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ON JORDAN ISOMORPHISMS OF 2-TORSION FREE PRIME GAMMA RINGS

Sujoy Chakraborty[1], **Akhil Chandra Paul**[2]

Abstract. This paper defines an isomorphism, an anti-isomorphism and a Jordan isomorphism in a gamma ring and develops some important results relating to these concepts. Using these results we prove Herstein's theorem of classical rings in case of prime gamma rings by showing that every Jordan isomorphism of a 2-torsion free prime gamma ring is either an isomorphism or an anti-isomorphism.

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1. Introduction

Let M and Γ be two additive abelian groups. If there is a mapping of $M \times \Gamma \times M$ to M (sending $(a, \alpha, b) \to a\alpha b$) satisfying for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:
(i) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$, and
(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$,
then M is called a Γ-ring. This definition is due to Barnes [1].

A gamma ring M is said to be a prime gamma ring if and only if $a\Gamma M\Gamma b = 0$ (with $a, b \in M$) implies $a = 0$ or $b = 0$.

A gamma ring M is called a semiprime gamma ring if and only if $a\Gamma M\Gamma a = 0$ (with $a \in M$) implies $a = 0$.

An element x of a Γ-ring M is said to be 2-torsion free if and only if $2x = 0$ implies $x = 0$. Thus, a Γ-ring M is called 2-torsion free if and only if $2x = 0$ implies $x = 0$ for all $x \in M$.

The concepts of isomorphisms, anti-isomorphisms and Jordan isomorphisms were introduced by I. N. Herstein [3], [4] in classical ring theory. Here, a well-known result states that every Jordan isomorphism of a prime ring of characteristic different from 2 is either an isomorphism or an anti-isomorphism. The fact is that if the ring is semiprime, then the said result does not hold; the arguments behind it were simplified by M. Bresar in [2].

In this paper, we define isomorphisms, anti-isomorphisms and Jordan isomorphisms in gamma rings. Relating to these concepts we prove some important

[1]Department of Mathematics, Shahjalal University of Science and Technology, Sylhet-3114, Bangladesh, e-mail: sujoy_chbty@yahoo.com
[2]Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh, e-mail: acpaul_math@yahoo.com

results in the form of lemmas. At last we prove the above well-known result of Herstein in case of prime gamma rings.

2. Jordan Isomorphisms of Prime Γ-Rings

Let M and N be two Γ-rings. A bijective additive map $\varphi : M \to N$ is called an isomorphism if and only if $\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$.

And, a bijective additive map $\varphi : M \to N$ is called an anti-isomorphism if and only if $\varphi(a\alpha b) = \varphi(b)\alpha\varphi(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Finally, a bijective additive map $\varphi : M \to N$ is called a Jordan isomorphism if and only if $\varphi(a\alpha a) = \varphi(a)\alpha\varphi(a)$ for all $a \in M$ and $\alpha \in \Gamma$.

Note that the isomorphisms and antiisomorphisms are the examples of Jordan isomorphisms.

Example 2.1. *Let D be a division ring and suppose $M_{p,q}(D)$ denotes the additive group of all $p \times q$ matrices whose entries are from D. If $\Gamma = M_{q,p}(D)$, then under the usual matrix multiplication, $M_{p,q}(D)$ is a Γ-ring. If $p = q = n$, then we write $M_n(D)$ for $M_{n,n}(D)$.*

Let $M = M_n(D) \oplus M_n(D)$. Define $\varphi : M \to M$ by $(A, B) \to (A, B^t)$ for all $A, B \in M_n(D)$, where B^t denotes the transpose of B. Since $A \to A$ is an isomorphism and $B \to B^t$ is an anti-isomorphism, φ is a Jordan isomorphism of M. However, it is neither an isomorphism nor an anti-isomorphism.

To prove that every Jordan isomorphism of a 2-torsion free prime Γ-ring is either an isomorphism or an anti-isomorphism, we develop some important results as follows.

Lemma 2.1. *Let M be a Γ-ring and let $\varphi : M \to M$ be a Jordan isomorphism. Then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:*

(a) $\varphi(a\alpha b + b\alpha a) = \varphi(a)\alpha\varphi(b) + \varphi(b)\alpha\varphi(a)$;

(b) $\varphi(a\alpha b\beta a + a\beta b\alpha a) = \varphi(a)\alpha\varphi(b)\beta\varphi(a) + \varphi(a)\beta\varphi(b)\alpha\varphi(a)$.

In particular, if M is 2-torsion free, then

(c) $\varphi(a\alpha b\alpha a) = \varphi(a)\alpha\varphi(b)\alpha\varphi(a)$;

(d) $\varphi(a\alpha b\alpha c + c\alpha b\alpha a) = \varphi(a)\alpha\varphi(b)\alpha\varphi(c) + \varphi(c)\alpha\varphi(b)\alpha\varphi(a)$.

Especially, if M is 2-torsion free and $a\alpha b\beta c = a\beta b\alpha c$, then

(e) $\varphi(a\alpha b\beta a) = \varphi(a)\alpha\varphi(b)\beta\varphi(a)$;

(f) $\varphi(a\alpha b\beta c + c\alpha b\beta a) = \varphi(a)\alpha\varphi(b)\beta\varphi(c) + \varphi(c)\alpha\varphi(b)\beta\varphi(a)$.

Proof. Computing $\varphi((a + b)\alpha(a + b))$ and cancelling the equal terms from both sides of the equality obtained from the above computations, we obtain the proof of (a). Then by replacing $a\beta b + b\beta a$ for b in (a), we get (b). Next, (c) is obtained by replacing α for β in (b), using the hypothesis. Again, by replacing $a + c$ for a in (c), we get (d). Since $a\alpha b\beta c = a\beta b\alpha c$ and M is 2-torsion free, we obtain (e) from (d). Finally, (f) is obtained by replacing $a + c$ for a in (e). □

Definition 2.1. *Let φ be a Jordan isomorphism of a Γ-ring M. We define $\phi_\alpha(a, b) = \varphi(a\alpha b) - \varphi(a)\alpha\varphi(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$.*

Then, we have $\phi_\alpha(b,a) = \varphi(b\alpha a) - \varphi(b)\alpha\varphi(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Lemma 2.2. *Let φ be a Jordan isomorphism of a Γ-ring M. Then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,*
(a) $\phi_\alpha(a,b) + \phi_\alpha(b,a) = 0$;
(b) $\phi_\alpha(a, b+c) = \phi_\alpha(a,b) + \phi_\alpha(a,c)$;
(c) $\phi_\alpha(a+b, c) = \phi_\alpha(a,c) + \phi_\alpha(b,c)$;
(d) $\phi_{\alpha+\beta}(a,b) = \phi_\alpha(a,b) + \phi_\beta(a,b)$.

Proof. (a) is easily obtained by Lemma 2.1(a). The proof of each of (b), (c) and (d) is obvious. □

It is clear that φ is an isomorphism if and only if $\phi_\alpha(a,b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 2.2. *Let M be a Γ-ring, and $\varphi : M \to M$ a Jordan isomorphism. Then we define $\psi_\alpha(a,b) = \varphi(a\alpha b) - \varphi(b)\alpha\varphi(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$.*

Thus, we have $\psi_\alpha(b,a) = \varphi(b\alpha a) - \varphi(a)\alpha\varphi(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Lemma 2.3. *Let M be a Γ-ring and let $\varphi : M \to M$ be a Jordan isomorphism. Then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,*
(a) $\psi_\alpha(a,b) + \psi_\alpha(b,a) = 0$;
(b) $\psi_\alpha(a, b+c) = \psi_\alpha(a,b) + \psi_\alpha(a,c)$;
(c) $\psi_\alpha(a+b, c) = \psi_\alpha(a,c) + \psi_\alpha(b,c)$;
(d) $\psi_{\alpha+\beta}(a,b) = \psi_\alpha(a,b) + \psi_\beta(a,b)$.

Proof. The proof is obvious. □

It is also clear that φ is an anti-isomorphism if and only if $\psi_\alpha(a,b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Lemma 2.4. *Let M be a 2-torsion free Γ-ring and let $\varphi : M \to M$ be a Jordan isomorphism. Then for all $a, b, m \in M$ and $\alpha, \beta \in \Gamma$,*
(a) $\phi_\alpha(a,b)\alpha\varphi(m)\alpha\psi_\alpha(a,b) + \psi_\alpha(a,b)\alpha\varphi(m)\alpha\phi_\alpha(a,b) = 0$;
(b) $\phi_\alpha(a,b)\beta\varphi(m)\beta\psi_\alpha(a,b) + \psi_\alpha(a,b)\beta\varphi(m)\beta\phi_\alpha(a,b) = 0$;
(c) $\phi_\beta(a,b)\alpha\varphi(m)\alpha\psi_\beta(a,b) + \psi_\beta(a,b)\alpha\varphi(m)\alpha\phi_\beta(a,b) = 0$.

Proof. Consider $F = \varphi(a\alpha b\alpha m\alpha b\alpha a + b\alpha a\alpha m\alpha a\alpha b)$. Using (c) in Lemma 2.1, we obtain

$$\begin{aligned}
F &= \varphi(a\alpha(b\alpha m\alpha b)\alpha a) + \varphi(b\alpha(a\alpha m\alpha a)\alpha b) \\
&= \varphi(a)\alpha\varphi(b\alpha m\alpha b)\alpha\varphi(a) + \varphi(b)\alpha\varphi(a\alpha m\alpha a)\alpha\varphi(b) \\
&= \varphi(a)\alpha\varphi(b)\alpha\varphi(m)\alpha\varphi(b)\alpha\varphi(a) + \varphi(b)\alpha\varphi(a)\alpha\varphi(m)\alpha\varphi(a)\alpha\varphi(b)
\end{aligned}$$

On the other hand, according to (d) in Lemma 2.1, we get

$$\begin{aligned}
F &= \varphi((a\alpha b)\alpha m\alpha(b\alpha a) + (b\alpha a)\alpha m\alpha(a\alpha b)) \\
&= \varphi(a\alpha b)\alpha\varphi(m)\alpha\varphi(b\alpha a) + \varphi(b\alpha a)\alpha\varphi(m)\alpha\varphi(a\alpha b)
\end{aligned}$$

Then, rearranging the terms in the equality obtained from F and using (a) in Lemma 2.1, we have the proof of (a).

Considering now $F = \varphi(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b)$ and applying the same procedure as in the proof of (a), we obtain (b).

Finally, (c) is obtained by interchanging α and β in (b). □

Lemma 2.5. *Let M be a 2-torsion free semiprime Γ-ring. If $a, b \in M$ such that $a\Gamma m\Gamma b + b\Gamma m\Gamma a = 0$ for all $m \in M$, then $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$.*

Proof. Let m and m' be two arbitrary elements of M. Then by using $a\Gamma m\Gamma b = -b\Gamma m\Gamma a$, we obtain

$$\begin{aligned} (a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b) &= -b\Gamma(m\Gamma a\Gamma m')\Gamma a\Gamma m\Gamma b \\ &= a\Gamma(m\Gamma a\Gamma m')\Gamma b\Gamma m\Gamma b \\ &= -(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b). \end{aligned}$$

Therefore, we get $2((a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)) = 0$. Since M is a 2-torsion free Γ-ring, then $a\Gamma m\Gamma b = 0$ for all $m \in M$. □

Lemma 2.6. *Let $G_1, \cdots, G_n$ be additive groups, and M a semiprime Γ-ring. Suppose that the mappings $f : G_1 \times \cdots \times G_n \to M$ and $g : G_1 \times \cdots \times G_n \to M$ are additive in each argument. If $f(a_1, \cdots, a_n)\Gamma m\Gamma g(a_1, \cdots, a_n) = 0$ for all $m \in M$ and $a_i \in G_i$, $i = 1, \cdots, n$, then $f(a_1, \cdots, a_n)\Gamma m\Gamma g(b_1, \cdots, b_n) = 0$ for all $m \in M$ and $a_i, b_i \in G_i$, $i = 1, \cdots, n$.*

Proof. It suffices to prove the case $n = 1$. The mappings are then $f : G_1 \to M$ and $g : G_1 \to M$ such that $f(a)\Gamma m\Gamma g(a) = 0$ and $f(b)\Gamma m\Gamma g(b) = 0$ for all $a, b \in G_1$ and $m \in M$. Thus, we have

$$\begin{aligned} 0 &= f(a+b)\Gamma m\Gamma g(a+b) \\ &= f(a)\Gamma m\Gamma g(a) + f(a)\Gamma m\Gamma g(b) + f(b)\Gamma m\Gamma g(a) + f(b)\Gamma m\Gamma g(b) \\ &= f(a)\Gamma m\Gamma g(b) + f(b)\Gamma m\Gamma g(a). \end{aligned}$$

Let $m' \in M$. Then by the assumption, we get

$$\begin{aligned} &(f(a)\Gamma m\Gamma g(b))\Gamma m'\Gamma(f(a)\Gamma m\Gamma g(b)) \\ &\quad= -f(a)\Gamma(m\Gamma g(b)\Gamma m'\Gamma f(b)\Gamma m)\Gamma g(a) \\ &\quad= 0. \end{aligned}$$

Hence, by the semiprimeness of M, we have $f(a)\Gamma m\Gamma g(b) = 0$. This completes the proof of the lemma. □

Now, from Lemma 2.2 and Lemma 2.3, we see that the both mappings $\phi_\alpha(a, b)$ and $\psi_\alpha(a, b)$ are additive in each argument. Hence, in view of Lemma 2.4, Lemma 2.5 and Lemma 2.6, we get the following:

Corollary 2.1. *Let φ be a Jordan isomorphism of a 2-torsion free semiprime Γ-ring M. Then for all $a, b, c, d, m \in M$ and $\alpha, \beta \in \Gamma$,*

(a) $\phi_\alpha(a, b)\alpha\varphi(m)\alpha\psi_\alpha(c, d) = 0$*;*

(b) $\phi_\alpha(a, b)\beta\varphi(m)\beta\psi_\alpha(c, d) = 0$*;*

(c) $\phi_\beta(a, b)\alpha\varphi(m)\alpha\psi_\alpha(c, d) = 0$.

Theorem 2.1. *Every Jordan isomorphism of a 2-torsion free prime Γ-ring is either an isomorphism or an anti-isomorphism.*

Proof. Let φ be a Jordan isomorphism of a 2-torsion free prime Γ-ring M. Taking any one of the equalities in Corollary 2.1 (say, (a)), we have

$$\phi_\alpha(a, b)\alpha\varphi(m)\alpha\psi_\alpha(c, d) = 0$$

for all $a, b, c, d, m \in M$ and $\alpha \in \Gamma$.

Since φ is onto and M is prime, therefore, either $\phi_\alpha(a, b) = 0$ or $\psi_\alpha(c, d) = 0$ for all $a, b, c, d \in M$ and $\alpha \in \Gamma$.

If $\phi_\alpha(a, b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$, then φ is an isomorphism. And, if $\psi_\alpha(c, d) = 0$ for all $c, d \in M$ and $\alpha \in \Gamma$, then φ is an anti-isomorphism. The proof of this theorem is thus completed. □

Remark 2.1. *The result of Theorem 2.1 does not hold in case of semiprime Γ-rings. In Example 2.1, we observe that M is a semiprime Γ-ring and the mapping $\varphi : M \to M$ is a Jordan isomorphism. But, φ is neither an isomorphism nor an anti-isomorphism of M.*

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