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THE IMPROVED SQUARE-ROOT METHODS FOR THE INCLUSION OF MULTIPLE ZEROS OF POLYNOMIALS[1]

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Abstract. Starting from a fixed point relation, we construct very fast iterative methods of Ostrowski-root's type for the simultaneous inclusion of all multiple zeros of a polynomial. The proposed methods possess a great computational efficiency since the acceleration of the convergence is attained with only a few additional calculations. Using the concept of the R-order of convergence of mutually dependent sequences, we present the convergence analysis of the total-step method with Schröder's and Halley's corrections under computationally verifiable initial conditions. Further acceleration is attained by the Gauss-Seidel approach (single-step mode). Numerical examples are given to demonstrate properties of the proposed inclusion methods.

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1. Introduction

We consider very fast iterative methods for the simultaneous inclusion of multiple (real or complex) zeros of a polynomial. This kind of methods works in circular complex interval arithmetic (arithmetic of disks) and produces disks in the complex plane in every iteration, each of them containing the sought zero of a polynomial. In this way, the automatic control of the upper bound of errors is provided. Besides, there exists the ability to incorporate rounding errors without altering the fundamental structure of interval method, which is another good property of interval methods. These very convenient features are the main advantages of interval methods in relation to iterative methods realized in ordinary complex arithmetic. An additional preference of interval methods

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is a relatively easy construction of computationally verifiable initial conditions that guarantee the convergence of the iterative process, which is a very hard problem working in ordinary real or complex arithmetic. The reviewed good properties launched interval methods in the last decades as a very powerful self-validated tool in solving many problems of applied mathematics, physics, engineering branches and other disciplines.

We concentrate on the interval simultaneous methods based on the Ostrowski-like fixed point relation. Our main goal is to give a precise convergence analysis that includes computationally verifiable initial conditions. Using an approach with corrections, proposed first in [2] and [11], we construct modified interval methods of Ostrowski's type with very fast convergence on the account of only a few additional numerical operations. In this way, a high computational efficiency of the proposed methods is achieved.

This paper is organized as follows. In Section 2 we give a fixed-point relation which makes the basis for the construction of the simultaneous Ostrowski-like interval method of the fourth order and the criterion for the choice of a proper square root of a disk. The new total-step methods of Ostrowski's type with the increased convergence speed are constructed in Section 3 using Schröder's and Halley's corrections. The convergence analysis of these methods is given in Section 4. The corresponding single-step methods are discussed in Section 5, while numerical results are given in Section 6.

The construction of inclusion methods and their convergence analysis, studied in this paper, require the basic definitions and properties of circular complex arithmetic (arithmetic of disks), introduced by Gargantini and Henrici [4]. A circular closed region (disk) $Z := \{z : |z-c| \le r\}$ with center $c := \operatorname{mid} Z$ and radius $r := \operatorname{rad} Z$ will be denoted by parametric notation $Z := \{c; r\}$. Throughout this paper, disks in the complex plane will be denoted by capital letters.

The inversion of a disk Z not containing 0 is defined by the Möbius transformation,

$$Z^{I_E} = \{c; r\}^{I_E} = \frac{\{\bar{c}; r\}}{|c|^2 - r^2} \quad (|c| > r, \text{ i.e., } 0 \notin Z). \tag{1}$$

The inversion Z^{I_E} is an exact operation, that is, $Z^{I_E} = \{z^{-1} : z \in Z\}$. Besides the exact inversion Z^{I_E} of a disk Z, it is sometime preferable to use the so-called *centered inversion* Z^{I_C} defined by

$$Z^{I_C} = \{c; r\}^{I_C} := \left\{ \frac{1}{c}; \frac{r}{|c|(|c| - r)} \right\} \supseteq Z^{I_E} \quad (0 \notin Z). \tag{2}$$

We will use the symbol INV to denote both mentioned types of inversion of a disk, that is, $\mathrm{INV} \in \{()^{I_E}, ()^{I_C}\}$.

Let $Z_k := \{c_k; r_k\}$ $(k = 1, 2)$, then

$$Z_1 \pm Z_2 = \{c_1 \pm c_2; r_1 + r_2\},$$
$$Z_1 \cdot Z_2 := \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\} \supseteq \{z_1 z_2 : z_1 \in Z_1, z_2 \in Z_2\},$$
$$Z_1 : Z_2 = Z_1 \cdot \mathrm{INV}\, Z_2 \quad (0 \notin Z_2, \ \mathrm{INV} \in \{()^{I_E}, ()^{I_C}\}.$$

The square root of a disk $\{c; r\}$ in the centered form, where $c = |c|e^{i\theta}$ and $|c| > r$, is defined as the union of two disjoint disks (see [3]):

$$\{c; r\}^{1/2} := \Big\{\sqrt{|c|}e^{i\theta/2}; \rho\Big\} \bigcup \Big\{-\sqrt{|c|}e^{i\theta/2}; \rho\Big\}, \quad \rho = \frac{r}{\sqrt{|c|} + \sqrt{|c| - r}}. \tag{3}$$

Throughout this paper we will use the following simple property

$$\{c_1; r_1\} \cap \{c_2; r_2\} = \emptyset \iff |c_1 - c_2| > r_1 + r_2. \tag{4}$$

We will also often apply very important inclusion property of circular interval arithmetic:

$$\text{if } \ z \in Z, \ \text{ then } \ f(z) \in f(Z) = \{f(z) \, : \, z \in Z\} \subseteq F(Z),$$

where F is a circular interval extension of a complex function f. More details about operations and properties of the circular arithmetic can be found in the books [1], [9], [10] and [13].

2. Fixed-point relation and the basic method

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial of degree $n \geq 3$ with multiple zeros $\zeta_1, \ldots, \zeta_k$ of the respective multiplicities $\mu_1, \ldots, \mu_k$, $\mu_1 + \cdots + \mu_k = n$ $(2 \leq k \leq n)$, that is,

$$P(z) = \prod_{j=1}^{k} (z - \zeta_j)^{\mu_j}, \tag{5}$$

and let

$$\delta_1(z) = \frac{P'(z)}{P(z)}, \quad \delta_2(z) = \frac{P'(z)^2 - P(z)P''(z)}{P(z)^2}.$$

From the factorization (5) we find

$$\begin{aligned} \delta_1(z) &= \frac{d}{dz}\big(\log P(z)\big) = \sum_{j=1}^{k} \frac{\mu_j}{z - \zeta_j} = \frac{\mu_i}{z - \zeta_i} + \sum_{\substack{j \in \boldsymbol{I}_k \\ j \neq i}} \frac{\mu_j}{z - \zeta_j}, \\ \delta_2(z) &= -\frac{d}{dz}\big(\delta_1(z)\big) = \sum_{j=1}^{k} \frac{\mu_j}{(z - \zeta_j)^2} = \frac{\mu_i}{(z - \zeta_i)^2} + \sum_{\substack{j \in \boldsymbol{I}_k \\ j \neq i}} \frac{\mu_j}{(z - \zeta_j)^2}. \end{aligned}$$

where $\boldsymbol{I}_k := \{1, \ldots, k\}$ is the index set. We single out the term $z - \zeta_i$ from the last relation and obtain the following fixed-point relation

$$\zeta_i = z - \frac{\sqrt{\mu_i}}{\left[\delta_2(z) - \sum\limits_{\substack{j \in \boldsymbol{I}_k \\ j \neq i}} \mu_j \Big(\dfrac{1}{z - \zeta_j}\Big)^2\right]_*^{1/2}} \quad (i \in \boldsymbol{I}_k). \tag{6}$$

The symbol $*$ indicates that one of the two complex numbers has to be selected. This value is chosen in such a way that the right-hand side reduces to ζ_i.

Suppose that k disjoint disks $Z_1, \ldots, Z_k$, such that $\zeta_j \in Z_j$ $(j \in \boldsymbol{I}_k)$, have been found. Let us put $z = z_i = \operatorname{mid} Z_i$ in (6). Since $\zeta_j \in Z_j$ $(j \in \boldsymbol{I}_k)$, according to the inclusion isotonicity property we obtain

$$(7)\quad \zeta_i \in z_i - \sqrt{\mu_i}\ \mathrm{INV}_2\Big[\delta_2(z_i) - \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j\Big(\mathrm{INV}_1(z_i - Z_j)\Big)^2\Big]_*^{1/2} =: \hat{Z}_i, \quad (i \in \boldsymbol{I}_k),$$

where $\mathrm{INV}_1, \mathrm{INV}_2 \in \{()^{I_E}, ()^{I_C}\}$. The subscripts 1 and 2 point to the order of application of the inversion of disks; first, the inversion INV_1 is applied to the addends, and then INV_2 is used in the final step of calculation in the circular complex arithmetic. This approach is assumed in all iterative formulae considered in this paper, concerned with interval methods.

Let us introduce the following vectors:

$$\begin{aligned}
\boldsymbol{Z}^{(m)} &= \big(Z_1^{(m)}, \ldots, Z_k^{(m)}\big) \quad \text{(current inclusion disks)},\\
\boldsymbol{Z}_N^{(m)} &= \big(Z_{N,1}^{(m)}, \ldots, Z_{N,k}^{(m)}\big),\ Z_{N,i}^{(m)} = Z_i^{(m)} - N\big(z_i^{(m)}\big) \quad \text{(Schröder's disks)},\\
\boldsymbol{Z}_H^{(m)} &= \big(Z_{H,1}^{(m)}, \ldots, Z_{H,k}^{(m)}\big),\ Z_{H,i}^{(m)} = Z_i^{(m)} - H\big(z_i^{(m)}\big) \quad \text{(Halley's disks)},
\end{aligned}$$

where

$$N(z_i) = \mu_i \frac{P(z_i)}{P'(z_i)}, \quad H(z_i) = \frac{P(z_i)}{\Big(\dfrac{1+1/\mu_i}{2}\Big)P'(z_i) - \dfrac{P(z_i)P''(z_i)}{2P'(z_i)}},$$

and $m = 0, 1, \ldots$ is the iteration index. The corrections $N(z)$ and $H(z)$ appear in Schröder's root-finding method $\hat{z} = z - N(z)$ of the second order, and Halley's method $\hat{z} = z - H(z)$ of the third order, respectively.

In order to ease the reading and writing, we will sometimes omit the iteration index and write, for example, $z_i,\ r_i,\ \hat{z}_i,\ \hat{r}_i,\ Z_i,\ \hat{Z}_i,\ Z_{N,i},\ Z_{H,i}$ instead of $z_i^{(m)}, r_i^{(m)}, z_i^{(m+1)}, r_i^{(m+1)},\ Z_i^{(m)},\ Z_i^{(m+1)},\ Z_{N,i}^{(m)},\ Z_{H,i}^{(m)}$. In what follows, we will write $w_1 \sim w_2$ or $w_1 = \mathcal{O}_M(w_2)$ (the same order of moduli) for two complex numbers w_1 and w_2 that satisfy $|w_1| = \mathcal{O}(|w_2|)$, where $\mathcal{O}$ is the Landau symbol.

Let us define the disk

$$(8)\qquad S_{q,i}(\boldsymbol{X}, \boldsymbol{Y}) := \sum_{j=1}^{i-1} \mu_j\Big(\mathrm{INV}_1(x_i - X_j)\Big)^q + \sum_{j=1+1}^{k} \mu_j\Big(\mathrm{INV}_1(x_i - Y_j)\Big)^q,$$

where $q \in \{1, 2\}$ and $\mathrm{INV}_1 \in \{()^{I_E}, ()^{I_C}\}$. $\boldsymbol{X} = (X_1, \ldots, X_k)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_k)$ are vectors whose components are disks.

Assume that we have found initial disjoint disks $Z_1^{(0)}, \ldots, Z_k^{(0)}$ containing the zeros $\zeta_1, \ldots, \zeta_k$, that is, $\zeta_i \in Z_i^{(0)}$ for all $i \in \boldsymbol{I}_k$. The relation (7) suggests the following *total-step* method for the simultaneous inclusion of all multiple zeros of a polynomial P,

$$(9)\quad Z_i^{(m+1)} = z_i^{(m)} - \sqrt{\mu_i}\ \mathrm{INV}_2\left(\left[\delta_2\big(z_i^{(m)}\big) - S_{2,i}\big(\boldsymbol{Z}^{(m)}, \boldsymbol{Z}^{(m)}\big)\right]_*^{1/2}\right) \quad (i \in \boldsymbol{I}_k),$$

where $\mathrm{INV}_2 \in \{()^{I_E}, ()^{I_C}\}$ and $m = 0, 1, \ldots$. Implementing the iterative formula (9) we first apply the inversion INV_1 under the sums (8), and then the inversion INV_2 in the final step. The convergence order of the interval method (9) is *four*, independently of the type of applied inversion (see [3]).

According to (3), the square root of a disk in (9) produces two disks; the symbol $*$ indicates that one of these disks has to be chosen. We can state the criterion for the choice of a proper disk in a similar manner as in [3]:

Let $\left[\delta_2\big(z_i^{(m)}\big) - S_{2,i}\big(\boldsymbol{Z}^{(m)}, \boldsymbol{Z}^{(m)}\big)\right]^{1/2} = D_{1,i}^{(m)} \bigcup D_{2,i}^{(m)}$. *Among the disks* $D_{1,i}^{(m)}$ *and* $D_{2,i}^{(m)}$, *one has to choose that disk whose center minimizes*

$$\left|\frac{P'\big(z_i^{(m)}\big)}{\mu_i P\big(z_i^{(m)}\big)} - \mathrm{mid}\, D_{q,i}^{(m)}\right| \quad (q = 1, 2).$$

The convergence of the method (9) can be accelerated by applying the Gauss-Seidel approach [9]. Thus, we use the already calculated circular approximations in the same iteration and obtain the single-step method

$$(10)\quad Z_i^{(m+1)} = z_i^{(m)} - \sqrt{\mu_i}\ \mathrm{INV}_2\left(\left[\delta_2\big(z_i^{(m)}\big) - S_{2,i}\big(\boldsymbol{Z}^{(m+1)}, \boldsymbol{Z}^{(m)}\big)\right]_*^{1/2}\right) \quad (i \in \boldsymbol{I}_k).$$

The R-order of convergence of the single-step method (10) is at least $3 + x_n$, where x_n is the unique positive root of the equation $x^n - x - 3 = 0$, see [9, Ch. 4]. The values of the R-order belong to the range $(4, 5.303)$ for $n \geq 2$.

Remark 1. Omitting the sum in the iterative formulae (9) and (10) we obtain the Ostrowski iterative formula [7]

$$z^{(m+1)} = z^{(m)} - \frac{\sqrt{\mu_i}}{\left[\delta_2\big(z^{(m)}\big)\right]_*^{1/2}}$$

with cubic convergence, also known as square-root method. For this reason, the methods (9) and (10), as well as their modifications which will be considered in this paper, are referred to as *Ostrowski-like* methods or *square-root methods*.

3. Inclusion methods with corrections

Let us introduce the following abbreviations

$$r^{(m)} = \max_{1\le i\le k} r_i^{(m)}, \quad \rho^{(m)} = \min_{\substack{1\le i,j\le k\\ i\ne j}} \left\{\left|z_i^{(m)} - z_j^{(m)}\right| - r_j^{(m)}\right\},$$

$$\varepsilon_i^{(m)} = z_i^{(m)} - \zeta_i, \quad \epsilon^{(m)} = \max_{1\le i\le k} \left|\varepsilon_i^{(m)}\right|, \quad \mu = \min_{1\le i\le k} \mu_i,$$

$$\Sigma_{q,i}^{(m)} = \sum_{\substack{j\in \boldsymbol{I}_k\\ j\ne i}} \frac{\mu_j}{\left(z_i^{(m)} - \zeta_j\right)^q} \quad \delta_{q,i}^{(m)} = \sum_{j=1}^{k} \frac{\mu_j}{\left(z_i^{(m)} - \zeta_j\right)^q}, \quad (q = 1, 2).$$

A Further increase of the convergence speed of the iterative method (9) can be achieved using Schröder's and Halley's corrections in a similar way as in [2, 8, 11]. In this construction we assume that the initial inclusion disks $Z_1^{(0)}, \ldots, Z_k^{(0)}$, containing the zeros $\zeta_1, \ldots, \zeta_k$, are chosen in such a way that each disk $Z_i^{(0)} - N\big(\mathrm{mid}\,(Z_i^{(0)})\big)$ or $Z_i^{(0)} - H\big(\mathrm{mid}\,(Z_i^{(0)})\big)$ also contains the zero ζ_i $(i \in \boldsymbol{I}_k)$. This is the subject of the following assertion.

Lemma 1. *Let $Z_1, \ldots, Z_k$ be inclusion disks for the zeros $\zeta_1, \ldots, \zeta_k,\ \zeta_i \in Z_i$, and let $z_i = \mathrm{mid}\ Z_i,\ r_i = \mathrm{rad}\ Z_i,\ r := \max\{r_1, \ldots, r_k\},\ \varepsilon_i = z_i - \zeta_i$. If the inclusion disks $Z_1, \ldots, Z_k$ are chosen so that the inequality*

$$\rho > 3(n - \mu)r \tag{11}$$

is valid, then the following is true for every $i \in \boldsymbol{I}_k$:

(i) $\zeta_i \in Z_i \quad \Rightarrow \quad \zeta_i \in Z_{N,i} := Z_i - N(z_i);$

(ii) $\zeta_i \in Z_i \quad \Rightarrow \quad \zeta_i \in Z_{H,i} := Z_i - H(z_i).$

Proof. *Of* (i): According to the properties of circular arithmetic, we should prove the implication

$$|z_i - \zeta_i| \le r_i \Rightarrow |z_i - N(z_i) - \zeta_i| \le r_i.$$

Using the triangle inequality

$$|z_i - \zeta_j| \ge |z_i - z_j| - |z_j - \zeta_j| \ge |z_i - z_j| - r_j \ge \rho,$$

we find

$$|\Sigma_{q,i}| = \left|\sum_{\substack{j\in \boldsymbol{I}_k\\ j\ne i}} \frac{\mu_j}{(z_i - \zeta_j)^q}\right| \le \sum_{\substack{j\in \boldsymbol{I}_k\\ j\ne i}} \frac{\mu_j}{|z_i - \zeta_j|^q} \le \frac{n - \mu_i}{\rho^q}, \quad (q = 1, 2). \tag{12}$$

Using (11) and (12) (for $q = 1$) we get

$$r_i < \frac{\rho}{3(n - \mu_i)} \le \frac{1}{3|\Sigma_{1,i}|},$$

wherefrom

$$\frac{r_i|\Sigma_{1,i}|}{\mu_i - r_i|\Sigma_{1,i}|} < \frac{1}{2}.$$

Since $|\varepsilon_i| = |z_i - \zeta_i| \le r_i$, according to the last inequality we get

$$\begin{aligned}
|z_i - N(z_i) - \zeta_i| &= |\varepsilon_i - N(z_i)| = |\varepsilon_i - \mu_i/\delta_{1,i}| = |\varepsilon_i|^2 \left| \frac{\Sigma_{1,i}}{\mu_i + \varepsilon_i \Sigma_{1,i}} \right| \\
&\le \frac{r_i^2 |\Sigma_{1,i}|}{\mu_i - r_i|\Sigma_{1,i}|} < \frac{1}{2} \cdot r_i < r_i.
\end{aligned}$$

Of (ii): Similarly as in the proof of assertion (i), we prove the implication

$$|z_i - \zeta_i| \le r_i \Rightarrow |z_i - H(z_i) - \zeta_i| \le r_i.$$

Using the denotation

$$\delta_{q,i} = \sum_{j=1}^{k} \frac{\mu_j}{(z_i - \zeta_j)^q} \quad (q = 1, 2),$$

we obtain

$$\begin{aligned}
H(z_i) &= \frac{2\delta_{1,i}}{\frac{1}{\mu_i}\delta_{1,i}^2 + \delta_{2,i}} = \frac{2\displaystyle\sum_{j=1}^{k} \frac{\mu_j}{z_i - \zeta_j}}{\displaystyle\frac{1}{\mu_i}\left(\sum_{j=1}^{k} \frac{\mu_j}{z_i - \zeta_j}\right)^2 + \sum_{j=1}^{k} \frac{\mu_j}{(z_i - \zeta_j)^2}} \\
&= \frac{2(\mu_i/\varepsilon_i + \Sigma_{1,i})}{(\mu_i/\varepsilon_i + \Sigma_{1,i})^2/\mu_i + \mu_i/\varepsilon_i^2 + \Sigma_{2,i}} \\
&= \frac{2\varepsilon_i(\mu_i + \varepsilon_i \Sigma_{1,i})}{2\mu_i + 2\varepsilon_i \Sigma_{1,i} + \varepsilon_i^2(\Sigma_{1,i}^2/\mu_i + \Sigma_{2,i})}.
\end{aligned}$$

Now we have

$$\begin{aligned}
z_i - H(z_i) - \zeta_i &= \varepsilon_i - \frac{2\varepsilon_i(\mu_i + \varepsilon_i \Sigma_{1,i})}{2\mu_i + 2\varepsilon_i \Sigma_{1,i} + \varepsilon_i^2(\Sigma_{1,i}^2/\mu_i + \Sigma_{2,i})} \\
&= \frac{\varepsilon_i^3(\Sigma_{1,i}^2/\mu_i + \Sigma_{2,i})}{2\mu_i + 2\varepsilon_i \Sigma_{1,i} + \varepsilon_i^2(\Sigma_{1,i}^2/\mu_i + \Sigma_{2,i})}
\end{aligned}$$

so that, by (11) and (12),

$$
\begin{aligned}
|z_i - H(z_i) - \zeta_i| &= \left| \frac{\varepsilon_i^3(\Sigma_{1,i}^2/\mu_i + \Sigma_{2,i})}{2\mu_i + 2\varepsilon_i\Sigma_{1,i} + \varepsilon_i^2(\Sigma_{1,i}^2/\mu_i + \Sigma_{2,i})} \right| \\
&< \frac{\dfrac{1}{\mu_i}\Big(\dfrac{n-\mu_i}{\rho}\Big)^2 + \dfrac{n-\mu_i}{\rho^2}}{2\mu_i - 2\dfrac{n-\mu_i}{\rho}r_i - \dfrac{1}{\mu_i}\Big(\dfrac{n-\mu_i}{\rho}\Big)^2 r_i^2 - \dfrac{n-\mu_i}{\rho^2}r_i^2} \cdot r_i^3 \\
&< \frac{\dfrac{(n-\mu_i)nr_i^2}{\mu_i\rho^2}}{\dfrac{4}{3} - \dfrac{n(n-\mu_i)r_i^2}{\mu_i\rho^2}} \cdot r_i < \frac{\dfrac{n}{9\mu_i(n-\mu_i)}}{\dfrac{4}{3} - \dfrac{n}{9\mu_i(n-\mu_i)}} \cdot r_i \\
&\le \frac{1}{7} \cdot r_i < r_i.
\end{aligned}
$$

□

Starting from the fixed-point relation (6), in this section we construct the total-step Ostrowski-like methods with Schröder's and Halley's corrections for the inclusion of multiple zeros of polynomials. To study the convergence analysis of both methods simultaneously, we indicate these methods with the superscripts $\lambda = 1$ (for Schröder's correction) and $\lambda = 2$ (for Halley's correction). Then the corresponding vectors of disk approximations are denoted by

$$
\begin{aligned}
\boldsymbol{Z}^{(0)} &= \big(Z_1, \ldots, Z_k\big), \\
\boldsymbol{Z}^{(1)} &= \big(Z_1^{(1)}, \ldots, Z_k^{(1)}\big) = \big(Z_{N,1}, \ldots, Z_{N,k}\big), \\
\boldsymbol{Z}^{(2)} &= \big(Z_1^{(2)}, \ldots, Z_k^{(2)}\big) = \big(Z_{H,1}, \ldots, Z_{H,k}\big).
\end{aligned}
$$

The corresponding corrections are $N(z_i) = C^{(1)}(z_i)$ and $H(z_i) = C^{(2)}(z_i)$. If we deal without corrections ($\lambda = 0$), then the vector of current disk approximations is denoted by $\boldsymbol{Z}^{(0)}$, as above.

For simplicity, we will omit the iteration index for all quantities at the m-th iteration and denote the quantities at the $(m+1)$-st iteration with the additional symbol ˆ ("hat"). Now, the Ostrowski-like methods with/without corrections can be presented in the unique form as

$$
(13) \quad \hat{Z}_i = z_i - \sqrt{\mu_i}\ \mathrm{INV}_2\Big(\Big[\delta_{2,i} - S_{2,i}\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big)\Big]_*^{1/2}\Big) \quad (i \in \boldsymbol{I}_k,\ \lambda = 0, 1, 2).
$$

4. Convergence analysis

Before stating the convergence theorem and initial convergence conditions for the simultaneous interval methods (13) with $\lambda = 1$ or 2, we give in Lemma 2 some necessary estimations.

First we find

$$z_i - Z_j + C^{(\lambda)}(z_j) = \{z_i - \zeta_j + \xi_j^{(\lambda)}\varepsilon_j^{\lambda+1}; r_j\},$$

where

$$\xi_j^{(1)} = -\frac{\Sigma_{1,j}}{\mu_j + \varepsilon_j \Sigma_{1,j}} \quad \text{and} \quad \xi_j^{(2)} = -\frac{\Sigma_{1,j}^2/\mu_j + \Sigma_{2,j}}{2\mu_j + 2\varepsilon_j \Sigma_{1,j} + \varepsilon_j^2(\Sigma_{1,j}^2/\mu_j + \Sigma_{2,j})}.$$

For simplicity, let us introduce the abbreviations (taking $\lambda = 1, 2$)

$$\begin{aligned}
h_{ij}^{(\lambda)} &= \operatorname{mid}\big(z_i - Z_j + C^{(\lambda)}(z_j)\big) = z_i - \zeta_j + \xi_j^{(\lambda)}\varepsilon_j^{\lambda+1},\\
d_{ij}^{(\lambda)} &= \frac{r_j}{|h_{ij}^{(\lambda)}|\big(|h_{ij}^{(\lambda)}| - r_j\big)}, \quad u_{ij}^{(\lambda)} = \frac{1}{h_{ij}^{(\lambda)}},\\
s_i^{(\lambda)} &= \sum_{\substack{j\in I_k\\ j\neq i}} \frac{1}{\big(z_i - z_j + C_j^{(\lambda)}\big)^2}, \quad c_i^{(\lambda)} = \delta_{2,i} - \sum_{\substack{j\in I_k\\ j\neq i}} \mu_j \big(u_{ij}^{(\lambda)}\big)^2,\\
\eta_i^{(\lambda)} &= \sum_{\substack{j\in I_k\\ j\neq i}} \mu_j \Big[2|u_{ij}^{(\lambda)}|d_{ij}^{(\lambda)} + \big(d_{ij}^{(\lambda)}\big)^2\Big], \quad \omega_i^{(\lambda)} = \frac{\eta_i^{(\lambda)}}{\sqrt{|c_i^{(\lambda)}|} + \sqrt{|c_i^{(\lambda)}| - \eta_i^{(\lambda)}}}.
\end{aligned}$$

In the proof of Lemma 1 we have found the bounds

$$|z_i - \zeta_j| \geq \rho \quad \text{and} \quad |\Sigma_{k,j}| \leq \frac{n-\mu_j}{\rho^q} \quad (q = 1, 2),$$

so that the moduli of $\xi_j^{(1)}$ and $\xi_j^{(2)}$ are bounded by

$$\big|\xi_j^{(1)}\big| \leq \frac{|\Sigma_{1,j}|}{\mu_j - |\varepsilon_j||\Sigma_{1,j}|} \leq \frac{\dfrac{n-\mu_j}{\rho}}{\mu_j - \dfrac{(n-\mu_j)r}{\rho}} < \frac{3(n-\mu_j)}{2\rho}$$

and

$$\begin{aligned}
\big|\xi_j^{(2)}\big| &\leq \frac{\dfrac{|\Sigma_{1,j}|^2}{\mu_j} + |\Sigma_{2,j}|}{2\mu_j - 2|\varepsilon_j||\Sigma_{1,j}| - |\varepsilon_j|^2\Big(\dfrac{|\Sigma_{1,j}|^2}{\mu_j} + |\Sigma_{2,j}|\Big)}\\
&\leq \frac{\dfrac{(n-\mu_j)n}{\mu_j\rho^2}}{2\mu_j - 2r\dfrac{n-\mu_j}{\rho} - r^2\dfrac{(n-\mu_j)n}{\mu_j\rho^2}} < \frac{6}{7}\frac{(n-\mu_j)n}{\mu_j\rho^2},
\end{aligned}$$

for all $j \in \mathbb{I}_k$. By the last two inequalities we estimate

$$\begin{aligned} |h_{ij}^{(1)}| &= |z_i - \zeta_j + \xi_j^{(1)}\varepsilon_j^2| \geq |z_i - \zeta_j| - |\xi_j^{(1)}||\varepsilon_j|^2 \\ &> \rho - \frac{3(n-\mu_j)r^2}{2\rho} > \rho - \frac{r}{2}, \\ |h_{ij}^{(2)}| &= |z_i - \zeta_j + \xi_j^{(2)}\varepsilon_j^3| \geq |z_i - \zeta_j| - |\xi_j^{(2)}||\varepsilon_j|^3 \\ &> \rho - \frac{6}{7}\frac{(n-\mu_j)nr^3}{\mu_j\rho^2} > \rho - \frac{1}{7}r. \end{aligned}$$

Hence, for $\lambda = 1, 2$,

$$|h_{ij}^{(\lambda)}| > \rho - \frac{r}{2} \tag{14}$$

and

$$|h_{ij}^{(\lambda)}|\big(|h_{ij}^{(\lambda)}| - r_j\big) > \Big(\rho - \frac{r}{2}\Big)\Big(\rho - \frac{3r}{2}\Big) = \rho^2\Big(1 - \frac{r}{2\rho}\Big)\Big(1 - \frac{3r}{2\rho}\Big) > \frac{3}{5}\rho^2. \tag{15}$$

Using (11) we bound

$$d_{ij}^{(\lambda)} = \frac{r_j}{|h_{ij}^{(\lambda)}|\big(|h_{ij}^{(\lambda)}| - r_j\big)} < \frac{5r}{3\rho^2}. \tag{16}$$

Similarly, by (11) and (14) we obtain

$$|u_{ij}^{(\lambda)}| = \frac{1}{|h_{ij}^{(\lambda)}|} < \frac{1}{\rho - \frac{r}{2}} < \frac{12}{11\rho}. \tag{17}$$

Lemma 2. *If* (11) *holds, then*

(i) $|\delta_{2,i}| > \dfrac{17\mu_i}{18|\varepsilon_i|^2} \geq \dfrac{17\mu_i}{18r_i^2}$;

(ii) $|c_i^{(\lambda)}| > \dfrac{4\mu_i}{5|\varepsilon_i|^2} \geq \dfrac{4\mu_i}{5r_i^2}$;

(iii) $|c_i^{(\lambda)}| > \eta_i^{(\lambda)}$;

(iv) $\omega_i^{(\lambda)} < \dfrac{12(n-\mu_i)|\varepsilon_i|r}{5\rho^3}$;

(v) $\omega_i^{(\lambda)}|\varepsilon_i| < \dfrac{1}{45}$.

Proof. *Of* (i): By (11) we estimate

$$\begin{aligned} |\delta_{2,i}| &= \Big|\frac{\mu_i}{\varepsilon_i^2} + \Sigma_{2,i}\Big| \geq \frac{\mu_i}{|\varepsilon_i|^2} - \sum_{\substack{j \in I_k \\ j \neq i}} \frac{\mu_j}{|z_i - \zeta_j|^2} \geq \frac{\mu_i}{|\varepsilon_i|^2}\Big(1 - \frac{(n-\mu_i)r^2}{\rho^2}\Big) \\ &> \frac{\mu_i}{|\varepsilon_i|^2}\Big(1 - \frac{n-\mu}{(3(n-\mu))^2}\Big) = \frac{\mu_i}{|\varepsilon_i|^2}\Big(1 - \frac{1}{9(n-\mu)}\Big) \geq \frac{17\mu_i}{18|\varepsilon_i|^2} \geq \frac{17\mu_i}{18r_i^2}. \end{aligned}$$

Of (ii): By virtue of (17) we get

$$\sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \big|u_{ij}^{(\lambda)}\big|^2 \le \frac{144(n-\mu)}{121\rho^2}.$$

Using this inequality, (11) and (i) we find

$$\begin{aligned}
\big|c_i^{(\lambda)}\big| &= \Big|\delta_{2,i} - \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j \big(u_{ij}^{(\lambda)}\big)^2\Big| \ge |\delta_{2,i}| - \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j \big|u_{ij}^{(\lambda)}\big|^2 \\
&> \frac{17\mu_i}{18|\varepsilon_i|^2} - \frac{144(n-\mu_i)}{121\rho^2} \ge \frac{\mu_i}{|\varepsilon_i|^2}\Big(\frac{17}{18} - \frac{144(n-\mu_i)}{121}\cdot\frac{r^2}{\rho^2}\Big) \\
&> \frac{\mu_i}{|\varepsilon_i|^2}\Big(\frac{17}{18} - \frac{144}{121\cdot 9(n-\mu)}\Big) \ge \frac{\mu_i}{|\varepsilon_i|^2}\Big(\frac{17}{18} - \frac{8}{121}\Big) > \frac{4}{5|\varepsilon_i|^2} \ge \frac{4}{5r_i^2}.
\end{aligned}$$

Of (iii): By virtue of (16) and (17) we estimate

$$\eta_i^{(\lambda)} < \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j \Big[2\frac{12}{11\rho}\cdot\frac{5r}{3\rho^2} + \Big(\frac{5r}{3\rho^2}\Big)^2\Big] < \frac{41(n-\mu)r}{10\rho^3} = \gamma_n r, \quad \gamma_n = \frac{41(n-\mu)}{10\rho^3}. \tag{18}$$

Using this bound, (11) and (ii), we obtain

$$\big|c_i^{(\lambda)}\big| - \eta_i^{(\lambda)} > \big|c_i^{(\lambda)}\big| - \gamma_n r > \frac{4\mu_i}{5r_i^2} - \frac{41(n-\mu)r}{10\rho^3} > \frac{27}{\rho^2},$$

which means that $\big|c_i^{(\lambda)}\big| > \eta_i^{(\lambda)}$.

Of (iv) and (v): According to (11), (18), (ii) and the bound

$$\gamma_n r|\varepsilon_i|^2 \le \frac{41(n-\mu)r^3}{10\rho^3} < \frac{41(n-\mu)}{10}\cdot\frac{1}{\big(3(n-\mu)\big)^3} = \frac{41}{10\cdot 27(n-\mu)^2} < \frac{1}{25},$$

we get

$$\omega_i^{(\lambda)} = \frac{\eta_i^{(\lambda)}}{\sqrt{\big|c_i^{(\lambda)}\big|} + \sqrt{\big|c_i^{(\lambda)}\big| - \eta_i^{(\lambda)}}} < \frac{12(n-\mu_i)|\varepsilon_i|r}{5\rho^3},$$

which yields

$$\omega_i^{(\lambda)}|\varepsilon_i| < \frac{12(n-\mu_i)|\varepsilon_i|^2 r}{5\rho^3} \le \frac{12(n-\mu)r^3}{5\rho^3} < \frac{4}{45(n-\mu)^2} \le \frac{1}{45}.$$

□

Let IM be a simultaneous iterative method for solving equations which produces k sequences $\{z_1^{(m)}\}, \ldots, \{z_k^{(m)}\}$ of the approximations of the solutions $z_1^*, \ldots, z_k^*$ to a polynomial equation $P(z) = 0$. To estimate the R-order of convergence of the iterative method IM, we introduce the error-sequences

$$v_i^{(m)} = ||z_i^{(m)} - z_i^*|| \quad (i \in \boldsymbol{I}_k)$$

and use the following assertion which is a special case of Theorem 3 given in [6].

Theorem 1. *Given the error-recursions*

$$v_i^{(m+1)} \le \alpha_i \prod_{j=1}^{k} \big(v_j^{(m)}\big)^{t_{ij}} \quad (i \in \boldsymbol{I}_k,\ m \ge 0), \tag{19}$$

where $t_{ij} \ge 0,\ \alpha_i > 0,\ 1 \le i, j \le k$. Denote the matrix of exponents appearing in (19) with T, that is $T = [t_{ij}]_{k\times k}$. If the non-negative matrix T has the spectral radius $\rho(T) > 1$ and the corresponding eigenvector $\boldsymbol{x}_\rho > 0$, then the R-order of all sequences $\{v_i^{(m)}\}$ $(i \in \boldsymbol{I}_k)$ is at least $\rho(T)$.

The matrix $T_k = [t_{ij}]$ is usually called the *R-matrix.*

Let $O_R(IM, k)$ denote the R-order of convergence of an iterative method IM. If the R-order does not depend on the number of distinct polynomial zeros k $(\le n)$, then we write simply $O_R(IM)$. For the total-step method (13), we can state the following convergence theorem under computationally verifiable initial conditions.

Theorem 2. *Suppose that the initial disks $Z_1^{(0)}, ..., Z_k^{(0)}$ are chosen so that $\zeta_i \in Z_i^{(0)}$ $(i \in \boldsymbol{I}_k)$ and the inequality*

$$\rho^{(0)} > 3(n-\mu) r^{(0)} \tag{20}$$

is satisfied. Then the interval method (13) *is convergent and the following is true for each $i \in \boldsymbol{I}_k$ and $m = 1, 2, \dots$:*

1° $\rho^{(m)} > 3(n-\mu) r^{(m)}$;

2° $\zeta_i \in Z_i^{(m)}$;

3° *the lower bound of R-order of convergence of the interval method* (13) *is*

$$O_R(13) \ge \begin{cases} \lambda + 4 \quad (\lambda = 1, 2), & \textit{if } \ \mathrm{INV}_1 = ()^{I_C}, \\[2ex] 2 + \sqrt{7} \cong 4.646, & \textit{if } \ \mathrm{INV}_1 = ()^{I_E}. \end{cases}$$

Proof. We first prove that the condition (20) provides disjunctivity of the initial disks $Z_1^{(0)}, \dots, Z_k^{(0)}$. Indeed, for an arbitrary pair $i, j \in \boldsymbol{I}_k$ $(i \ne j)$ we have

$$\big|z_i^{(0)} - z_j^{(0)}\big| > \rho^{(0)} > 3(n-\mu) r^{(0)} > 2r^{(0)} \ge r_i^{(0)} + r_j^{(0)},$$

which means that $Z_i^{(0)} \cap Z_j^{(0)} = \emptyset$ (according to (4)).

The assertions of Theorem 2 will be proved by induction. Most frequently, we omit the iteration index and assume that the superscript $\lambda \in \{1, 2\}$ points to the type of correction. First, let $m = 0$ and let us take into consideration the initial condition (20). Then, according to Lemma 1, we immediately obtain the implication

$$\zeta_i \in Z_i \Rightarrow \zeta_i \in Z_i^{(\lambda)} := Z_i - C^{(\lambda)}(z_i) \quad (i \in \boldsymbol{I}_k,\ \lambda = 1, 2),$$

which is necessary to keep the inclusion isotonicity property of the interval method (13). We should also prove that the new inclusion disks $Z_1^{(\lambda)}, \ldots, Z_k^{(\lambda)}$ are nonintersecting. Using (20) we find

$$|N(z_i)| = \left|\frac{\mu_i \varepsilon_i}{\mu_i + \varepsilon_i \Sigma_{1,i}}\right| \le \frac{r_i}{1 - r_i|\Sigma_{1,i}|/\mu_i} \le \frac{3}{2} r_i < 2r_i \le 2r$$

and

$$\begin{aligned} |H(z_i)| &= \left|\frac{2\varepsilon_i(\mu_i + \varepsilon_i\Sigma_{1,i})}{2\mu_i + 2\varepsilon_i\Sigma_{1,i} + \varepsilon_i^2(\Sigma_{1,i}^2/\mu_i + \Sigma_{2,i})}\right| \\ &\le \frac{1}{\left|1 + \dfrac{\varepsilon_i^2(\Sigma_{1,i}^2/\mu_i + \Sigma_{2,i})}{2(\mu_i + \varepsilon_i\Sigma_{1,i})}\right|} \cdot r_i < \frac{1}{1 - \dfrac{1}{8}} \cdot r_i = \frac{8}{7} r_i < 2r_i \le 2r, \end{aligned}$$

so that

$$\begin{aligned} \left|\operatorname{mid} Z_i^{(\lambda)} - \operatorname{mid} Z_j^{(\lambda)}\right| &= \left|z_i - C^{(\lambda)}(z_i) - z_j + C^{(\lambda)}(z_j)\right| \\ &\ge |z_i - z_j| - \left|C^{(\lambda)}(z_i)\right| - \left|C^{(\lambda)}(z_j)\right| \\ &> \rho - 4r > 3(n-\mu)r - 4r \ge r_i + r_j. \end{aligned}$$

Thus, $Z_i^{(\lambda)} \cap Z_j^{(\lambda)} = \emptyset$ $(i \ne j)$ in regard to (4). The above facts are necessary for the inclusion method (13) to be well defined.

We recall that we combine two types of inversions in the iterative formula (13). For this reason, we use super(sub)scripts "E" and "C" to indicate the type of the used inversion in (13).

1) *The case* $INV_1 = ()^{I_C}$

Let us consider first the case INV_1, $\mathrm{INV}_2 = ()^{I_C}$. Applying circular arithmetic operations and the centered inversion (2), we obtain

$$\begin{aligned} S_{2,i}\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big) &= \sum_{\substack{j \in \boldsymbol{I}_k \\ j \ne i}} \mu_j \left(\frac{1}{z_i - Z_j + C^{(\lambda)}(z_j)}\right)^2 = \sum_{\substack{j \in \boldsymbol{I}_k \\ j \ne i}} \mu_j \left\{u_{ij}^{(\lambda)}; d_{ij}^{(\lambda)}\right\}^2 \\ &= \left\{\sum_{\substack{j \in \boldsymbol{I}_k \\ j \ne i}} \mu_j \big(u_{ij}^{(\lambda)}\big)^2; \eta_i^{(\lambda)}\right\}. \end{aligned}$$

Now, the iterative formula (13) can be rewritten in the form

$$\hat{Z}_i = z_i - \frac{\sqrt{\mu_i}}{\left\{\delta_{2,i} - \sum\limits_{\substack{j \in \boldsymbol{I}_k \\ j \ne i}} \mu_j \big(u_{ij}^{(\lambda)}\big)^2; \eta_i^{(\lambda)}\right\}_*^{1/2}} = z_i - \frac{\sqrt{\mu_i}}{\left\{\big(c_i^{(\lambda)}\big)^{1/2}; \omega_i^{(\lambda)}\right\}}. \tag{21}$$

Applying once again the centered inversion (2) ($\mathrm{INV}_2 = ()^{I_C}$), from (21) we obtain

$$\hat{Z}_i = z_i - \sqrt{\mu_i} \left\{\frac{1}{\big(c_i^{(\lambda)}\big)^{1/2}}; \frac{\omega_i^{(\lambda)}}{\big|c_i^{(\lambda)}\big|^{1/2}\big(\big|c_i^{(\lambda)}\big|^{1/2} - \omega_i^{(\lambda)}\big)}\right\}. \tag{22}$$

By virtue of (ii), (iv) and (v) of Lemma 2, from (22) we find

$$\begin{aligned}
\hat{r}_i &= \operatorname{rad}\hat{Z}_i = \frac{\sqrt{\mu_i}\,\omega_i^{(\lambda)}}{\big|c_i^{(\lambda)}\big|^{1/2}\big(\big|c_i^{(\lambda)}\big|^{1/2}-\omega_i^{(\lambda)}\big)} < \frac{12\sqrt{\mu_i}\,(n-\mu_i)|\varepsilon_i|r/5\rho^3}{\dfrac{\sqrt{4/5}}{|\varepsilon_i|}\Big(\dfrac{\sqrt{4/5}}{|\varepsilon_i|}-\omega_i^{(\lambda)}\Big)}\\
&= \frac{12\sqrt{\mu_i}\,(n-\mu_i)|\varepsilon_i|^3 r}{5\rho^3\sqrt{4/5}\big(\sqrt{4/5}-|\varepsilon_i|\omega_i^{(\lambda)}\big)} < \frac{7\sqrt{\mu_i}\,(n-\mu_i)|\varepsilon_i|^3 r}{2\rho^3}.
\end{aligned}$$

Hence we have the estimations

$$\hat{r} = \mathcal{O}(\epsilon^3 r) \tag{23}$$

and

$$\hat{r}_i < \frac{r_i}{25}. \tag{24}$$

From (22) we get

$$\hat{z}_i = \operatorname{mid}\hat{Z}_i = z_i - \frac{\sqrt{\mu_i}}{\big(c_i^{(\lambda)}\big)^{1/2}} = z_i - \frac{\sqrt{\mu_i}}{\Big[\delta_{2,i} - \sum\limits_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j\big(u_{ij}^{(\lambda)}\big)^2\Big]_*^{1/2}}, \tag{25}$$

and hence, by (ii) of Lemma 2, we find

$$|\hat{z}_i - z_i| = \frac{\sqrt{\mu_i}}{\big|c_i^{(\lambda)}\big|^{1/2}} < \frac{|\varepsilon_i|}{\sqrt{4/5}} < \frac{3}{2} r_i. \tag{26}$$

Let $A_i = \Sigma_{2,i} - \sum\limits_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j\big(u_{ij}^{(\lambda)}\big)^2$, then

$$\begin{aligned}
A_i &= \Sigma_{2,i} - \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j\big(u_{ij}^{(\lambda)}\big)^2 = \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j\left[\frac{1}{(z_i-\zeta_j)^2} - \frac{1}{\big(h_{ij}^{(\lambda)}\big)^2}\right]\\
&= \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j\left(\frac{1}{z_i-\zeta_j} - \frac{1}{h_{ij}^{(\lambda)}}\right)\left(\frac{1}{z_i-\zeta_j} + \frac{1}{h_{ij}^{(\lambda)}}\right)\\
&= \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j\left(\frac{\xi_j^{(\lambda)}\varepsilon_j^{\lambda+1}}{(z_i-\zeta_j)h_{ij}^{(\lambda)}}\right)\left(\frac{1}{z_i-\zeta_j} + \frac{1}{h_{ij}^{(\lambda)}}\right),
\end{aligned}$$

and we estimate

$$|A_i| = \mathcal{O}(\epsilon^{\lambda+1}) \quad (\lambda = 1, 2). \tag{27}$$

Since

$$\delta_{2,i} - \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j\big(u_{ij}^{(\lambda)}\big)^2 = \frac{\mu_i}{\varepsilon_i^2} + \Sigma_{2,i} - \sum_{\substack{j\in \boldsymbol{I}_k\\ j\neq i}} \mu_j\big(u_{ij}^{(\lambda)}\big)^2 = \frac{\mu_i}{\varepsilon_i^2}\big(1+\varepsilon_i^2 A_i\big),$$

from (25) it follows

$$\hat{\varepsilon}_i = \hat{z}_i - \zeta_i = z_i - \zeta_i - \sqrt{\mu_i}/\big(c^{(\lambda)}\big)^{1/2} = \varepsilon_i - \frac{\sqrt{\mu_i}\,\varepsilon_i}{\Big[1+\varepsilon_i^2 A_i\Big]_*^{1/2}}. \tag{28}$$

Considering that the approximations are close to the zeros (which is provided by (20)) and taking into account (by (27)) that $|\varepsilon_i^2 A_i| = \mathcal{O}(|\varepsilon_i|^{\lambda+3})$ is a very small quantity, we can use the approximation

$$\Big[1+\varepsilon_i^2 A_i\Big]_*^{1/2} \cong 1 + \frac{\varepsilon_i^2 A_i}{2}.$$

According to this we get from (28)

$$\hat{\varepsilon}_i \cong \varepsilon_i - \frac{\varepsilon_i}{1+\dfrac{\varepsilon_i^2 A_i}{2}} = \frac{\varepsilon_i^3 A_i}{2+\varepsilon_i^2 A_i}.$$

The denominator is obviously bounded and tends to 2 when $\varepsilon_i \to 0$. In regard to this fact and (27), we find

$$|\hat{\varepsilon}_i| = |\varepsilon_i|^3 \mathcal{O}(\epsilon^{\lambda+1}). \tag{29}$$

The use of the inequalities (20), (24) and (26) yields

$$\begin{aligned} |\hat{z}_i - \hat{z}_j| &\geq |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| > \rho + r_j - \frac{3}{2}r_i - \frac{3}{2}r_j \\ &> 3(n-\mu)r - 2r > 25\hat{r}\Big[3(n-\mu)-2\Big]. \end{aligned}$$

According to the last inequality we get for any pairs $i,j \in \boldsymbol{I}_k$ $(i \neq j)$

$$|\hat{z}_i - \hat{z}_j| > 2\hat{r} \geq \hat{r}_i + \hat{r}_j \quad (i \neq j),$$

meaning that the disks $\widehat{Z}_1, \ldots, \widehat{Z}_k$ are mutually disjoint (in view of (4)). Furthermore, we have for an arbitrary pair $i,j \in \boldsymbol{I}_k$ $(i \neq j)$

$$|\hat{z}_i - \hat{z}_j| - \hat{r}_j > 25\hat{r}\big[3(n-\mu)-2\big] - \hat{r} > 3(n-\mu)\hat{r}.$$

Hence

$$\hat{\rho} > 3(n-\mu)\hat{r}.$$

Therefore, we have proved that the initial condition (20) leads to the inequality of the same form but for the index $m = 1$. Besides, we note that the inequality (24) of the form $r^{(1)} < r^{(0)}/25$ points to the rigorous contraction of the new circular approximations $\widehat{Z}_1, \ldots, \widehat{Z}_k$.

Repeating the above analysis and the argumentation for an arbitrary index $m \geq 0$, we prove that previously derived relations hold for the index $m+1$. Since these relations have already been proved for $m = 0$, according to induction it follows that, under the condition (20), they are valid for all $m \geq 1$. In particular, we have

$$\rho^{(m)} > 3(n-\mu)r^{(m)} \tag{30}$$

(the assertion 1°) and

$$r^{(m+1)} < \frac{r^{(m)}}{25}. \tag{31}$$

From the inequality (31) we conclude that the sequence of maximal radii $\{r^{(m)}\}$ tends to 0, which means that the inclusion method (13) is *convergent.* Furthermore, since the inequality (30) holds, the implications considered in Lemma 1 are valid for arbitrary m. Therefore, the Ostrowski-like method (13) with corrections is well defined at each iterative step.

Suppose that $\zeta_i \in Z_i^{(m)}$ for each $i \in \boldsymbol{I}_k$. Then from (7) and (13) we obtain that $\zeta_i \in Z_i^{(m+1)}$ (according to the inclusion isotonicity). Since $\zeta_i \in Z_i^{(0)}$ (the assumption of Theorem 2) it follows by mathematical induction that $\zeta_i \in Z_i^{(m)}$ for each $i \in \boldsymbol{I}_k$ and $m = 0, 1, \ldots$ (the assertion 2°).

It remains to determine the lower bound of the R-order of convergence of the method (13) (the assertion 3°). From (23) we observe that the sequences $\{r_i^{(m)}\}$ and $\{\varepsilon_i^{(m)}\}$ of the radii and errors are mutually dependent. For simplicity, as usual in this type of analysis, we adopt $1 > \epsilon^{(0)} = r^{(0)} > 0$, which means that we deal with the "worst case" model. This assumption does not have the influence on the final result since the lower bound of the R-order of convergence is obtained in a limit process.

From (23) and (29) we observe that the sequences $\{\epsilon^{(m)}\}$ and $\{r^{(m)}\}$ behave as follows

$$\epsilon^{(m+1)} \sim \left(\epsilon^{(m)}\right)^{\lambda+4}, \quad r^{(m+1)} \sim \left(\epsilon^{(m)}\right)^3 r^{(m)} \quad (\lambda = 1, 2).$$

Using these relations we form the R-matrix $T_2^{(C)} = \begin{bmatrix} \lambda+4 & 0 \\ 3 & 1 \end{bmatrix}$ (in regard to (19)) and find its spectral radius $\rho\big(T_2^{(C)}\big) = \lambda + 4$ and the corresponding eigenvector $\boldsymbol{x}_\rho = \big((\lambda+3)/3, 1\big) > 0$. Hence, according to Theorem 1, we get

$$O_R\big((13)_C\big) \geq \rho\left(T_2^{(C)}\right) = \lambda + 4 \quad (\lambda = 1, 2).$$

In a similar way we can prove that the lower bound of the R-order of convergence of the inclusion method (13) when $\mathrm{INV}_1 = ()^{I_C}$, $\mathrm{INV}_2 = ()^{I_E}$ is the same as in the case when $\mathrm{INV}_1, \mathrm{INV}_2 = ()^{I_C}$.

2) *The case* $INV_1 = ()^{I_E}$

Let us apply the exact inversion (1) (that is, $\mathrm{INV}_1 = ()^{I_E}$) under the sums

(8) involved in the iterative formula (13), then we obtain

$$\begin{aligned}
A_i &= \Sigma_{2,i} - \sum_{\substack{j\in \boldsymbol{I}_k \\ j\neq i}} \mu_j \big(u_{ij}^{(\lambda)}\big)^2 = \sum_{\substack{j\in \boldsymbol{I}_k \\ j\neq i}} \mu_j \left[\frac{1}{(z_i-\zeta_j)^2} - \left(\frac{\bar{h}_{ij}^{(\lambda)}}{\big|h_{ij}^{(\lambda)}\big|^2 - r_j^2}\right)^2\right] \\
&= \sum_{\substack{j\in \boldsymbol{I}_k \\ j\neq i}} \mu_j \left(\frac{1}{z_i-\zeta_j} - \frac{\bar{h}_{ij}^{(\lambda)}}{\big|h_{ij}^{(\lambda)}\big|^2 - r_j^2}\right)\left(\frac{1}{z_i-\zeta_j} + \frac{\bar{h}_{ij}^{(\lambda)}}{\big|h_{ij}^{(\lambda)}\big|^2 - r_j^2}\right) \\
&= \sum_{\substack{j\in \boldsymbol{I}_k \\ j\neq i}} \mu_j \left(\frac{\xi_j^{(\lambda)}\bar{h}_{ij}^{(\lambda)}\varepsilon_j^2 - r_j^2}{(z_i-\zeta_j)\big(\big|h_{ij}^{(\lambda)}\big|^2 - r_j^2\big)}\right)\left(\frac{1}{z_i-\zeta_j} + \frac{\bar{h}_{ij}^{(\lambda)}}{\big|h_{ij}^{(\lambda)}\big|^2 - r_j^2}\right) \\
&= \mathcal{O}_M(\alpha'\epsilon^2 + \beta' r^2),
\end{aligned}$$

where α' and β' are real constants. Hence

$$|\hat{\varepsilon}_i| = |\varepsilon_i|^3 \mathcal{O}_M(\alpha'\epsilon^2 + \beta' r^2).$$

In a similar way as in the derivation of the relation (23), we can show that $\hat{r} = \mathcal{O}(\epsilon^3 r)$. Therefore, the sequences $\{\epsilon^{(m)}\}$ and $\{r^{(m)}\}$ behave as follows

$$\epsilon^{(m+1)} \sim \big(\epsilon^{(m)}\big)^3 \big(r^{(m)}\big)^2, \quad r^{(m+1)} \sim \big(\epsilon^{(m)}\big)^3 r^{(m)}$$

and the corresponding R-matrix is $T_2^{(E)} = \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix}$, independently of the type of used corrections (Schröder's or Halley's one). The spectral radius of this matrix is $\rho(T_2^{(E)}) = 2+\sqrt{7}$ and the corresponding eigenvector $\boldsymbol{x}_\rho = \big((1+\sqrt{5})/2, 1\big)$ has positive components. Hence, according to Theorem 1, we obtain

$$O_R\big((13)_E\big) \geq \rho\left(T_2^{(E)}\right) = 2+\sqrt{7} \approx 4.646.$$

□

Remark 2. Theorem 2 and a number of numerical examples show that the application of the exact inversion (1) produces less acceleration of the convergence of the interval method (13) than the centered inversion (2) (compare Tables 1 and 2). The explanation of this paradoxical outcome, having in mind that the exact inversion gives smaller disks (see (2)), lies in the fact that the centered inversion provides the better convergence of the midpoints of disks generated by (13), which further forces faster convergence of the radii of disks. More details can be found in [2].

Remark 3. Theorem 2 is proved under computationally verified initial condition (formula (20))

$$\rho^{(0)} > 3(n-\mu)r^{(0)}.$$

Conditions of this form are quite reasonable and natural since the maximal radius $r^{(0)}$ gives an information on the size of initial inclusion disks, while the quantity $\rho^{(0)}$ is a measure of the separation of disks $Z_1^{(0)}, \ldots, Z_k^{(0)}$ from each other. The above condition (20) and conditions of similar form are only sufficient; in practice, simultaneous interval methods can converge although the corresponding initial conditions are not satisfied. In other words, in practice the ratio $r^{(0)}/\rho^{(0)}$ may be greater (or sometimes considerably greater in the case of polynomials of higher degree) than $1/(3(n-\mu)r^{(0)})$, which can be observed from Examples 1 and 2. Actually, stronger conditions are needed in the convergence analysis because a number of inequalities and inclusion properties must be fulfilled. Theoretically, an interval method could be feasible if the initial disks are nonintersecting, that is, if $\rho^{(0)} > r^{(0)}$. However, some other conditions (for example, division by a zero-disk has to be avoided) force more rigorous condition of the form $\rho^{(0)} > \alpha r^{(0)}$, where $\alpha > 1$ is a real constant. The determination of α is an open problem yet; we can only say that well-spaced disks provide a smaller α, and opposite; very close disks, similarly as very close (complex) approximations in complex arithmetic, slow down the convergence of any iterative method. An efficient approach in finding approximate value of α is given in [14].

5. Single step methods with corrections

Applying the Gauss-Seidel approach to the methods (13) we obtain the single-step methods

$$(32)\qquad \hat{Z}_i = z_i - \sqrt{\mu_i}\ \mathrm{INV}_2\Big(\Big[\delta_{2,i} - S_{2,i}\big(\widehat{\boldsymbol{Z}}, \boldsymbol{Z}^{(\lambda)}\big)\Big]_*^{1/2}\Big) \quad (i \in \boldsymbol{I}_k,\ \lambda = 0,1,2),$$

where $\mathrm{INV}_2 \in \{()^{I_E}, ()^{I_C}\}$.

It is very difficult to find the R-order of convergence of the single step method (32) with corrections ($\lambda = 1,2$) as a function of the number of zeros k. However, we can estimate the limit bounds of the R-order taking the limits cases $k = 2$ and very large k.

Using the well-known fact that the convergence rate of a single-step method becomes almost the same to the one of the corresponding total-step method when the number of different zeros is large, according to Theorem 2 we have for a very large k

$$O_R((32,k)) \cong O_R(13) \geq \begin{cases} \lambda + 4 \quad (\lambda = 1,2), & \text{if } \mathrm{INV}_1 = ()^{I_C}, \\[2ex] 2 + \sqrt{7} \cong 4.646, & \text{if } \mathrm{INV}_1 = ()^{I_E}. \end{cases}$$

Consider now the single-step methods (32) for $k = 2$ and assume that $|\varepsilon_1^{(0)}| = |\varepsilon_2^{(0)}| = r_1^{(0)} = r_2^{(0)} < 1$ (the "worst case" model). After an extensive calculation we derive the following estimates:

(i) Case $\mathrm{INV}_1 = ()^{I_E}$:

$$\begin{aligned} |\hat{\varepsilon}_1| &\sim |\varepsilon_1|^3 r_2^2, \quad & |\hat{\varepsilon}_2| &\sim |\varepsilon_1|^3 |\varepsilon_2|^3 r_2^2, \\ \hat{r}_1 &\sim |\varepsilon_1|^3 r_2, \quad & \hat{r}_2 &\sim |\varepsilon_1|^3 |\varepsilon_2|^3 r_2. \end{aligned}$$

(ii) Case $\mathrm{INV}_1 = ()^{I_C}$:

$$\begin{aligned} |\hat{\varepsilon}_1| &\sim |\varepsilon_1|^3 |\varepsilon_2|^{\lambda+1}, \quad & |\hat{\varepsilon}_2| &\sim |\varepsilon_1|^3 |\varepsilon_2|^{\lambda+4}, \\ \hat{r}_1 &\sim |\varepsilon_1|^3 r_2, \quad & \hat{r}_2 &\sim |\varepsilon_1|^3 |\varepsilon_2|^3 r_2. \end{aligned}$$

The corresponding R-matrices and their spectral radii and eigenvectors are:

(i) Case $\mathrm{INV}_1 = ()^{I_E}$:

$$T_4^{(E)} = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 3 & 3 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix}, \qquad \begin{array}{l} \rho\big(T_4^{(E)}\big) = 6.2965, \\ \boldsymbol{x}_\rho^{(E)} = (1, 1.91, 0.7382, 1.6483) > 0. \end{array}$$

(ii) Case $\mathrm{INV}_1 = ()^{I_C}$:

$$T_4^{(C)} = \begin{bmatrix} 3 & \lambda+1 & 0 & 0 \\ 3 & \lambda+4 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix}, \quad \rho\big(T_4^{(C)}\big) = \begin{cases} 6.6457, & \lambda = 1, \\ 7.8541, & \lambda = 2, \end{cases}$$

$$\boldsymbol{x}_\rho^{(C)} = \begin{cases} (1, 1.8229, 0.6771, 1.5) > 0, & \lambda = 1, \\ (1, 1.6180, 0.5279, 1.1459) > 0, & \lambda = 2. \end{cases}$$

According to the previous results and the range of the R-order of the method (32) without corrections ($\lambda = 0$, see the end of Section 2), we can formulate the following assertion.

Theorem 3. *The ranges of the lower bounds of the R-order of convergence of the single-step method* (32) *are*

$$\begin{aligned} O_R(32) &\in (4, 5.303) \quad (\lambda = 0), \\ O_R(32) &\in (4.646, 6.297), \quad (\lambda = 1, 2), \quad \textit{if} \ \ \mathrm{INV}_1 = ()^{I_E}, \\ O_R(32) &\in \begin{cases} (5, 6.646) & (\lambda = 1), \\ (6, 7.855) & (\lambda = 2), \end{cases} \quad \textit{if} \ \ \mathrm{INV}_1 = ()^{I_C}. \end{aligned}$$

6. Numerical examples

The presented inclusion methods of Ostrowski's type have been tested in solving a number of polynomial equations, applying a multi-stage *globally convergent* composite algorithm described in detail in [12] and [5].

For comparison purpose, we have also tested the following methods:

Total-step Halley-like inclusion method:

$$\hat{Z}_i = z_i - \mathrm{INV}_2\left(H(z_i)^{-1} - \frac{P(z_i)}{2P'(z_i)}\left[\frac{1}{\mu_i}S_{1,i}^2\big(\boldsymbol{Z}^{(\lambda)},\boldsymbol{Z}^{(\lambda)}\big) + S_{2,i}\big(\boldsymbol{Z}^{(\lambda)},\boldsymbol{Z}^{(\lambda)}\big)\right]\right); \tag{33}$$

Single-step Halley-like inclusion method:

$$\hat{Z}_i = z_i - \mathrm{INV}_2\left(H(z_i)^{-1} - \frac{P(z_i)}{2P'(z_i)}\left[\frac{1}{\mu_i}S_{1,i}^2\big(\widehat{\boldsymbol{Z}},\boldsymbol{Z}^{(\lambda)}\big) + S_{2,i}\big(\widehat{\boldsymbol{Z}},\boldsymbol{Z}^{(\lambda)}\big)\right]\right); \tag{34}$$

Total-step Laguerre-like inclusion method:

$$\begin{aligned}\widehat{Z}_i &= z_i - n\,\mathrm{INV}_2\Big(\delta_{1,i} + \Big[\frac{n-\mu_i}{\mu_i}\Big(n\delta_{2,i} - \delta_{1,i}^2 - nS_{2,i}\big(\boldsymbol{Z}^{(\lambda)},\boldsymbol{Z}^{(\lambda)}\big)\\ &\quad + \frac{n}{n-\mu_i}S_{1,i}^2\big(\boldsymbol{Z}^{(\lambda)},\boldsymbol{Z}^{(\lambda)}\big)\Big)\Big]_*^{1/2}\Big);\end{aligned} \tag{35}$$

Single-step Laguerre-like inclusion method:

$$\begin{aligned}\widehat{Z}_i &= z_i - n\,\mathrm{INV}_2\Big(\delta_{1,i} + \Big[\frac{n-\mu_i}{\mu_i}\Big(n\delta_{2,i} - \delta_{1,i}^2 - nS_{2,i}\big(\widehat{\boldsymbol{Z}},\boldsymbol{Z}^{(\lambda)}\big)\\ &\quad + \frac{n}{n-\mu_i}S_{1,i}^2\big(\widehat{\boldsymbol{Z}},\boldsymbol{Z}^{(\lambda)}\big)\Big)\Big]_*^{1/2}\Big);\end{aligned} \tag{36}$$

Total-step Euler-like inclusion method:

$$\begin{aligned}\widehat{Z}_i &= z_i - 2\mu_i\,\mathrm{INV}_2\Big(\delta_{1,i} + \Big[2\mu_i\delta_{2,i} - \delta_{1,i}^2 - 2\big(\mu_i S_{2,i}\big(\boldsymbol{Z}^{(\lambda)},\boldsymbol{Z}^{(\lambda)}\big)\\ &\quad - S_{1,i}^2\big(\boldsymbol{Z}^{(\lambda)},\boldsymbol{Z}^{(\lambda)}\big)\big)\Big]_*^{1/2}\Big);\end{aligned} \tag{37}$$

Single-step Euler-like inclusion method:

$$\widehat{Z}_i = z_i - 2\mu_i\,\mathrm{INV}_2\Big(\delta_{1,i} + \Big[2\mu_i\delta_{2,i} - \delta_{1,i}^2 - 2\big(\mu_i S_{2,i}\big(\widehat{\boldsymbol{Z}},\boldsymbol{Z}^{(\lambda)}\big) - S_{1,i}^2\big(\widehat{\boldsymbol{Z}},\boldsymbol{Z}^{(\lambda)}\big)\big)\Big]_*^{1/2}\Big). \tag{38}$$

For all methods (33)–(38) $i \in \boldsymbol{I}_k$, $\mathrm{INV}_1, \mathrm{INV}_2 \in \{()^{I_E}, ()^{I_C}\}$ and the correction $C(z)$ is taken to be $C(z) = 0$ (the associated code $\lambda = 1$), $C(z) = N(z)$

($\lambda = 1$) or $C(z) = H(z)$ ($\lambda = 2$). The R-order of convergence of the total-step methods (33), (35) and (37) is the same as that of the method (13), given in Theorem 2. The ranges of the lower bounds of the R-order of convergence of the single-step methods (34), (36) and (38) is the same as that of the method (32), given in Theorem 3.

Example 1. We have implemented total-step interval methods (13), (33), (35) and (37) for $\lambda = 0, 1, 2$ to enclose multiple zeros of the polynomial

$$\begin{aligned} P(z) &= z^{20} - 3z^{19} - 3z^{18} - z^{17} - 34z^{16} + 144z^{15} + 184z^{14} + 184z^{13} + 448z^{12} \\ &\quad -1648z^{11} - 2992z^{10} - 5392z^9 - 8352z^8 - 20864z^7 - 33536z^6 \\ &\quad -52224z^5 - 98304z^4 - 47104z^3 - 73728z^2 - 110592z - 221184. \end{aligned}$$

The zeros of P are $\zeta_1 = 3$, $\zeta_2 = -2$, $\zeta_3 = 1 + i$, $\zeta_4 = 1 - i$, $\zeta_5 = -1 - i$, $\zeta_6 = -1 + i$, $\zeta_7 = -2i$, $\zeta_8 = 2i$ of the multiplicity $\mu_1 = \mu_2 = 3$, $\mu_3 = \mu_4 = \mu_5 = \mu_6 = 2$, $\mu_7 = \mu_8 = 3$, respectively. The initial disks were selected to be $Z_i^{(0)} = \{z_i^{(0)}; 0.5\}$, with the centers

$$\begin{aligned} z_1^{(0)} &= 3.2 + 0.1i, \quad z_2^{(0)} = -2.1 + 0.2i, \; z_3^{(0)} = 0.9 + 1.2i, \; z_4^{(0)} = 0.8 - 1.2i, \\ z_5^{(0)} &= -1.2 - 0.9i, \; z_6^{(0)} = -0.9 + 0.8i, \; z_7^{(0)} = 0.1 - 2.2i, \; z_8^{(0)} = 0.2 + 2.1i. \end{aligned}$$

We have applied separately the exact and centered inversion under the corresponding sums in all tested methods.

method	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(13) $\lambda = 0$	$2.32(-2)$	$2.41(-9)$	$1.69(-38)$
(13) $\lambda = 1$	$3.31(-2)$	$7.66(-9)$	$1.04(-42)$
(13) $\lambda = 2$	$3.45(-2)$	$1.01(-8)$	$1.08(-43)$
(33) $\lambda = 0$	$1.05(-2)$	$1.05(-5)$	$3.25(-25)$
(33) $\lambda = 1$	$1.43(-1)$	$2.96(-5)$	$2.26(-26)$
(33) $\lambda = 2$	$1.47(-1)$	$3.07(-5)$	$3.05(-26)$
(35) $\lambda = 0$	$2.96(-2)$	$4.25(-9)$	$5.66(-39)$
(35) $\lambda = 1$	$4.34(-2)$	$2.72(-8)$	$9.16(-41)$
(35) $\lambda = 2$	$4.59(-2)$	$3.75(-8)$	$1.70(-40)$
(37) $\lambda = 0$	$7.59(-2)$	$1.10(-6)$	$1.47(-28)$
(37) $\lambda = 1$	$1.42(-1)$	$5.78(-5)$	$8.21(-23)$
(37) $\lambda = 2$	diverges		

Table 1 Maximal radii of inclusion disks – exact inversions

method	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(13) $\lambda = 0$	3.15(−2)	1.67(−9)	1.04(−40)
(13) $\lambda = 1$	4.63(−2)	6.61(−11)	1.03(−57)
(13) $\lambda = 2$	4.84(−2)	1.96(−13)	5.41(−82)
(33) $\lambda = 0$	2.09(−1)	5.47(−7)	3.70(−32)
(33) $\lambda = 1$	3.46(−1)	1.57(−8)	3.19(−45)
(33) $\lambda = 2$	3.70(−1)	1.38(−10)	5.85(−73)
(35) $\lambda = 0$	4.25(−2)	2.96(−9)	3.56(−41)
(35) $\lambda = 1$	6.56(−2)	1.03(−10)	1.48(−58)
(35) $\lambda = 2$	6.96(−2)	4.50(−13)	6.32(−82)
(37) $\lambda = 0$	1.96(−1)	2.18(−7)	3.14(−34)
(37) $\lambda = 1$	diverges		
(37) $\lambda = 2$	diverges		

Table 2 Maximal radii of inclusion disks – centered inversions

The maximal radii of the inclusion disks produced in the three first iterative steps are given in Tables 1 and 2, where $A(-t)$ is the abbreviation of $A \times 10^{-t}$. Let us note that the Euler-like method (37) with corrections break down or work with difficulties in this example.

Comparing the results from Tables 1 and 2, we observe that the centered inversion enables considerably smaller inclusion disks than the exact inversion, which coincides with the assertions of Theorem 2. The entries from Tables 1 and 2 and a number of numerical examples show that the Ostrowski-like methods possess better convergence behavior than the Halley-like and Euler-like methods.

Example 2. To find circular inclusion approximations to the zeros of the polynomial

$$\begin{aligned} P(z) \;=\;& z^{12} - (2-3i)z^{11} + (16-6i)z^{10} - (26-38i)z^9 + (101-58i)z^8 \\ & -(120-131i)z^7 + (250-76i)z^6 - (72+20i)z^5 - (84-432i)z^4 \\ & +(864-292i)z^3 - 504z^2 + 432iz + 864, \end{aligned}$$

we have implemented the total-step interval methods (13) (with $\lambda = 0, 1, 2$) and the single-step methods (32) (with $\lambda = 0, 1, 2$). For comparison purpose, we have also tested the Halley-like, the Laguerre-like and the Euler-like methods (33)–(38). Total-step methods without corrections and with Newton's and Halley's corrections are additionally denoted by TS, TSN and TSH, respectively (see Table 3), while the corresponding single-step methods are marked with SS, SSN and SSH. All tested methods used the centered inversion which gives considerably better inclusions than the exact inversion, as stated in Theorems 2 and 3.

The zeros of P are $\zeta_1 = -1,\ \zeta_2 = 2i,\ \zeta_3 = 1+i,\ \zeta_4 = 1-i,\ \zeta_5 = -3i$ of the multiplicities $\mu_1 = 2,\ \mu_2 = 3,\ \mu_3 = 2,\ \mu_4 = 2,\ \mu_5 = 3$, respectively. The initial disks were selected to be $Z_i^{(0)} = \{z_i^{(0)}; 0.6\}$, with the centers:

$$\begin{aligned} z_1^{(0)} &= -1.1 + 0.2i, \quad z_2^{(0)} = -0.1 + 2.3i, \quad z_3^{(0)} = 0.9 + 1.1i, \\ z_4^{(0)} &= 0.8 - 1.2i, \quad z_5^{(0)} = 0.2 - 2.8i. \end{aligned}$$

The maximal radii of the inclusion disks produced in the first three iterative steps are given in Table 3.

	method	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
TS	(13) $\lambda = 0$	1.29(−2)	6.31(−12)	5.95(−50)
	(33) $\lambda = 0$	4.33(−2)	1.50(−9)	2.18(−41)
	(35) $\lambda = 0$	1.81(−2)	1.54(−11)	1.91(−50)
	(37) $\lambda = 0$	5.20(−2)	7.77(−10)	6.19(−45)
SS	(32) $\lambda = 0$	8.42(−3)	5.85(−13)	3.36(−54)
	(34) $\lambda = 0$	2.24(−2)	7.16(−11)	1.21(−45)
	(36) $\lambda = 0$	1.39(−2)	5.12(−13)	3.88(−56)
	(38) $\lambda = 0$	3.60(−2)	8.81(−12)	1.15(−50)
TSN	(13) $\lambda = 1$	1.01(−2)	2.60(−14)	6.07(−71)
	(33) $\lambda = 1$	3.74(−2)	3.80(−12)	9.32(−62)
	(35) $\lambda = 1$	1.51(−2)	1.45(−13)	6.10(−72)
	(37) $\lambda = 1$	3.79(−2)	9.23(−12)	1.45(−64)
SSN	(32) $\lambda = 1$	5.60(−3)	3.57(−15)	7.46(−75)
	(34) $\lambda = 1$	1.53(−2)	7.13(−13)	3.41(−64)
	(36) $\lambda = 1$	1.01(−2)	2.78(−15)	5.36(−77)
	(38) $\lambda = 1$	2.59(−2)	2.02(−13)	7.04(−68)
TSH	(13) $\lambda = 2$	1.03(−2)	5.39(−16)	7.69(−99)
	(33) $\lambda = 2$	3.76(−2)	3.18(−14)	4.45(−89)
	(35) $\lambda = 2$	1.52(−2)	2.09(−15)	1.29(−98)
	(37) $\lambda = 2$	3.74(−2)	5.83(−14)	1.90(−89)
SSH	(32) $\lambda = 2$	5.75(−3)	8.72(−18)	4.59(−104)
	(34) $\lambda = 2$	1.52(−2)	1.74(−15)	8.94(−93)
	(36) $\lambda = 2$	1.03(−2)	6.82(−17)	1.85(−102)
	(38) $\lambda = 2$	2.64(−2)	9.66(−15)	4.00(−92)

Table 3 Maximal radii of inclusion disks – centered inversions

We observe that the proposed methods with corrections produce very small disks. This is especially expressed when the centered inversion is employed, compare Tables 1 and 2. In some particular cases the proposed methods with corrections give in the first iterative step disks not smaller compared to those produced by the inclusion methods without corrections. The explanation lies in the fact that Schröder's and Halley's method do not necessarily improve the initial (point) approximations at the beginning of iterative process. In later iterations the convergence order of the inclusion methods with corrections increases and approaches thetheoretical one obtained in the presented convergence analysis.

According to results of the presented examples given in Tables 1, 2 and 3, and a number of tested polynomials, we can conclude that the proposed iterative methods of Ostrowski's type for the simultaneous inclusion of polynomial zeros either compete the existing methods of the same order or they produce better results, in concrete cases compared with the Halley-like methods (33) and (34) and the Euler-like methods (37) and (38). A simple analysis of computational efficiency shows that, in general, inclusion methods with corrections and centered inversion are more efficient than the corresponding methods without corrections. This advantage is a direct consequence of significantly increased convergence order achieved without additional function evaluations.

A comparison of simultaneous interval methods and the corresponding "point" versions (realized in ordinary complex arithmetic) was the subject of some previous papers. Here we wanted only to emphasize that these methods of different structures are essentially incomparable. Both classes have their own advantages and shortcomings, interval methods produce self-validated results; on the other hand, methods implemented in ordinary (real or complex) arithmetic have lower computational cost. In fact, the choice of a most convenient method from these two classes depends on the nature of the problem to be solved and specific requirements (for example, information on the error of computation, control of rounding errors, handling with uncertain data, and so on).

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