, \tag{1.7}$$

(ii) the differential polynomial

$$g_f(z)=d_2f^{''}+d_1f^{'}+d_0f \tag{1.8}$$

has infinitely many fixed points and satisfies $\overline{\tau}(g_f)=\infty$.

The first main purpose of this paper is to study the growth, the oscillation and the relation between small functions and differential polynomials generated by solutions of second-order linear differential equation (1.5).

Before we state our results, we denote by

$$\alpha_1=d_1-d_2A_1,\ \beta_0=d_2A_0A_1-(d_2A_0)^{'}-d_1A_0+d_0^{'}, \tag{1.9}$$

$$\alpha_0=d_0-d_2A_0,\ \ \beta_1=d_2A_1^2-(d_2A_1)^{'}-d_1A_1-d_2A_0+d_0+d_1^{'}, \tag{1.10}$$

$$h=\alpha_1\beta_0-\alpha_0\beta_1 \tag{1.11}$$

and

$$\psi=\frac{\alpha_1\left(\varphi^{'}-(d_2F)^{'}-\alpha_1F\right)-\beta_1(\varphi-d_2F)}{h}, \tag{1.12}$$

where $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F, d_j $(j=0,1,2)$ and φ are meromorphic functions of finite order.

Theorem 1.1. *Let $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F be meromorphic functions of finite order. Let $d_0(z), d_1(z), d_2(z)$ be meromorphic functions that are not all equal to zero with $\rho(d_j)<\infty$ $(j=0,1,2)$ such that $h\not\equiv 0$, and let $\varphi(z)$ be a meromorphic function with finite order such that $\psi(z)$ is not a solution of* (1.5).

(i) If f is an infinite order meromorphic solution of (1.5)*, then the differential polynomial $g_f(z)=d_2f^{''}+d_1f^{'}+d_0f$ satisfies*

$$\overline{\lambda}(g_f-\varphi)=\rho(f)=\infty. \tag{1.13}$$

(ii) If f is an infinite order meromorphic solution of (1.5) *with* $\rho_2(f) = \rho$, *then we have*

$$\overline{\lambda}_2(g_f - \varphi) = \rho_2(f) = \rho. \tag{1.14}$$

Theorem 1.2. *Let* $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F *be meromorphic functions of finite order such that all meromorphic solutions of equation* (1.5) *are of infinite order. Let* $d_0(z), d_1(z), d_2(z)$ *be meromorphic functions that are not all equal to zero with* $\rho(d_j) < \infty$ $(j = 0, 1, 2)$ *such that* $h \not\equiv 0$. *Let* φ *be a finite order meromorphic function.*

(i) If f is a meromorphic solution of equation (1.5), *then the differential polynomial* $g_f(z)$ *satisfies* (1.13).

(ii) If f is a meromorphic solution of equation (1.5) *with* $\rho_2(f) = \rho$, *then the differential polynomial* $g_f(z)$ *satisfies* (1.14).

Theorem 1.3. *Let* $A_1(z), A_0(z), F, d_0(z), d_1(z), d_2(z), \varphi$ *satisfy the hypotheses of Theorem 1.1, such that* $A_1(z)$ *or* $A_0(z)$ *is transcendental. If all solutions of equation* (1.5) *are meromorphic, then equation* (1.5) *has meromorphic solution f that* $g_f(z)$ *satisfies* (1.13).

From Theorem 1.2, we obtain the following corollary:

Corollary 1.1. *[3] Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *and* $Q(z) = \sum_{i=0}^{n} b_i z^i$ *be nonconstant polynomials where* a_i, b_i $(i = 0, 1, ..., n)$ *are complex numbers,* $a_n b_n \neq 0$ *such that* $\arg a_n \neq \arg b_n$ *or* $a_n = c b_n$ $(0 < c < 1)$ *and* $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ *be meromorphic functions with* $\rho(A_j) < n$ $(j = 0, 1)$. *Let* $d_0(z), d_1(z), d_2(z)$ *be meromorphic functions that are not all equal to zero with* $\rho(d_j) < n$ $(j = 0, 1, 2)$, *and let* $\varphi(z) \not\equiv 0$ *be a meromorphic function with finite order. If* $f(z) \not\equiv 0$ *is a meromorphic solution of the differential equation*

$$f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = 0, \tag{1.15}$$

then the differential polynomial $g_f(z) = d_2 f'' + d_1 f' + d_0 f$ *satisfies* $\overline{\lambda}(g_f - \varphi) = \infty$.

The other main purpose of this paper is to investigate the relation between infinite order solutions of higher order linear differential equations with meromorphic coefficients and meromorphic functions of finite order. We will prove the following theorem.

Theorem 1.4. *Let* A_0, $A_1, ..., A_{k-1}, F$ *be finite order meromorphic functions, and let* φ *be a finite order meromorphic function which is not a solution of equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_1 f' + A_0 f = F. \tag{1.16}$$

(i) *If f is an infinite order meromorphic solution of the equation* (1.16), *then we have* $\overline{\lambda}(f-\varphi)=\rho(f)=\infty$.

(ii) *If f is an infinite order meromorphic solution of the equation* (1.16) *with* $\rho_2(f)=\rho$, *then we have* $\overline{\lambda}_2(f-\varphi)=\rho_2(f)=\rho$.

Applying Theorem 1.4 for $\varphi(z)=z$, we obtain the following result:

Corollary 1.2. *Let A_0, $A_1, ..., A_{k-1}$, F be finite order meromorphic functions such that $zA_0+A_1\not\equiv F$. Then every infinite order meromorphic solution f of equation* (1.16) *has infinitely many fixed points and satisfies* $\overline{\tau}(f)=\rho(f)=\infty$. *If f is an infinite order meromorphic solution of the equation* (1.16) *with* $\rho_2(f)=\rho$, *then we have* $\overline{\tau}_2(f)=\rho_2(f)=\rho$.

Corollary 1.3. *Let* $P_j(z)=\sum\limits_{i=0}^{n}a_{i,j}z^i$ $(j=0,...,k-1)$ *be nonconstant polynomials where* $a_{0,j},...,a_{n,j}$ $(j=0,1,...,k-1)$ *are complex numbers such that* $a_{n,j}a_{n,0}\neq 0$ $(j=1,...,k-1)$, *let* $A_j(z)(\not\equiv 0)$ $(j=0,...,k-1)$ *be meromorphic functions. Suppose that* $\arg a_{n,j}\neq \arg a_{n,0}$ *or* $a_{n,j}=ca_{n,0}$ $(0<c<1)$ $(j=1,...,k-1)$, $\rho(A_j)<n$ $(j=0,...,k-1)$. *Let $\varphi\not\equiv 0$ be a finite order meromorphic function. Then every meromorphic solution $f(z)\not\equiv 0$ of the equation*

$$(1.17)\quad f^{(k)}+A_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)}+...+A_1(z)e^{P_1(z)}f'+A_0(z)e^{P_0(z)}f=0,$$

where $k\geqslant 2$, satisfies $\overline{\lambda}(f-\varphi)=\rho(f)=\infty$. *In particular, every meromorphic solution $f(z)\not\equiv 0$ of equation* (1.17) *has infinitely many fixed points and satisfies* $\overline{\tau}(f)=\rho(f)=\infty$.

Remark 1.1. *In Theorem 1.1 and Theorem 1.2, if we don't have the condition $h\not\equiv 0$, then the differential polynomial can be of finite order. For example, if $d_2(z)\not\equiv 0$ is finite order meromorphic function and $d_0(z)=A_0d_2(z)$, $d_1(z)=A_1d_2(z)$, then we have $g_f(z)=d_2(z)F$ is of finite order.*

2. Auxiliary Lemmas

Lemma 2.1. *(see [7, p. 412]) Let the differential equation*

$$(2.1)\qquad f^{(k)}+a_{k-1}f^{(k-1)}+...+a_0f=0$$

be satisfied in the complex plane by the linearly independent meromorphic functions $f_1,f_2,...,f_k$. Then the coefficients $a_{k-1},...,a_0$ are meromorphic in the plane with the following properties:

$$(2.2)\qquad m(r,a_j)=O\{\log[\max(T(r,f_s):s=1,...,k)]\}\quad (j=0,...,k-1).$$

Lemma 2.2. *Suppose that $A_0,A_1,...,A_{k-1},F$ are meromorphic functions with at least one A_s $(0\leq s\leq k-1)$ being transcendental. If all solutions of*

$$(2.3)\qquad f^{(k)}+A_{k-1}f^{(k-1)}+...+A_1f'+A_0f=F$$

are meromorphic, then (2.3) *has an infinite order solution.*

Proof. Since all the solutions of (2.3) are meromorphic, all the solutions of the corresponding homogeneous differential equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_1 f^{'} + A_0 f = 0 \tag{2.4}$$

of (2.3) are meromorphic. Now assume that $\{f_1, ..., f_k\}$ is a fundamental solution set of (2.4). Then by Lemma 2.1, we have for $j = 0, ..., k-1$

$$m(r, A_j) = O\{\log[\max(T(r, f_s) : s = 1, ..., k)]\}. \tag{2.5}$$

Since A_s is transcendental, at least one of $f_1, ..., f_k$ is of infinite order of growth. Suppose f_1 satisfies $\rho(f_1) = \infty$.

If f_0 is a solution of (2.3), then every solution f of (2.3) can be written in the form

$$f = C_1 f_1 + C_2 f_2 + ... + C_k f_k + f_0, \tag{2.6}$$

where $C_1, C_2, ..., C_k$ are arbitrary constants. If $\rho(f_0) = \infty$, then Lemma 2.2 holds. If $\rho(f_0) < \infty$, then $f = f_1 + f_0$ is a meromorphic solution of (2.3) and $\rho(f) = \infty$. □

Lemma 2.3. *[6] Let A_0, $A_1, ...,$ A_{k-1}, $F(\not\equiv 0)$ be finite order meromorphic functions. If f is a meromorphic solution with $\rho(f) = +\infty$ of the equation* (2.3), *then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.*

Lemma 2.4. *[2] Let A_0, $A_1, ...,$ A_{k-1}, $F(\not\equiv 0)$ be finite order meromorphic functions. If f is a meromorphic solution of the equation* (2.3) *with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho$.*

Lemma 2.5. *Suppose that $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F are meromorphic functions of finite order. Let $d_0(z), d_1(z), d_2(z)$ be meromorphic functions that are not all equal to zero with $\rho(d_j) < \infty$ $(j = 0, 1, 2)$ such that $h \not\equiv 0$, where h is defined in* (1.11).

(*i*) *If f is an infinite order meromorphic solution of* (1.5), *then the differential polynomial $g_f(z) = d_2 f^{''} + d_1 f^{'} + d_0 f$ satisfies*

$$\rho(g_f) = \rho(f) = \infty. \tag{2.7}$$

(*ii*) *If f is an infinite order meromorphic solution of* (1.5) *with $\rho_2(f) = \rho$, then the differential polynomial $g_f(z) = d_2 f^{''} + d_1 f^{'} + d_0 f$ satisfies*

$$\rho_2(g_f) = \rho_2(f) = \rho. \tag{2.8}$$

Proof. (i) Suppose that f is a meromorphic solution of equation (1.5) with $\rho(f) = \infty$. Substituting $f^{''} = F - A_1 f^{'} - A_0 f$ into g_f, we get

$$g_f - d_2 F = (d_1 - d_2 A_1) f^{'} + (d_0 - d_2 A_0) f. \tag{2.9}$$

Differentiating both sides of equation (2.9) and replacing $f^{''}$ with $f^{''} = F - A_1 f^{'} - A_0 f$, we obtain

$$g_f^{'} - (d_2 F)^{'} - (d_1 - d_2 A_1) F = \left[d_2 A_1^2 - (d_2 A_1)^{'} - d_1 A_1 - d_2 A_0 + d_0 + d_1^{'}\right] f^{'}$$

$$+ \left[d_2 A_0 A_1 - (d_2 A_0)^{'} - d_1 A_0 + d_0^{'}\right] f. \tag{2.10}$$

Set

$$\alpha_1 = d_1 - d_2 A_1, \quad \alpha_0 = d_0 - d_2 A_0, \tag{2.11}$$

$$\beta_1 = d_2 A_1^2 - (d_2 A_1)^{'} - d_1 A_1 - d_2 A_0 + d_0 + d_1^{'}, \tag{2.12}$$

$$\beta_0 = d_2 A_0 A_1 - (d_2 A_0)^{'} - d_1 A_0 + d_0^{'}. \tag{2.13}$$

Then, we have

$$\alpha_1 f^{'} + \alpha_0 f = g_f - d_2 F, \tag{2.14}$$

$$\beta_1 f^{'} + \beta_0 f = g_f^{'} - (d_2 F)^{'} - (d_1 - d_2 A_1) F. \tag{2.15}$$

Set

$$h = \alpha_1 \beta_0 - \alpha_0 \beta_1 = (d_1 - d_2 A_1) \left(d_2 A_0 A_1 - (d_2 A_0)^{'} - d_1 A_0 + d_0^{'}\right)$$

$$- (d_0 - d_2 A_0) \left(d_2 A_1^2 - (d_2 A_1)^{'} - d_1 A_1 - d_2 A_0 + d_0 + d_1^{'}\right). \tag{2.16}$$

By $h \not\equiv 0$ and $(2.14) - (2.16)$, we obtain

$$f = \frac{\alpha_1 \left(g_f^{'} - (d_2 F)^{'} - \alpha_1 F\right) - \beta_1 (g_f - d_2 F)}{h}. \tag{2.17}$$

If $\rho(g_f) < \infty$, then by (2.17) we get $\rho(f) < \infty$ and this is a contradiction. Hence $\rho(g_f) = \infty$.

(ii) Suppose that f is a meromorphic solution of equation (1.5) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$. Then, by (2.9) we get $\rho_2(g_f) \leqslant \rho_2(f)$ and by (2.17) we have $\rho_2(f) \leqslant \rho_2(g_f)$. Thus $\rho_2(g_f) = \rho_2(f) = \rho$. $\square$

Lemma 2.6. *[1] Let* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, ..., k-1)$ *be nonconstant polynomials where* $a_{0,j}, ..., a_{n,j}$ $(j = 0, 1, ..., k-1)$ *are complex numbers such that* $a_{n,j} a_{n,0} \neq 0$ $(j = 1, ..., k-1)$, *let* $A_j(z) (\not\equiv 0)$ $(j = 0, ..., k-1)$ *be meromorphic functions. Suppose that* $\arg a_{n,j} \neq \arg a_{n,0}$ *or* $a_{n,j} = c a_{n,0}$ $(0 < c < 1)$ $(j = 1, ..., k-1)$, $\rho(A_j) < n$ $(j = 0, ..., k-1)$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of the equation*

$$f^{(k)} + A_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + A_1(z) e^{P_1(z)} f' + A_0(z) e^{P_0(z)} f = 0, \tag{2.18}$$

where $k \geq 2$, *is of infinite order.*

3. Proof of Theorem 1.1

(i) Suppose that f is a meromorphic solution of equation (1.5) with $\rho(f) = \infty$. Set $w(z) = d_2 f'' + d_1 f' + d_0 f - \varphi$. Since $\rho(\varphi) < \infty$, then by Lemma 2.5 (i) we have $\rho(w) = \rho(g_f) = \rho(f) = \infty$. In order to prove $\overline{\lambda}(g_f - \varphi) = \infty$, we need to prove only $\overline{\lambda}(w) = \infty$. By $g_f = w + \varphi$, we get from (2.17)

$$f = \frac{\alpha_1 w' - \beta_1 w}{h} + \psi, \tag{3.1}$$

where α_1, β_1, h, ψ are defined in $(1.9) - (1.12)$. Substituting (3.1) into equation (1.5), we obtain

$$\frac{\alpha_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w$$

$$= F - \left(\psi'' + A_1(z)\psi' + A_0(z)\psi\right) = A, \tag{3.2}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho(\phi_j) < \infty$ $(j = 0, 1, 2)$. Since $\psi(z)$ is not a solution of (1.5), it follows that $A \not\equiv 0$ and by Lemma 2.3, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, i.e., $\overline{\lambda}(g_f - \varphi) = \infty$.

(ii) Suppose that f is a meromorphic solution of equation (1.5) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$. Then, by Lemma 2.5 (ii) we have $\rho_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = \rho$. Using a similar reasoning to that above and by Lemma 2.4, we get (1.14).

4. Proof of Theorem 1.2

By the hypotheses of Theorem 1.2 all meromorphic solutions of equation (1.5) are of infinite order. From (1.12), we see that $\psi(z)$ is a meromorphic function of finite order, then $\psi(z)$ is not a solution of (1.5). By Theorem 1.1, we obtain Theorem 1.2.

5. Proof of Theorem 1.3

By Lemma 2.2, we know that equation (1.5) has an infinite order meromorphic solution f. Then, by Theorem 1.1, $g_f(z)$ satisfies (1.13).

6. Proof of Theorem 1.4

(i) Let f be an infinite order meromorphic solution of equation (1.16). Set $w = f - \varphi$. Then, we have $\rho(w) = \rho(f-\varphi) = \rho(f) = \infty$. Substituting $f = w + \varphi$ into equation (1.16), we obtain

$$\begin{aligned} & w^{(k)} + A_{k-1}w^{(k-1)} + ... + A_1 w^{'} + A_0 w \\ = {} & F - \left(\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + ... + A_1\varphi^{'} + A_0\varphi\right) = W. \end{aligned}$$

Since φ is not a solution of equation (1.16), then we have $W \not\equiv 0$. By Lemma 2.3, we get $\overline{\lambda}(w) = \overline{\lambda}(f-\varphi) = \rho(w) = \rho(f-\varphi) = \infty$.

(ii) Suppose that f is a meromorphic solution of equation (1.16) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$. By using a similar reasoning to that above and Lemma 2.4, we obtain $\overline{\lambda}_2(f-\varphi) = \rho_2(f) = \rho$.

7. Proof of Corollary 1.3

Suppose that $f(z) \not\equiv 0$ is a meromorphic solution of equation (1.17). Then by Lemma 2.6, we have $\rho(f) = \infty$. Using Theorem 1.4 (i), we obtain Corollary 1.3.

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