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INTERVAL BASIC ALGEBRAS[1]

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Abstract. We show that for any basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ and elements $a, b \in A$, there can be introduced operations $\oplus_a$, $\neg_a$ and $\oplus^b$, $\neg^b$ such that $([a, 1]; \oplus_a, \neg_a, a)$ and $([0, b]; \oplus^b, \neg^b, 0)$ are basic algebras again. It is shown that the interval basic algebras on a given basic algebra $\mathcal{A}$ satisfy the so-called patchwork condition.

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By a **basic algebra** (see e.g.[1, 2]) is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following axioms

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$ (double negation);

(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Łukasiewicz axiom);

(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

As usual, we will denote $\neg 0$ by 1. We will need the following two results from [1].

Proposition 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Then*

(1) the relation $\leq$ defined by

$$x \leq y \quad \textit{iff} \quad \neg x \oplus y = 1$$

is a partial order on A such that 0 and 1 are the least and greatest element of A, respectively;

(2) $x \leq y$ iff $\neg y \leq \neg x$;

(3) $x \leq y$ implies $x \oplus z \leq y \oplus z$;

(4) $y \leq x \oplus y$;

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(5) $0 \oplus x = x$.

Call $\leq$ the **induced order** of $\mathcal{A}$.

Proposition 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $\leq$ the induced order. Then $(A; \leq)$ is a lattice where*

$$x \vee y = \neg(\neg x \oplus y) \oplus y \qquad \text{and} \qquad x \wedge y = \neg(\neg x \vee \neg y).$$

Hence $\vee, \wedge$ are term operations of $\mathcal{A}$.

The concept of interval MV-algebra was introduced in [3] and for GMV-algebras (alias pseudo MV-algebras) in [4, 6]. For BL-algebras and pseudo BL-algebras, interval algebras were treated in [5]. Since basic algebras are generalizations of MV-algebras, it is a natural question how the operations can be defined on a given interval to obtain a basic algebra again. This is answered in the paper.

Theorem 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $\leq$ the induced order and $a \in A$. Define the operations $\neg_a$ and $\oplus_a$ on the interval $[a, 1]$ as follows:*

$$\neg_a x = \neg x \oplus a, \qquad x \oplus_a y = \neg(\neg x \oplus a) \oplus y.$$

Then $([a, 1]; \oplus_a, \neg_a, a)$ is a basic algebra.

Proof. If $x, y \in [a, 1]$ then $a \leq y \leq \neg(\neg x \oplus a) \oplus y = x \oplus_a y$ thus $\oplus_a$ is really binary operation on $[a, 1]$. Since $a \leq \neg x \oplus a$, $\neg_a x$ is a unary operation on $[a, 1]$. Moreover, $\neg_a a = \neg a \oplus a = 1$ and $\neg_a 1 = \neg 1 \oplus a = 0 \oplus a = a$. We must check the axioms (BA1)–(BA4).
(BA1) and (BA2): For $x \in [a, 1]$ we have $x \oplus_a a = \neg(\neg x \oplus a) \oplus a = x \vee a = x$ and, analogously, $\neg_a \neg_a x = \neg(\neg x \oplus a) \oplus a = x \vee a = x$.
(BA3): Assume $x, y \in [a, 1]$. Since $y \leq \neg x \oplus y$, thus also $a \leq \neg x \oplus y$. Further, we have $\neg_a x \oplus_a y = \neg x \oplus y$. Hence, we compute

$$\begin{aligned}\neg_a(\neg_a x \oplus_a y) \oplus_a y &= \neg_a(\neg(\neg(\neg x \oplus a) \oplus a) \oplus y) \oplus_a y = \\ &= \neg_a(\neg(x \vee a) \oplus y) \oplus_a y = \neg_a(\neg x \oplus y) \oplus_a y = \\ &= \neg(\neg x \oplus y) \oplus y = x \vee y\end{aligned}$$

and, by symmetry, also

$$\neg_a(\neg_a y \oplus_a x) \oplus_a x = y \vee x = x \vee y.$$

(BA4): Let $x, y, z \in [a, 1]$. Since $a \leq x \oplus_a y$, $a \leq y \leq \neg(x \oplus_a y) \oplus y$ and

$\oplus$	p	a	b	1
p	p	a	b	1
a	a	1	b	1
b	b	a	1	1
1	1	1	1	1

x	p	a	b	1
$\neg x$	1	a	b	p

Tab. 1

$a \leq z \leq \neg(\neg(x \oplus_a y) \oplus y) \oplus z$, we obtain

$$\begin{aligned}\neg_a(\neg_a(\neg_a(x \oplus_a y) \oplus_a y) \oplus_a z) \oplus_a (x \oplus_a z) &= \\ &= \neg_a(\neg_a(\neg(x \oplus_a y) \oplus y) \oplus_a z) \oplus_a (x \oplus_a z) = \\ &= \neg_a(\neg(\neg(x \oplus_a y) \oplus y) \oplus z) \oplus_a (x \oplus_a z) = \\ &= \neg(\neg(\neg(x \oplus_a y) \oplus y) \oplus z) \oplus (x \oplus_a z) = \\ &= \neg(\neg(\neg(w \oplus y) \oplus y) \oplus z) \oplus (w \oplus z) = 1,\end{aligned}$$

where $w = \neg(\neg x \oplus a)$. □

Since every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is a lattice (with respect to the induced order) and for any interval $[a, 1]$ we can derive the operations $\neg_a$ and $\oplus_a$ by means of the operations $\neg$ and $\oplus$ of $\mathcal{A}$ to reach the interval algebra, we consider the situation when $a, b \in A$ with $a \leq b$ and ask what is the relationship between the coresponding operations $\neg_a$ and $\neg_b$ or $\oplus_a$ and $\oplus_b$. By Theorem 1, $([a, 1]; \oplus_a, \neg_a, a)$ is a basic algebra again and $[b, 1]$ is an interval within $[a, 1]$. Hence, we have the following conclusion

Corollary. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $a, b \in A$ and $a \leq b$. Then for $x, y \in [b, 1]$ we have*

(PC) $\qquad \neg_b x = \neg_a x \oplus_a b \quad$ *and* $\quad x \oplus_b y = \neg_a(\neg_a x \oplus_a b) \oplus_a y.$

Since the whole lattice induced by $\mathcal{A} = (A; \oplus, \neg, 0)$ is patched up from its intervals, we can call (PC) the **patchwork condition**.

Now, we can consider the converse problem. Let $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ be a bounded lattice and on intervals $[a, 1]$ (not necessary for all $a \in L$) are defined basic algebras $([a, 1]; \oplus_a, \neg_a, a)$. Do there exist operations $\oplus$ and $\neg$ such that $(L; \oplus, \neg, 0)$ is a basic algebra and $([a, 1]; \oplus_a, \neg_a, a)$ are its interval basic algebras? The answer is straightforward. Of course, $\oplus$ must be $\oplus_0$, $\neg$ must be $\neg_0$ and, due to Theorem 1, the patchwork condition must be satisfied. This is illustrated by the following example.

Example. Consider the basic algebra $\mathcal{A}_p = (A; \oplus, \neg, p)$ with four elements whose tables are in Tab. 1. The induced lattice is depicted in Fig. 1.

Now, consider the lattice as shown in Fig. 2. We can define the operations

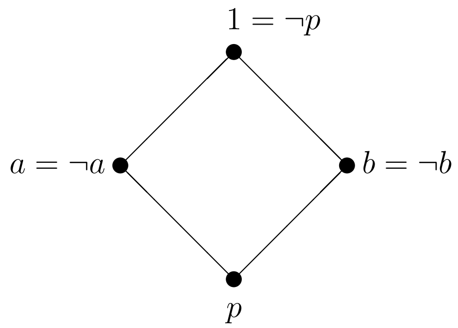

Fig. 1

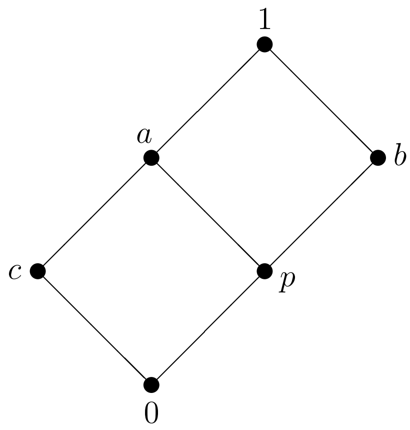

Fig. 2

$\oplus$	0	c	p	a	b	1
0	0	c	p	a	b	1
c	c	c	a	a	1	1
p	p	a	b	1	b	1
a	a	a	1	1	1	1
b	b	1	b	1	b	1
1	1	1	1	1	1	1

x	0	c	p	a	b	1
$\neg x$	1	b	a	p	c	0

Tab. 2

$\oplus$	0	c	p	a	b	1
0	0	c	p	a	b	1
c	c	c	b	a	1	1
p	p	a	a	1	b	1
a	a	a	1	1	1	1
b	b	1	a	1	b	1
1	1	1	1	1	1	1

Tab. 3

$\oplus, \neg$ as in Tab. 2. Then for $A = \{0, c, p, a, b, 1\}$ we have that $\mathcal{A} = (A; \oplus, \neg, 0)$ is a basic algebra (which is even an MV-algebra). However, the previous basic algebra $\mathcal{A}_p$ is not its interval algebra since the patchwork condition does not hold:

$$a \oplus_p b = \neg(\neg a \oplus p) \oplus b = \neg(p \oplus p) \oplus b = c \oplus b = 1 \text{ in } \mathcal{A} \text{ but}$$

$$a \oplus b = b \text{ in } \mathcal{A}_p.$$

On the contrary, if the operation $\oplus$ on A is defined as in Tab. 3 (and the operation $\neg$ is the same as in Tab. 2) then we obtain a basic algebra again but the patchwork condition is satisfied. For example, for the above elements we have

$$a \oplus_p b = \neg(\neg a \oplus p) \oplus b = \neg(p \oplus p) \oplus b = p \oplus b = b.$$

Now, we can ask about operations on the lower interval $[0, b]$ of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ to obtain another type of an interval basic algebra. It turns out that this is possible and, similarly as for upper intervals $[a, 1]$, also here the operations can be expressed as polynomial operations of $\mathcal{A}$.

Theorem 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $\leq$ the induced order and $b \in A$. Define the operations $\neg^b$, $\oplus^b$ on the interval $[0, b]$ as follows:*

$$\neg^b x = \neg(x \oplus \neg b), \qquad x \oplus^b y = \neg(\neg(x \oplus (y \oplus \neg b)) \oplus \neg b).$$

Then $([0, b]; \oplus^b, \neg^b, 0)$ is a basic algebra.

Proof. Clearly, $\neg(x \oplus^b y) = \neg(x \oplus (y \oplus \neg b)) \oplus \neg b \geq \neg b$ thus $x \oplus^b y \leq b$, i.e. for $x, y \in [0, b]$ we have $x \oplus^b y \in [0, b]$ thus $\oplus^b$ is correctly defined on $[0, b]$. The unary operation $\neg^b x$ is also well-defined, because $\neg b \leq x \oplus \neg b$ yields $\neg^b x = \neg(x \oplus \neg b) \leq b$. Further, we can see that $\neg^b 0 = b$ and $\neg^b b = 0$. Now, we check the axioms of basic algebras.
(BA1): $x \oplus^b 0 = \neg(\neg(x \oplus (0 \oplus \neg b)) \oplus \neg b) = \neg(\neg(\neg\neg x \oplus \neg b) \oplus \neg b) = \neg(\neg x \vee \neg b) = x \wedge b = x$ for $x \in [0, b]$.
(BA2): $\neg^b \neg^b x = \neg^b(\neg(x \oplus \neg b)) = \neg(\neg(\neg\neg x \oplus \neg b) \oplus \neg b) = \neg(\neg x \vee \neg b) = x \wedge b = x$ for $x \in [0, b]$.

For the proof of (BA3) and (BA4), we will firstly prove the following claims.

Claim 1. *Let* $x, y \in [0, b]$, *then* $\neg^b(x \oplus^b y) = \neg(x \oplus (y \oplus \neg b))$.

Proof. Since $x \oplus (y \oplus \neg b) \geq y \oplus \neg b \geq \neg b$, by (4) of Proposition 1, we have $\neg(x \oplus (y \oplus \neg b)) \leq b$. Hence, we compute

$$\begin{aligned}\neg^b(x \oplus^b y) &= \neg(\neg(\neg(x \oplus (y \oplus \neg b)) \oplus \neg b) \oplus \neg b) = \\ &= \neg(x \oplus (y \oplus \neg b)) \wedge b = \neg(x \oplus (y \oplus \neg b)).\end{aligned}$$

□

Claim 2. *Let* $x, y, z \in [0, b]$. *If* $x \leq y$ *then* $x \oplus^b z \leq y \oplus^b z$.

Proof. If $x \leq y$ then, by (2) and (3) of Proposition 1,

$$x \oplus (z \oplus \neg b) \leq y \oplus (z \oplus \neg b)$$

$$\neg(x \oplus (z \oplus \neg b)) \geq \neg(y \oplus (z \oplus \neg b))$$

$$\neg(x \oplus (z \oplus \neg b)) \oplus \neg b \geq \neg(y \oplus (z \oplus \neg b)) \oplus \neg b$$

$$\neg(\neg(x \oplus (z \oplus \neg b)) \oplus \neg b) \leq \neg(\neg(y \oplus (z \oplus \neg b)) \oplus \neg b)$$

$$x \oplus^b z \leq y \oplus^b z.$$

□

Claim 3. *Let* $x, y \in [0, b]$. *If* $x \leq y$ *then* $\neg^b x \oplus^b y = b$.

Proof. By Proposition 1, if $x \leq y$ then $x \oplus \neg b \leq y \oplus \neg b$ thus $\neg(x \oplus \neg b) \oplus (y \oplus \neg b) = 1$. Hence, $\neg^b x \oplus^b y = \neg(\neg(\neg(x \oplus \neg b) \oplus (y \oplus \neg b)) \oplus \neg b) = \neg(\neg 1 \oplus \neg b) = \neg(0 \oplus \neg b) = \neg\neg b = b$. □

Now, we are ready to prove the remaining axioms.
(BA3): Using Claim 1, we compute

$$\begin{aligned}\neg^b(\neg^b x \oplus^b y) \oplus^b y &= \neg(\neg^b x \oplus (y \oplus \neg b)) \oplus^b y = \\ &= \neg(\neg(x \oplus \neg b) \oplus (y \oplus \neg b)) \oplus^b y = \\ &= \neg(\neg(\neg(\neg(x \oplus \neg b) \oplus (y \oplus \neg b)) \oplus (y \oplus \neg b)) \oplus \neg b).\end{aligned}$$

Analogously,

$$\neg^b(\neg^b y \oplus^b x) \oplus^b x = \neg(\neg(\neg(\neg(y \oplus \neg b) \oplus (x \oplus \neg b)) \oplus (x \oplus \neg b)) \oplus \neg b).$$

Since the algebra $\mathcal{A}$ satisfies (BA3), we conclude

$$\neg^b(\neg^b x \oplus^b y) \oplus^b y = \neg^b(\neg^b y \oplus^b x) \oplus^b x.$$

(BA4): We apply Claim 1 to reduce

$$\neg^b(\neg^b(x \oplus^b y) \oplus^b y) = \neg^b(\neg(x \oplus (y \oplus \neg b)) \oplus^b y) =$$

$$= \neg(\neg(x \oplus (y \oplus \neg b)) \oplus (y \oplus \neg b)) = \neg(\neg x \vee (y \oplus \neg b)).$$

Thus,

$$\neg^b(\neg^b(\neg^b(x \oplus^b y) \oplus^b y) \oplus^b z) \oplus^b (x \oplus^b z) =$$

$$(*) \qquad = \neg^b(\neg(\neg x \vee (y \oplus \neg b)) \oplus^b z) \oplus^b (x \oplus^b z).$$

Then $\neg x \vee (y \oplus \neg b) \geq \neg x$ implies $\neg(\neg x \vee (y \oplus \neg b)) \leq \neg\neg x = x$ and, by Claim 2, we have

$$(**) \qquad \neg(\neg x \vee (y \oplus \neg b)) \oplus^b z \leq x \oplus^b z.$$

Applying $(**)$ in $(*)$ and using Claim 3, we infer

$$\neg^b(\neg(\neg x \vee (y \oplus \neg b)) \oplus^b z) \oplus^b (x \oplus^b z) = b$$

whence (BA4) is evident. □

Finally, we can ask whether every interval $[a, b]$ of a basic algebra $\mathcal{A}$ can be made into a basic algebra again. The answer is as follows.

Theorem 3. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $\leq$ its induced order and $a, b \in A$ with $a \leq b$. Define the operations $\neg_a^b$, $\oplus_a^b$ on the interval $[a, b]$ as follows:*

$$\neg_a^b x = \neg(\neg(\neg x \oplus a) \oplus (\neg b \oplus a)) \oplus a,$$
$$x \oplus_a^b y = \neg(\neg(\neg(\neg(\neg(\neg x \oplus a) \oplus (\neg(\neg y \oplus a) \oplus (\neg b \oplus a))) \oplus a) \oplus a) \oplus (\neg b \oplus a)) \oplus a.$$

Then $\mathcal{A}(a, b) = ([a, b]; \oplus_a^b, \neg_a^b, a)$ is a basic algebra.

Proof. By Theorem 1, $([a, 1]; \oplus_a, \neg_a, a)$ is a basic algebra and $b \in [a, 1]$. Hence, we can apply Theorem 2 to operations $\oplus_a$ and $\neg_a$ to obtain

$$\neg_a^b x = \neg_a(x \oplus_a \neg_a b) = \neg(x \oplus_a \neg_a b) \oplus a = \neg(x \oplus_a (\neg b \oplus a)) \oplus a =$$
$$= \neg(\neg(\neg x \oplus a) \oplus (\neg b \oplus a)) \oplus a.$$

For $\oplus_a^b$ we have

$$x \oplus_a^b y = \neg_a(\neg_a(x \oplus_a (y \oplus_a \neg_a b)) \oplus_a \neg_a b) =$$
$$= \neg_a(\neg_a(x \oplus_a (\neg(\neg y \oplus a) \oplus (\neg b \oplus a))) \oplus_a \neg_a b) =$$
$$= \neg_a((\neg(\neg(\neg x \oplus a) \oplus (\neg(\neg y \oplus a) \oplus (\neg b \oplus a))) \oplus a) \oplus_a \neg_a b) =$$
$$= \neg(\neg(\neg(\neg(\neg(\neg x \oplus a) \oplus (\neg(\neg y \oplus a) \oplus (\neg b \oplus a))) \oplus a) \oplus a) \oplus (\neg b \oplus a)) \oplus a.$$

□

Remark. Dually as in the proof of Theorem 3, we can firstly consider the interval algebra on $[0, b]$ and, since $a \in [0, b]$, we can define the operations $\oplus_a^b$ and $\neg_a^b$ as follows

$$\neg_a^b x = \neg^b x \oplus^b a \qquad \text{and} \qquad x \oplus_a^b y = \neg^b(\neg^b x \oplus^b a) \oplus^b y.$$

These operations differ from that of Theorem 3.

References

[1] Chajda I., Halaš R., Kühr J., Semilattice Structures. Heldermann Verlag, Lemgo (Germany), 2007, ISBN 978-3-88538-230-0.

[2] Chajda I., Kolařík M., Independence of axiom system of basic algebras. Soft Computing13, 1 (2009), 41–43.

[3] Chajda I., Kühr J., A note on interval MV-algebras. Math. Slovaca 56 (2006), 47–52.

[4] Chajda I., Kühr J., GMV-algebras and meet-semilattices with sectionally antitone permutations. Math. Slovaca 56 (2006), 275–288.

[5] Dvurečenskij A., Hyčko M., Algebras on subintervals of BL-algebras, pseudo BL-algebras and bounded residuated ℓ-monoids. Math. Slovaca 56 (2006), 125–144.

[6] Jakubík J., On intervals and the dual of pseudo MV-algebras. Math. Slovaca 56 (2006), 213–221.

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