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SOME PROPERTIES OF JACOBI POLYNOMIALS

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Abstract. A main motivation for this paper is the search for the sufficient condition of the primality of an integer n in order that the congruence $1^{n-1} + 2^{n-1} + 3^{n-1} + \cdots + (n-1)^{n-1} \equiv -1 \pmod{n}$ holds. Some properties of Jacobi polynomials were investigated using certain Kummer results. Certain properties of Bernoulli polynomials as well as the Staudt–Clausen theorem for prime factors were also used. In this paper, several new properties of the coefficients of the polynomial

$$d(m,k) = (-1)^k \cdot \frac{2^k}{k} \cdot \sum_{s \geq \frac{k}{2}}^{k-1} \binom{2m-1}{2k-2s-1} \cdot \binom{2s-1}{k-1} \cdot B_{2m-2k+2s},$$

have been obtained, and they are formulated in Theorem 1 and Theorem 2.

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1. Introduction

For almost three centuries, the properties of Bernoulli numbers and polynomials have been investigated (they were introduced in the work of Jacob Bernoulli in 1713 [1]), and yet there are still important unsolved problems in this theory.

Bernoulli numbers and polynomials give important connections between various solved and unsolved problems in different mathematical theories. This fact is rather attractive for further research.

We begin with the remark that in the year 1950, G. Ging gave the following conjecture:

n is a prime number if and only if the following congruence holds

$$1^{n-1} + 2^{n-1} + 3^{n-1} + \cdots + (n-1)^{n-1} \equiv -1 \pmod{n}. \tag{1}$$

That this condition is necessary for the primality of n has been proved, but whether it is sufficient is still unknown, although it is known that the hypothesis holds for all numbers which do exceed 10^{1000}.

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It is easy to conclude (see [7]) that the natural number n will satisfy the condition (1) if and only if for every prime factor p of n one has

$$n - 1 \equiv 0 \pmod{p} - 1, \tag{2}$$

$$n \equiv p \pmod{p}^2. \tag{3}$$

The condition (2) points out to the fact that if the number n is not prime it has to be a number with Fermat properties, see p.p. 331–338.

Therefore, the composite number n, has to be an odd number. We prove that the conditions (2) and (3) are equivalent to the condition

$$n \cdot B_{n-1} \equiv -1 \pmod{n}, \tag{4}$$

where B_{n-1} is a Bernoulli number in the so-called "system of symbols with even indices" (see [6]). It is not difficult to see that the condition (4) is a congruence among the n-integers. The rational number $\frac{a}{b}$ is an n-integer if $(b, n) = 1$. In this paper we will often use congruences in the ring of n-integers. For two n-integers $\frac{a}{b}$ and $\frac{c}{d}$ we write $\frac{a}{b} \equiv \frac{c}{d} \pmod{n}$ if and only if $ad - bc \equiv 0 \pmod{n}$. It is obvious that the n-integer $a^{-1} = \frac{1}{a}$ may be seen as the solution of the congruence

$$ax \equiv 1 \pmod{n}$$

Let us consider the congruence (4). Suppose p ia any prime factor of n. Obviously, B_{n-1} cannot be a p-integer, therefore from the Staudt-Clausen theorem it follows the condition (2), and then, by repeated application, the condition (3). Conversely, from the relation (2) it follows that $n - 1 = k(p - 1)$, where $k \in \mathbb{Z}$. It is known (see [2]) that for the Bernoulli numbers one has

$$p \cdot B_{k\cdot(p-1)} \equiv -1 \pmod{p},$$

and so we may represent the condition (3) in the following form

$$\frac{n}{p} \equiv 1 \pmod{p}.$$

By multiplying the corresponding sides of the last two congruences, we arrive at the condition (4). In this way, we have shown that the aforementioned conjecture reduces to the hypotheses that the condition (4) may be satisfied then and only then when n is a prime number.

If we try to give an indirect proof of the Ging conjecture, we quickly arrive at the conclusion that it is very important to know an estimate for the following sum

$$W_m(r) = \sum_{i=1}^{r} a_i^m,$$

where m is a natural number, a_i are natural numbers such that $a_i < a_{i+1}$, for $i \in \{1, \ldots, r - 1\}$ (see [8]). These estimates should be made in such a way that the error is the smallest when $a_i = i$, $i \in \{1, \dots, r\}$. If we introduce notation $s = W_1(r)$, we prove the following relations

$$W_3(r) \geq C_2(s), \quad W_5(r) \geq C_3(s),$$

where $C_2(s)$ and $C_3(s)$ are Jacobi polynomials.

A prime number $p \geq 3$ is regular if and only if it is not a factor of any numerator of the Bernoulli numbers $B_2, B_4, B_6 \dots B_{p-3}$. If p is a regular prime number, the equation

$$x^p + y^p = z^p$$

does not have solutions in $\mathbb{Z}$. Kumer [4] proved the following statement:

If not both of the numerators of B_{p-3} and B_{p-5} are divisible by p, then one of the numbers in a solution (x, y, z) of the above Diophantine equation must be divisible by p.

2. Jacobi polynomials

The following polynomials

$$C_m(t) = \sum_{k=1}^{m} \alpha(m,k) \cdot t^k, \quad m \in \{1, 2, 3, \dots\} \tag{5}$$

$$\alpha(m,k) = \frac{1}{k} \cdot 2^{k-1} \cdot \sum_{s=0}^{2m-2k} \binom{2m-1}{2} \cdot \binom{2m-2-k-1}{k-1} \cdot B_s \tag{6}$$

where $k \in \{1, 2, \dots, m\}$ are known as Jacobi polynomials since they were first discovered by Jacobi in 1834 (see [3]).

It is important to emphasize that the formula (6), which establishes the connection between the coefficients $\alpha(m,k)$ and Bernoulli numbers, is not determined uniquely. It is not hard to use this formula for computation of the coefficients if k is close to m. One also proves that $\alpha(1,1) = 1$ and $\alpha(m,1) = 0$ for $m \in \{2, 3, \dots\}$. Nevertheless, the formula (6) may be transformed so as to be appropriate for computation of the coefficients $\alpha(m,k)$ when k is close to 2.

We start with the Bernoulli polynomial

$$B_{2m-1}(x) = \sum_{s=0}^{2m-1} \binom{2m-1}{s} \cdot x^{2m-s-1} \cdot B_s$$

which we multiply by x^{-k} and then find its $k-1$st derivation. We then put $x = 1$ and, using the fact that $B_s(1) = B_s$ we get the relation

$$\sum_{s=0}^{2m-1} \binom{2m-1}{2} \cdot \binom{2m-k-s-1}{k-1} \cdot B_s =$$
$$= \sum_{s=0}^{k-1} (-1)^{k-s-1} \cdot \binom{2m-1}{s} \cdot \binom{2k-s-2}{k-1} \cdot B_{2m-s-1}.$$

Similarly, one proves the following relation

$$\begin{aligned}
&\sum_{s=0}^{2m-1}\binom{2m-1}{2}\binom{2m-k-s-1}{k-1}B_s\\
=\ &\sum_{s=0}^{2m-2k}\binom{2m-1}{2}\binom{2m-k-s-1}{k-1}B_s\sum_{s=0}^{k-1}(-1)^{k-1}\binom{2k-s-2}{k-1}B_{2m-s-1}.
\end{aligned}$$

From the formula (6) and the last two relations we get

$$\alpha(m,k)=(-1)^k\frac{2^{k-1}}{k}\sum_{s=0}^{k-1}(1-(-1)^s)\binom{2m-1}{s}\binom{2k-s-2}{k-1}B_{2m-s-1}, \tag{7}$$

where $m\in\{1,2,3,\ldots\}$, $k\in\{1,2,\ldots,m\}$. If $m\geq 1$ and $k\in(1,m)$, we have

$$\alpha(m,k)=(-1)^k\cdot\frac{2^k}{k}\cdot\sum_{s\geq\frac{k}{2}}^{k-1}\binom{2m-1}{2k-2s-1}\cdot\binom{2s-1}{k-1}\cdot B_{2m-2k+2s}. \tag{8}$$

We use (6) and (8) to calculate the coefficients of the Jacobi polynomials (5):

$$\begin{array}{lcl}
\alpha(m,m) & = & \frac{1}{m}\cdot 2^{m-1}, m\geq 1\\
\alpha(m,m-1) & = & -2^{m-2}\cdot(m-2)\cdot\frac{1}{3!}, m\geq 2\\
\alpha(m,m-2) & = & \frac{2^{m-2}}{3!5!}\cdot(m-1)\cdot(m-3)\cdot(7m-8), m\geq 3\\
\alpha(m,m-3) & = & \frac{-2^{m-3}}{3!7!}\cdot(m-1)(m-2)(m-4)(31m^2-89m+48), m\geq 4\\
\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\alpha(m,3) & = & -8\cdot(2m-1)\cdot B_{2m-2}, m\geq 3\\
\alpha(m,2) & = & 2\cdot(2m-1)\cdot B_{2m-2}, m\geq 2.
\end{array}$$

Let us list a few Jacobi polynomials.

$$\begin{array}{lcl}
c_1(t) & = & t\\
c_2(t) & = & t^2\\
c_3(t) & = & \frac{1}{3}(4t^3-t^2)\\
c_4(t) & = & \frac{1}{3}(6t^4-4t^3+t^2)\\
c_5(t) & = & \frac{1}{5}(16t^5-20t^4+12t^3-3t^2).
\end{array}$$

3. Some properties of Jacobi polynomials

By using the relation between the sum of odd degrees of natural numbers $S_{2m-1}(r)$ and Jacobi polynomials $C_m(t)$, i.e.

$$S_{2m-1}(r)=C_m(t) \tag{9}$$

we can discover one particular property of these polynomials. Namely, by putting $r=1$ we have $t=1$ and

$$C_m(1)=1,\quad m\in\{1,2,3,\ldots\} \tag{10}$$

The coefficients $\alpha(m,k)$ in general are not integers. In order to apply congruence theory to these rational numbers, one should know more about prime factors of the denominator. From the Staudt-Clausen theorem (see [5]) it follows that p is a prime factor of the denominator N_{2m} of the Bernoulli number B_{2m} if and only if $(p-1) \mid 2m$.

Theorem 1. *Denominators of the coefficients $\alpha(m,k)$, $k \in \{2,3,4,\dots,m\}$ are odd integers for all prime factors less than $2m-1$.*

Proof. In the formula (8), the factor in front of the sum is $\frac{2^k}{k}$. Let a be the maximum integer such that $2^a \mid k$. Then we have $k \geq 2^a > a$. Therefore, $2^{k-a} \geq 2$. The denominator of the coefficient $\alpha(m,k)$ is consequently an odd integer. The Bernoulli number with the highest index in the formula (8) is B_{2m-2}, which would have $2m-1$ as its highest prime factor, but this Bernoulli number is multiplied by $2m-1$. $\square$

For arbitrary real numbers a and b one has

$$\sum_{n=1}^{r}(an+b)^{2m-1} = a^{2m-1}\left(C_m\left(\frac{v}{a}+c\right) - C_m(c)\right),$$

where

$$v = at + br = \sum_{n=1}^{r}(an+b), \quad c = \frac{b(a+b)}{2a^2}.$$

Theorem 2. *Let p be a prime number. Then*

$$\sum_{1\leq i_1\leq\cdots\leq i_k}^{\frac{p-1}{2}} t_{i_1}\cdot t_{i_2}\cdots t_{i_k} \equiv -\frac{1}{k\cdots 2^{5k-1}}\cdot\binom{2k-2}{k-1} \pmod p, \tag{$\blacktriangle$}$$

$$\sum_{1\leq i_1\leq\cdots\leq i_k}^{\frac{p-1}{2}} (t_{i_1}\cdot t_{i_2}\cdots t_{i_k})^{-1} \equiv \frac{-2}{k+1}\binom{2k}{k} \pmod p, \tag{$\blacksquare$}$$

where $k \in \{1,2,3,\dots,\frac{p-1}{2}\}$.

Proof. Let $p = 2m+1$. From the relation

$$\sum_{n=1}^{r} n^{2m} = S_{2m}(r) = \frac{2r+1}{2(2m+1)}\cdot C'_{m+1}(t),$$

by taking into account Theorem 1, we derive the congruence of mth degree

$$C'_{m+1}(u) \equiv 0 \pmod p,$$

which holds for $u \in \{t_0, t_1, t_2, \dots, t_{\frac{p-3}{2}}\}$. Let us suppose that some of these m solutions are congruent

$$t_i \equiv t_k \pmod p.$$

In this case it follows that $i-k \equiv 0 \pmod p$ or $i+k+1 \equiv 0 \pmod p$, which is impossible.

Furthermore, from the known results, we have

$$C'_{m+1}(u) \equiv d(u-t_0)(u-t_1)\cdots(u-t_{\frac{p-3}{2}}) \pmod p, \qquad (*)$$

where

$$d \equiv (m+1)\alpha(m+1,m+1) \equiv 2^m \equiv 2^{\frac{p-1}{2}} \equiv \binom{2}{p} \equiv (-1)^{\frac{p^2-1}{8}} \pmod p.$$

For the coefficients of the polynomial $C'_{m+1}(u)$, we may, using formula (6), derive the congruence

$$k\cdot\alpha(m+1,k) \equiv 2^{k-1}\cdot\binom{-k}{k-1} \equiv (-2)^{k-1}\cdot\binom{2k-2}{k-1} \pmod p,$$

since $k>1$, and $\binom{p}{s}$ is divisible by p for $s \in \{1,2,\ldots,p-1\}$.

Comparing the coefficients in the conguence $(*)$ we find

$$\alpha\cdot(-1)^k\cdot\sum_{1\le i_1\le\cdots\le i_k}^{\frac{p-3}{2}} t_{i_1}\cdot t_{i_2}\cdots t_{i_k} \equiv (-2)^{m-k-1}\cdot\binom{2m-2k-2}{m-k-1} \pmod p,$$

where $k \le \frac{p-3}{2}$, from which, after simplification, we get

$$\sum_{1\le i_1\le\cdots\le i_k}^{\frac{p-3}{2}} t_{i_1}\cdot t_{i_2}\cdots t_{i_k} \equiv \frac{1}{2^{5k}}\cdot\binom{2k}{k} \pmod p.$$

If we denote the last sum by s_k, it is clear that

$$\sum_{1\le i_1\le\cdots\le i_k}^{\frac{p-1}{2}} t_{i_1}\cdot t_{i_2}\cdots t_{i_k} = s_k + s_{k-1}\cdot t_{\frac{p-1}{2}}.$$

From this the result (▲) directly follows. After dividing the corresponding sides of the congruence (▲) and the congruence

$$t_1\cdot t_2\cdots t_{\frac{p-1}{2}} \equiv \frac{1}{2}\cdot(-1)^{\frac{p^2-1}{8}+\frac{p+1}{2}} \pmod p,$$

then by changing the index and simplifying we arrive at the congruence (■). □

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