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A NEW CHARACTERIZATION OF $PSL_2(7)$[1]

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Abstract. Let G be a group and $\tau_e(G)$ the set of numbers of elements of G of the same order. In this note it is proved that a group G is isomorphic to $PSL_2(7)$ if and only if $\tau_e(G) = \{1, 21, 56, 42, 48\}$.

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1. Introduction

Let G be a group. The set of orders of elements of G and one of numbers of elements of G of the same order are denoted by $\pi_e(G)$ and $\tau_e(G)$, respectively. Let $\pi(G)$ be the set of prime divisors of $|G|$ if G is finite. In 1980s, J.G. Thompson posed a very interesting problem related to algebraic number fields as follows (see Problem 12.37 of [1]),

Problem. *Let $T(G) = \{(n, s_n) \mid n \in \pi_e(G)$ and $s_n \in \tau_e(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$. If G is solvable, is it true that H is also necessarily solvable?*

In the paper [2], W. Shi studied the case of the simple group $PSL_2(7)$ of above Thompson Problem. Even if the restricted condition $\tau_e(G)$ is removed, he also proves a strong result that a finite group G is isomorphic to $PSL_2(7)$ if and only if $\pi_e(G) = \{1, 2, 3, 4, 7\}$. Can the word 'finite' of this result be removed? But it still remains an open research problem (see Problem 16.57 of [1]). As this motivation, we will study on the influence of the condition $\tau_e(G)$ on the group structure. In this note we also get a parallel result as Shi's for the case of $PSL_2(7)$. Also, the word 'finite' can be left out. Our main result is the following.

Theorem. *A group G is isomorphic to $PSL_2(7)$ if and only if $\tau_e(G) = \{1, 21, 56, 42, 48\}$.*

Before starting the proof of theorem, we will mention a well-known result of Frobenius (see [3]), which is quoted frequently in the sequel.

Lemma. *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

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2. Proof of Theorem

Let $\tau_e(G)$ be the set $\{1, 21, 56, 42, 48\}$ and s_m number of elements of order m. We divide it into seven assertions to complete the proof.

1. G is finite.

G is obvious a periodic group. Since each prime divisor p of $|G|$ is less than 56, we have $|\pi(G)| < 56$. Also, if $p^s \in \pi_e(G)$, then the number of elements of order p^s is a multiple of $\phi(p^s)$, so we have $\phi(p^s) < 56$ (where $\phi(n)$ Euler totient function), this can lead to $|\pi_e(G)| < \infty$. Therefore, $|G| < 56|\pi_e(G)|$.

2. $\pi(G) \subseteq \{2, 3, 7\}$.

Note that $s_m = k\phi(m)$, where k is the number of cyclic subgroups of order m and $\phi(m)$ Euler totient function. If $m > 2$, then $\phi(m)$ is always even, so we have $s_2 = 21$ and $2 \in \pi(G)$. Obviously, 5 does not belong to $\pi(G)$. Otherwise, 5 divides $1+s_5$ for some $s_5 \in \{56, 42, 48\}$ by the Lemma, it is impossible. Suppose that there is a prime $p > 7$ and $p \in \pi(G)$. Use the same result of the Lemma, and we will get $p \mid 1 + s_p$ for some $s_p \in \{56, 42, 48\}$, thus a possible value of p is 43. On the other hand, if $43 \in \pi(G)$, then G has no order 86 element. In fact, if $86 \in \pi_e(G)$, then $86 \mid 1 + s_2 + s_{43} + s_{86}$. Also $s_{86} = 42$ since $\phi(86) \mid s_{86}$, thus $86 \mid 106$, contradicts. Now we consider the subgroup N of order 43. Clearly, it is normal in G. Let t be an element of order 2, then the group $N\langle t\rangle$ is Frobenius, which has 43 elements of order 2 exactly. It contradicts the fact that $s_2 = 21$. Also, if 3 or 7 is in $\pi(G)$, again by the Lemma it is easy to see $s_3 = 56$ and $s_7 = 48$.

3. 21 is not in $\pi_e(G)$, and if $3 \mid |G|$, then $3 \| \, |G|$.

If $21 \in \pi_e(G)$, then $s_{21} = 48$ since $\phi(21) = 12$. By the Lemma, we have $21 \mid 1 + s_3 + s_7 + s_{21} = 156$, it is impossible. Now, consider Sylow 3-subgroup P_3 acts fixed point freely on the set of elements of order 7, and we will get $|P_3| \mid s_7(= 48)$. Hence we must have $|P_3| = 3$. If 7 is not in $\pi(G)$, then there does not exist an element of order 27 since $\phi(27) = 18$, which does not divide one of 56, 42 and 48. If $9 \in \pi_e(G)$, again by the Lemma, then we have $9 \mid 1 + s_3 + s_9 = 57 + s_9$, so $s_9 = 42$. This can imply that the number of 3-elements of G is $1 + s_3 + s_9 = 99$. Clearly, the Sylow 3-subgroup P_3 of G is not normal. Denote by k the number of Sylow 3-subgroups, and we will get $k \equiv 1(\text{mod}3)$. Let $|P_3| = 3^n$. Then the number of 3-elements l of G is not less than $3^n + (k-1)(3^n - 3^{n-1})$. Also if $n \geq 4$, or $n \geq 3$ and $k > 4$, then $l > 99$, which is not possible. If $n = 2$ and $k = 4$, then $l \leq k(3^n - 1) + 1 < 99$, which is also impossible. For the remaining case $n = 3$ and $k = 4$, we combine 4 Sylow 3-subgroups subject to $l = 99$, and then the only possibility is that the order of intersection of every two Sylow 3-subgroup is 3. Since $9 \in \pi_e(G)$

and 27 is not in, we must have $P_3 \cong Z_3 \times Z_9$, so the number of elements of order 9 in every Sylow 3-subgroup P_3 is 18. Thus the number of elements of order 9 of G is $18 \cdot 4 = 72$, it contradicts the fact that $s_9 = 42$. Therefore, 9 is not in $\pi_e(G)$. Now, using the Lemma we will get $|P_3| \mid 1+s_3 = 57$, then $|P_3| = 3$.

4. $4 \in \pi_e(G)$ and $s_4 = 42$.

If G has no order 4 element, then G has an element of order 6 or order 14. Otherwise, $|\pi_e(G)| \leq 4$, it contradicts the facts that $|\pi_e(G)| \geq 5$. We consider three cases.

(i) If $6 \in \pi_e(G)$ and 14 is not in $\pi_e(G)$, since $6 \mid 1 + s_2 + s_3 + s_6 = 78 + s_6$ for some $s_6 \in \{56, 42, 48\}$, then $s_6 = 48$. Also, since $|\pi_e(G)| \geq 5$, we must have $\pi_e(G) = \{1, 2, 3, 6, 7\}$, and hence $|G| = 1+21+56+42+48 = 168$. But $24 \mid |G|$, by the Lemma we have $24 \mid 1 + s_2 + s_3 + s_6 = 126$, it is a contradiction.

(ii) If $14 \in \pi_e(G)$ and 6 is not in, since $\phi(14) = 6$, then we have $s_{14} = 42$ or 48. In addition, since $14 \mid 1 + s_2 + s_7 + s_{14} = 70 + s_{14}$, we have $s_{14} = 42$. Similarly, $\pi_e(G) = \{1, 2, 3, 14, 7\}$ and $|G| = 168$, then $24 \mid 1 + s_2 + s_3 = 78$, a contradiction.

(iii) If 6 and 14 are both in $\pi_e(G)$, then $s_6 = 48$ and $s_{14} = 42$ by the above. Similarly, we have $\pi_e(G) = \{1, 2, 3, 6, 14, 7\}$ and $|G| = 216$. But $9 \mid 216$, it contradicts the fact that $3 \| \, |G|$ in the assertion 3.

On collecting the results of $(i) - (iii)$, we can draw a conclusion that $4 \in \pi_e(G)$. Finally, using the Lemma, we get $4 \mid 1 + s_2 + s_4 = 22 + s_4$ for some $s_4 \in \{56, 42, 48\}$, hence $s_4 = 42$.

5. 14 is not in $\pi_e(G)$.

If $14 \in \pi_e(G)$, then we have $s_{14} = 42$ by above (ii) of assertion 4. Moreover, we claim that 28 does not belong to $\pi_e(G)$. In fact, if $28 \in \pi_e(G)$, then $s_{28} = 48$ since $\phi(28) = 12$. By the result of Lemma we have $28 \mid 1 + s_2 + s_4 + s_7 + s_{14} + s_{28} = 202$, a contradiction. Also, since $28 \mid |G|$, then by the Lemma we have $28 \mid 1+s_2+s_4+s_7+s_{14} = 154$, which is also a contradiction.

6. $|G| = 168$ or 336.

Firstly, one claims that G is not a 2-group. Otherwise, if $|G| = 2^m$, then we can assume $\pi_e(G) = \{1, 2, 2^2, \cdots, 2^t\}$. Since $|\pi_e(G)| \geq 5$, we have $t \geq 4$. In addition, since $\phi(2^t) \mid s_{2^t}$ for $s_{2^t} \in \{56, 42, 48\}$, then we must have $t \leq 5$. If $t = 4$, then the order of G is just 168, which contradicts the fact that $|G|$ is a power of 2. Also, if $t = 5$, then we have $|G| = 360 + s_{2^i}$ for some $s_{2^i} \in \{56, 42, 48\}$, which also contradicts the fact that G is a 2-group.

Secondly, one claims that $\pi(G) \neq \{2, 7\}$. If not, assume that $|G| = 2^m 7^n$ $7 \mid |G|$, then we consider the Sylow 7-subgroup P_7 acts fixed point freely on the set of elements of order 2, and thus we get $|P_7| \mid s_2 = 21$, then $|P_7| = 7$. Similarly, the Sylow 2-subgroup P_2 acts fixed point freely on the set of the ones of order 7, then $|P_2| \mid s_7 = 48$, besides $|G| \geq 168$, so we have $|P_2| = 2^4$ or 2^3. On the other hand, since G has no element of order 14, we must have G is a Frobenius group. But both the cases $m = 3$ and $m = 4$ do not lead to a Frobenius group.

Finally, if $\pi(G) = \{2, 3\}$ and $|G| = 2^m 3^n$, then we can assume that $\pi_e(G) = \{1, 2, 2^2, \cdots, 2^t\} \cup \{3, 2 \cdot 3, 2^2 \cdot 3, \cdots, 2^t \cdot 3\}$, where $2 \leq t \leq 5$ and $m \geq 2$. Also, since $\phi(2^5 \cdot 3)$ does not divide one of 56, 42 and 48, we must have $5 \leq |\pi_e(G)| \leq 11$. Next we assume that

$$|G| = 168 + 56k_1 + 42k_2 + 48k_3$$

where $0 \leq k_1 + k_2 + k_3 \leq 6$. Then we get an equation

$$84 + 28k_1 + 21k_2 + 24k_3 = 2^{m-1}3^n.$$

It is easy to see $3 \mid k_1$ and $2 \mid k_2$. In addition, if $i \geq 3$, then s_{2^i} and $s_{2^{i-1}\cdot 3}$ are not equal to 42 since both $\phi(2^i)$ and $\phi(2^{i-1} \cdot 3)$ are divided by 4, so we have $k_2 \leq 1$, and hence this leads to $k_2 = 0$. It is not hard to work out that the solutions of this equation are

$$\begin{cases} k_1 = 3 \\ k_3 = 2 \\ m = 4 \\ n = 3 \ . \end{cases}$$

Thus $|G| = 2^4 3^3$ and $|\pi_e(G)| = 10$. From the solution, we know that there are 4 elements of $\pi_e(G)$ such that their number is 56. Assume that these orders are m_1, m_2, m_3 and m_4, and we will get $\{m_1, m_2, m_3, m_4\} \subseteq \{6, 8, 12, 16, 24\}$ since $2^4 \| \, |G|$ and $\phi(m_i) \mid 56$ for $1 \leq i \leq 4$. But if $16 \in \pi_e(G)$, then the Sylow 2-subgroup of G is cyclic, and hence the number of Sylow 2-subgroup n_2 is $s_{16}/\phi(16) = 7$ or 6, which contradicts the facts that 7 is not in $\pi(G)$ and $(2, n_2) = 1$. Hence $\{m_1, m_2, m_3, m_4\} = \{6, 8, 12, 24\}$, and then $|\pi_e(G)| = 8$, which contradicts the fact $|\pi_e(G)| = 10$.

Therefore, by the above arguments we have $|G| = 168$ or 336.

7. G is isomorphic to $PSL_2(7)$.

Since G has no elements of order 14 and 21, these groups were described well, which is close to so-called prime graph on $\pi(G)$ with the following adjacency relation: vertices p and q in $\pi(G)$ are joined by edge if and only if $pq \in \pi_e(G)$ (see [4]). The structure of groups with disconnected prime is due to Gruenberg and Kegel, which is stated that if G is solvable with more than one prime graph components, then G is either Frobenius or 2-Frobenius, i.e., $G = ABC$, where A and AB are normal subgroups of G, AB and BC are Frobenius group with kernel A, B and complements B, C respectively (see the Corollary of [4]). Now we come back to our question. Clearly, G has a disconnected prime graph. It is not hard to see that G is not Frobenius. Also, if G is a 2-Frobenius group, then

$G = ABC$, where A, B and C are the same to the above. In [5], it is shown that B is cyclic of odd order and C is cyclic (see Theorem 2 of [5]). Since AB and BC are both Frobenius groups, we have $|A| = 2^3$, $|B| = 7$ and $|C| = 3$. Thus A must be the unique Sylow 2-group of G since A is normal in G, and then the number of 2-elements of G is 8, which contradicts the fact that $s_2 = 21$.

If G is non-solvable and $|G| = 336$, suppose that N is a minimal normal subgroup of G, then N is isomorphic to Z_2 or $PSL_2(7)$. At the first case N must be included in the central subgroup $Z(G)$, this can imply that there is an element of order 14, that is a contradiction. If $N \cong PSL_2(7)$, note that $C_G(N) = 1$ since the prime graph of G is disconnected, then $G = G/C_G(N) \leq Aut(N)$. But $|Aut(PSL_2(7))| = 336$, so we have $G \cong Aut(PSL_2(7))$. Finally, we refer to page 3 of Atlas [6], and find that $Aut(PSL_2(7))$ has two conjugacy classes of involutions, which contradicts the fact that $s_2 = 21$. If G is non-solvable and $|G| = 168$, then G is isomorphic to $PSL_2(7)$.

References

[1] Mazurov V.D., Khukhro E.I., Unsolved Problems in Group Theory. The Kourovka Notebook, 16 ed., Inst. Mat. Sibirsk. Otdel. Akad. Novosibirsk, 2006.

[2] Shi W., A characteristic property of $PSL_2(7)$. J. Austral. Math.Soc. Ser.A 36 (1984), 354-356.

[3] Frobenius G., Verallgemeinerung des Sylowschen Satze. Berliner Sitz. (1895), 981-993.

[4] Williams J.S., Prime graph components of finite groups. J. Alg. 69 (1981), 487-513

[5] Chen G., On structure of Frobenius group and 2-Frobenius group. J. Southwest China Normal University, 20(5)(1995), 485-487. (in Chinese)

[6] Conway J.H., Curtis R.T., Norton S.P., etc., Atlas of finite groups. Oxford: Clarendon Press, 1985.

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