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INFINITE DIMENSIONAL STOCHASTIC EQUATION WITH MULTIPLICATIVE NOISE IN SPACES OF STOCHASTIC DISTRIBUTIONS

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Abstract. Abstract stochastic distributions are used to investigate existence of strong solutions of a stochastic equation with multiplicative noise on a separable Hilbert space.

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1. Introduction

Analysis of white noise functionals (see [5, 4] and references therein), introduced by Hida [3] and developed further by many authors, proved to be a powerful framework for studying stochastic differential equations. It continues to find new extensions and applications [7, 8, 9]. In [2] and [6] the ideas of white noise analysis were used to introduce the Hilbert space valued stochastic distributions (abstract stochastic distributions), which were used to study differential-operator equations with additive noise.

In the present work we investigate the existence of general solutions to the Cauchy problem for the equation with multiplicative noise. Within the framework of the Ito calculus of H-valued random processes, where H is a Hilbert space, the problem can be written as

$$dX(t) = AX(t) + B(X(t))dW(t)\,,\ t \geq 0\,, \quad X(0) = \zeta\,, \tag{1}$$

where $A : H \to H$ and $B(\cdot) : H \to \mathcal{L}(H)$ are linear operators, ζ is an H-valued random variable and $W(t)$ is an H-valued Wiener process. It arises in different applications and draws interest of many researchers (see [1] and references therein). By introducing the Hitsuda–Skorohod integral for the processes with values in spaces of abstract stochastic distributions we rewrite equation (1) in differential form and apply the Hermite transform to reduce it to a deterministic equation in the Hilbert space.

In section 2 we briefly describe the spaces of abstract stochastic distributions and the framework we use for investigation of the problem. The details can be found in [6]. Section 3 contains the main result.

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2. Framework

We will use the white noise probability space $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$, where $\mathcal{S}'$ is the space of tempered distributions, $\mathcal{B}(\mathcal{S}')$ is the Borel σ-algebra of $\mathcal{S}'$ and μ is the unique probability measure on $\mathcal{B}(\mathcal{S}')$ with

$$\int_{\mathcal{S}'} e^{i\langle \omega\,,\,\theta\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\theta\|^2_{L^2(\mathbb{R})}}\,, \quad \theta \in \mathcal{S}\,. \tag{2}$$

Let $(L^2) = L^2(\mathcal{S}', \mu; \mathbb{R})$ be the space of $\mathbb{R}$-valued functions on $\mathcal{S}'$ square integrable with respect to μ. Let $\{\xi_k\}_{k=1}^{\infty}$ be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the Hermite functions $\xi_k(x) = \pi^{-\frac{1}{4}}\big((k-1)!\big)^{-\frac{1}{2}}\, e^{-\frac{x^2}{2}} h_{k-1}(x)$, where $\{h_k(x)\}_{k=0}^{\infty}$ is the set of Hermite polynomials $h_k(x) = (-1)^n e^{\frac{x^2}{2}} (d/dx)^k e^{-\frac{x^2}{2}}$. Let $\mathcal{T} \subset \big(\mathbb{N} \cup \{0\}\big)^{\mathbb{N}}$ be the set of all finite multi-indices. Stochastic Hermite polynomials are defined by $\mathbf{h}_\alpha(\omega) := \prod_k h_{\alpha_k}(\langle \omega\,,\,\xi_k\rangle)\,, \omega \in \mathcal{S}'\,, \alpha \in \mathcal{T}$. The family $\{\mathbf{h}_\alpha\}_{\alpha\in\mathcal{T}}$ constitutes an orthogonal basis in (L^2) with

$$(\mathbf{h}_\alpha\,,\,\mathbf{h}_\beta)_{(L^2)} = \begin{cases} 0\,, & \alpha \neq \beta\,, \\ \alpha!\,, & \alpha = \beta\,, \end{cases} \qquad \alpha! := \prod_k \alpha_k!\,.$$

Let H be a separable Hilbert space over $\mathbb{C}$. Denote by $(L^2)(H)$ the space of all H-valued functions Bochner square integrable with respect to μ on $\mathcal{S}'$. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in H. The family $\{\mathbf{h}_\alpha e_j\}_{\alpha\in\mathcal{T}, j\in\mathbb{N}}$ is an orthogonal basis of $(L^2)(H)$. For any $f \in (L^2)(H)$ we have the following expansions:

$$f = \sum_{\alpha\in\mathcal{T}, j\in\mathbb{N}} f_{\alpha,j}\mathbf{h}_\alpha e_j = \sum_{\alpha\in\mathcal{T}} f_\alpha \mathbf{h}_\alpha\,, \quad f_{\alpha,j} \in \mathbb{R}\,,\ f_\alpha = \sum_j f_{\alpha,j} e_j \in H$$

with $\|f\|^2_{(L^2)(H)} = \sum_{\alpha\in\mathcal{T}, j\in\mathbb{N}} \alpha!|f_{\alpha,j}|^2 = \sum_{\alpha\in\mathcal{T}} \alpha!\|f_\alpha\|^2_H$. For any $p \in \mathbb{N}$, $\rho \in [0;1]$ denote by $(\mathcal{S}_p)_\rho(H)$ the space of all $\varphi = \sum_{\alpha\in\mathcal{T}} \varphi_\alpha \mathbf{h}_\alpha \in (L^2)(H)$ with

$$\|\varphi\|^2_{\rho,p} := \sum_{\alpha\in\mathcal{T}} (\alpha!)^{1+\rho} \|\varphi_\alpha\|^2_H \big(2\mathbb{N}\big)^{p\alpha} < \infty\,,$$

where $\big(2\mathbb{N}\big)^{p\alpha} := \prod_{n\in\mathbb{N}} (2n)^{p\alpha_n}$. It is a Hilbert space with the scalar product

$$(\varphi\,,\,\psi)_{\rho,p} = \sum_{\alpha\in\mathcal{T}} (\alpha!)^{1+\rho} (\varphi_\alpha, \psi_\alpha)_H \big(2\mathbb{N}\big)^{p\alpha}\,.$$

Let $(\mathcal{S})_\rho(H) = \cap_{p\in\mathbb{N}} (\mathcal{S}_p)_\rho(H)$ with the projective limit topology. We will call it the space of H-valued stochastic test functions. Denote by $(\mathcal{S})_{-\rho}(H)$ its dual, which coincides with $\cup_{p\in\mathbb{N}} (\mathcal{S}_{-p})_{-\rho}(H)$ with the inductive limit topology, where for any $p \in \mathbb{N}$ the space $(\mathcal{S}_{-p})_{-\rho}(H)$ is the dual of $(\mathcal{S}_p)_\rho(H)$. It can be represented as the totality of all formal expansions of the form

$$\Phi = \sum_{\alpha\in\mathcal{T}, j\in\mathbb{N}} \Phi_{\alpha,j}\mathbf{h}_\alpha e_j = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha\,, \quad \Phi_\alpha = \sum_{j\in\mathbb{N}} \Phi_{\alpha,j} e_j \in H\,,$$

such that there exists $p \in \mathbb{N}$ with

$$\|\Phi\|^2_{-\rho,-p} := \sum_{\alpha\in\mathcal{T}} (\alpha!)^{1-\rho} \|\Phi_\alpha\|^2_H \big(2\mathbb{N}\big)^{-p\alpha} < \infty\,.$$

We will call $(\mathcal{S})_{-\rho}(H)$ the space of H-valued stochastic distributions over $(\mathcal{S})_\rho(H)$. So we have the next triple:

$$(\mathcal{S})_\rho(H) \subset (L^2)(H) \subset (\mathcal{S})_{-\rho}(H)\,. \tag{3}$$

Let $n(\cdot,\cdot) : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection such that

$$n(i,j) \geq ij\,, \quad i\,,j \in \mathbb{N}\,. \tag{4}$$

Define the sequence of independent Brownian motions $\{\beta_j(t)\}_{j=1}^\infty$ by $\beta_j(t) = \sum_{i=1}^\infty \int_0^t \xi_i(s)\,ds\,\mathbf{h}_{\epsilon_{n(i,j)}}$, where $\epsilon_i := (0,0,\ldots,\underset{i}{1},0,\ldots)$, and a cylindrical H-valued Wiener process by

$$W(t) = \sum_{j\in\mathbb{N}} \beta_j(t) e_j = \sum_{n\in\mathbb{N}} W_{\epsilon_n}(t)\,\mathbf{h}_{\epsilon_n}\,, \quad W_{\epsilon_n}(t) = \int_0^t \xi_{i(n)}(s)\,ds\,e_{j(n)} \in H\,,$$

where $i(n)\,,\,j(n) \in \mathbb{N}$ are such that $n(i(n),j(n)) = n$. It is easy to see that $W(t) \notin (L^2)(H)$ for any $t \in \mathbb{R}$. At the same time, using the estimate $\int_0^t \xi_i(s)\,ds = O(i^{-\frac{3}{4}})$ and (4), one can easily see that $\|W(t)\|^2_{-1,-2} < \infty$.

So, $W(t) \in (\mathcal{S}_{-2})_{-1}(H) \subset (\mathcal{S})_{-1}(H)$.

Define the H-valued cylindrical white noise by

$$\mathbb{W}(t) := \sum_{i,j\in\mathbb{N}} \xi_i(t)\,(\mathbf{h}_{\epsilon_{n(i,j)}} e_j) = \sum_{n\in\mathbb{N}} \mathbb{W}_{\epsilon_n}(t)\,\mathbf{h}_{\epsilon_n}\,,$$

where $\mathbb{W}_{\epsilon_n}(t) = \xi_{i(n)}(t)\,e_{j(n)} \in H$, i.e. as a formal derivative of $W(t)$. Since $\xi_i(t) = O\big(i^{-\frac{1}{4}}\big)$, one can see that $\|\mathbb{W}(t)\|^2_{-1,-2} < \infty$, which means that $\mathbb{W}(t) \in (\mathcal{S}_{-2})_{-1}(H) \subset (\mathcal{S})_{-1}(H)$.

We call a sequence $\{\Phi_n\} \subset (\mathcal{S})_{-1}(H)$ *strongly convergent* to $\Phi \in (\mathcal{S})_{-1}(H)$, if $\langle \Phi_n, \phi\rangle \underset{n\to\infty}{\longrightarrow} \langle \Phi, \phi\rangle$ uniformly on all bounded subsets of $(\mathcal{S})_1(H)$. The next proposition gives a characterization of strong convergence.

Proposition 2.1. *Let $\Phi_n = \sum_\alpha \Phi^{(n)}_\alpha \mathbf{h}_\alpha\,, \Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})_{-1}(H)$. The following assertions are equivalent:*

(i) *$\{\Phi_n\}$ strongly converges to Φ;*

(ii) *There exists $p \in \mathbb{N}$ such that $\Phi_n \in (\mathcal{S}_{-p})_{-1}(H)$ for all $n \in \mathbb{N}$, $\Phi \in (\mathcal{S}_{-p})_{-1}(H)$ and $\lim_{n\to\infty} \|\Phi_n - \Phi\|_{-1,-p} = 0$;*

(iii) *$\{\Phi_n\}$ belongs to some $(\mathcal{S}_{-p})_{-1}(H)$, is bounded with respect to $\|\cdot\|_{-1,-p}$ and for any $\alpha \in \mathcal{T}$ $\lim_{n\to\infty} \|\Phi^{(n)}_\alpha - \Phi_\alpha\|_H = 0$.*

Let $\Phi(t) \in (\mathcal{S})_{-1}(H)$ for all $t \in \mathbb{R}$. We will call $\Theta \in (\mathcal{S})_{-1}(H)$ a *strong derivative* of $\Phi(t)$ at a point $t \in \mathbb{R}$ and denote it $\frac{d}{dt}\Phi(t)$, if $\frac{\Phi(s)-\Phi(t)}{s-t} \underset{s\to t}{\longrightarrow} \Theta$ strongly in $(\mathcal{S})_{-1}(H)$. From Proposition 2.1 it follows the next proposition.

Proposition 2.2. *The derivative $\frac{d}{dt}\Phi(t)$ will exist if and only if there exist $p \in \mathbb{N}$, $\delta > 0$ and $M > 0$ such that for any $s \in \mathbb{R}$ $|t-s| < \delta \Rightarrow \|\Phi(t)-\Phi(s)\|_{-1,-p} \leq M|t-s|$ and for any $\alpha \in \mathcal{T}$ there exists $\Phi'_\alpha(t) = \lim_{s\to t} \frac{\Phi_\alpha(s)-\Phi_\alpha(t)}{s-t}$ in H. In this case $\frac{d}{dt}\Phi(t) = \sum_\alpha \Phi'_\alpha(t)\mathbf{h}_\alpha$.*

It is easy to see that for the H-valued cylindrical Wiener process and white noise we have $\frac{d}{dt}W(t) = \mathbb{W}(t)$.

We will call a function $\Phi(\cdot) : \mathbb{R} \to (\mathcal{S})_{-1}(H)$ integrable on a measurable $C \subseteq \mathbb{R}$ if there exists $p \in \mathbb{N}$ such that $\Phi(t) \in (\mathcal{S}_{-p})_{-1}(H)$ for any $t \in C$ and Φ is Bochner integrable on C as an $(\mathcal{S}_{-p})_{-1}(H)$-valued function.

Proposition 2.3. *Let for any $\alpha \in \mathcal{T}$ $\Phi_\alpha(t)$ be a Bochner integrable on C H-valued function and $\sum_\alpha \left(\int_C \|\Phi_\alpha(t)\|_H dt \right)^2 (2\mathbb{N})^{-p\alpha} < \infty$ for some $p \in \mathbb{N}$. Then $\Phi(\cdot) : \mathbb{R} \to (\mathcal{S})_{-1}(H)$ is integrable on C and*

$$\int_C \Phi(t)\, dt = \sum_\alpha \int_C \Phi_\alpha(t)\, dt\, \mathbf{h}_\alpha .$$

Let H and H_0 be separable Hilbert spaces and HS_0 be the space of all linear Hilbert–Schmidt operators acting from H_0 to H. It is a separable Hilbert space, therefore we can define the triple

$$(\mathcal{S})_\rho(\mathrm{HS}_0) \subset (L^2)(\mathrm{HS}_0) \subset (\mathcal{S})_{-\rho}(\mathrm{HS}_0)$$

in the same manner as the triple (3).

Definition 2.4. Let $\Psi \in (\mathcal{S})_{-1}(\mathrm{HS}_0)$, $\Psi = \sum_{\alpha\in\mathcal{T}} \Psi_\alpha \mathbf{h}_\alpha$, where $\Psi_\alpha \in \mathrm{HS}_0$, $\alpha \in \mathcal{T}$ and $\Phi \in (\mathcal{S})_{-1}(H_0)$, $\Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha$, where $\Phi_\alpha \in H_0$, $\alpha \in \mathcal{T}$. The Wick product of Ψ and Φ is defined by

$$\Psi \diamond \Phi := \sum_{\gamma\in\mathcal{T}} \left(\sum_{\alpha+\beta=\gamma} \Psi_\alpha \Phi_\beta \right) \mathbf{h}_\gamma .$$

The next proposition is proved by the same scheme as its real-valued analog in [4].

Proposition 2.5. *For any $\Psi \in (\mathcal{S})_{-1}(\mathrm{HS}_0)$, and $\Phi \in (\mathcal{S})_{-1}(H_0)$ their Wick product is well defined as an element of $(\mathcal{S})_{-1}(H)$.*

For any linear symmetric positive nuclear operator Q, denote by H_Q the space $Q^{\frac{1}{2}}(H)$ endowed with the scalar product $(u\,,\, v)_{H_Q} = (Q^{-\frac{1}{2}}u\,,\, Q^{-\frac{1}{2}}v)_H$. For the above defined H-valued cylindrical white noise we have:

Proposition 2.6. *$\mathbb{W}(t) \in (\mathcal{S})_{-1}(H_Q)$, $t \in \mathbb{R}$ for any $Q = \sum_{j=1}^{\infty} \sigma_j^2 (e_j \otimes e_j)$,*[3] *with $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$, satisfying*

$$\sum_{j=1}^{\infty} \sigma_j^{-2} j^{-p} < \infty \quad \text{for some } p \in \mathbb{N}\,. \tag{5}$$

Proof. We have

$$\begin{aligned} \|\mathbb{W}_{\epsilon_{n(i,j)}}\|_{H_Q}^2 \big(2\mathbb{N}\big)^{-p\epsilon_{n(i,j)}} &= |\xi_i(t)|^2 \sigma_j^{-2} \big(2n(i,j)\big)^{-p} \\ &\le \frac{|\xi_i(t)|^2}{\sigma_j^2 \big(2ij\big)^p} \\ &= O\big(\sigma_j^{-2} i^{-p-\frac{1}{2}} j^{-p}\big) \end{aligned}$$

Hence the assertion follows. □

Let HS_Q be the space of all Hilbert–Schmidt operators acting from H_Q to H. From Proposition 2.6 it follows that if the operator Q satisfies condition (5), then for any $(\mathcal{S})_{-1}(\mathrm{HS}_Q)$-valued random process $\Psi(t)$, the $(\mathcal{S})_{-1}(H)$-valued random process $\Psi(t) \diamond \mathbb{W}(t)$ is well defined.

The next definition is a generalization of the Ito integral $\int_0^T \Psi(t)\, dW(t)$ with respect to a cylinder Wiener process.

Definition 2.7. We will call an $(\mathcal{S})_{-1}(\mathrm{HS}_Q)$-valued random process $\Psi(t)$ *Hitsuda–Skorohod integrable* on $[0;T]$ if $\Psi(t)\diamond\mathbb{W}(t)$ is integrable on $[0;T]$ as an $(\mathcal{S})_{-1}(H)$-valued function. In that case we will call the integral

$$\int_0^T \Psi(t) \diamond \mathbb{W}(t)\, dt$$

Hitsuda–Skorohod integral of $\Psi(t)$.

Definition 2.8. For any $\Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})_{-1}(H)$, where $\Phi_\alpha \in H$, denote its Hermite transform by $\mathcal{H}\Phi$, and defines it by

$$\mathcal{H}\Phi(z) := \sum_{\alpha\in\mathcal{T}} \Phi_\alpha z^\alpha\,, \tag{6}$$

where $z = (z_1, z_2, \dots) \in \mathbb{C}^{\mathbb{N}}$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdot \dots$ for $\alpha \in \mathcal{T}$, if the series (6) converge in $H_{\mathbb{C}}$.

We will use the next neighborhoods of zero in $\mathbb{C}^{\mathbb{N}}$:

$$\overline{\mathbb{K}}_p := \left\{ z \in \mathbb{C}^{\mathbb{N}}\,,\ |z_i| \le (2i)^{-p}\,,\ i \in \mathbb{N} \right\}\,.$$

[3] For any $v \in V$, $u \in U$, where V and U are Hilbert spaces, we denote by $v \otimes u$ the operator, belonging to $\mathcal{L}(U,V)$, defined by $(v \otimes u)h := v(u\,,\, h)_U$.

Now we list the basic properties of the Hermite transform that will be used later. They are proved in a similar manner as the corresponding properties in the real valued case. See details for the H-valued case in [6].

In the sequel we denote $A(p) := \sum_{\alpha\in\mathcal{T}} (2\mathbb{N})^{-p\alpha}$ which is convergent for $p > 1$ (see [4]).

Proposition 2.9. *For any* $\Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})_{-1}(H)$ *there exists* $p \in \mathbb{N}\setminus\{1\}$ *such that* $\mathcal{H}\Phi(z)$ *converges absolutely for all* $z \in \overline{\mathbb{K}}_p$ *and*

$$\sum_{\alpha\in\mathcal{T}} \|\Phi_\alpha\|_H |z^\alpha| \le \|\Phi\|_{-1,-p}\sqrt{A(p)}\,.$$

Proposition 2.10. *Let* $\hat{\Phi}(z) = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha z^\alpha$ *with* $\Phi_\alpha \in H$*, where the series is absolutely convergent in* $H_{\mathbb{C}}$ *for* $z \in \overline{\mathbb{K}}_p$ *for some* $p \in \mathbb{N}$*. Then* $\Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha$ *defines an element of* $(\mathcal{S})_{-1}(H)$ *with* $\mathcal{H}\Phi = \hat{\Phi}$*.*

Proposition 2.11. *Let* $X(\cdot), F(\cdot) : [a,b] \to (\mathcal{S})_{-1}(H)$*. The next assertions are equivalent:*

1. $X(\cdot)$ *is differentiable on* $[a,b]$ *and* $X'(t) = F(t)$ *for* $t \in [a,b]$*;*
2. *There exists* $p \in \mathbb{N}\setminus\{1\}$ *such that for all* $t \in [a,b]$ *the Hermite transforms* $\mathcal{H}X(t,z)$ *and* $\mathcal{H}F(t,z)$ *exist and are absolutely convergent for all* $z \in \overline{\mathbb{K}}_p$ *and* $\frac{\partial}{\partial t}\mathcal{H}X(t,z) = \mathcal{H}F(t,z)\,, \quad (t,z) \in [a,b]\times\overline{\mathbb{K}}_p\,.$

Proposition 2.12. *Let* $\Phi \in (\mathcal{S})_{-1}(H_0)$*,* $\Psi \in (\mathcal{S})_{-1}(\mathrm{HS}_0)$*. Then, for all* $z \in \mathbb{C}^{\mathbb{N}}$ *such that both* $\mathcal{H}\Phi(z)$ *and* $\mathcal{H}\Psi(z)$ *exist, it holds*

$$\mathcal{H}\big(\Psi \diamond \Phi\big)(z) = \mathcal{H}\Psi(z)\mathcal{H}\Phi(z)\,.$$

3. The Cauchy problem for a linear operator-differential equation with multiplicative noise

Let A be a linear operator acting from Hilbert space H_1 to Hilbert space H_2. We define its action on $(\mathcal{S})_{-1}(H_1)$ in the next way. If $A \in \mathcal{L}(H_1, H_2)$, define

$$A\Phi := \sum_{\alpha\in\mathcal{T}} A\Phi_\alpha \mathbf{h}_\alpha\,, \quad \Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})_{-1}(H_1)\,. \tag{7}$$

This evidently defines a linear continuous operator from $(\mathcal{S})_{-1}(H_1)$ to $(\mathcal{S})_{-1}(H_2)$. If A is unbounded, define $(\operatorname{dom} A)$ as the set of all $\sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})_{-1}(H_1)$ such that $\Phi_\alpha \in \operatorname{dom} A$ for all $\alpha \in \mathcal{T}$ and

$$\sum_{\alpha\in\mathcal{T}} \|A\Phi_\alpha\|^2_{H_2} (2\mathbb{N})^{-2p\alpha} < \infty$$

for some $p \in \mathbb{N}$. Then formula (7) defines on $(\operatorname{dom} A)$ a linear operator acting from $(\mathcal{S})_{-1}(H_1)$ to $(\mathcal{S})_{-1}(H_2)$. It's easy to see that it is closed if A is a closed operator from H_1 to H_2. It is easy to prove the next statement.

Proposition 3.1. *Let A be a linear closed operator acting from Hilbert space H_1 to Hilbert space H_2. For any $\Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha \in$ (dom$A$) there exists $p \in \mathbb{N}\setminus\{1\}$ such that for all $z \in \overline{K}_p$ $\mathcal{H}\Phi(z) \in \mathrm{dom}A$ and*

$$A\mathcal{H}\Phi(z) = \sum_{\alpha\in\mathcal{T}} A\Phi_\alpha z^\alpha .$$

Let A be a linear closed operator in a Hilbert space H, $B(\cdot) \in \mathcal{L}\big(H, \mathcal{L}(H)\big)$, $\Phi \in (\mathrm{dom}A) \subseteq (\mathcal{S})_{-1}(H)$. Consider the Cauchy problem

$$\frac{dX(t)}{dt} = AX(t) + B\big(X(t)\big) \diamond \mathbb{W}(t)\,,\ t \geq 0\,, \quad X(0) = \Phi\,. \tag{8}$$

It can be obtained by introducing the Hitsuda–Skorohod integral instead of the Ito one in (1) and differentiating with respect to t. We will study the existence of a strong solution of (8) in $(\mathcal{S})_{-1}(H)$, i.e. an $(\mathcal{S})_{-1}(H)$-valued differentiable function satisfying (8). Note that if Q is a nuclear operator in H satisfying the condition of Proposition 2.6 for some $p \in \mathbb{N}$, then since $B\big(X(t)\big) \in (\mathcal{S})_{-1}(\mathrm{HS}_Q)$ for any $X(t) \in (\mathcal{S})_{-1}(H)$, the Wick product in (8) is well defined.

We will further suppose that the operator B in equation (8) satisfies the following assumption

Assumption **B** *For any $x \in H$*

$$B(\mathrm{dom}A)x \subseteq \mathrm{dom}A \tag{9}$$

and the commutator of A and $B(\cdot)x$ defined by the equality $[A, B(\cdot)x]\, y := AB(y)x - B(Ay)x\,,\ y \in \mathrm{dom}A$ is bounded.

Example 1. Let $H = L^2(\mathbb{R})$, $A = \frac{d}{dx}$ with $\mathrm{dom}A = \{y \in L^2(\mathbb{R})\,, \frac{dy}{dx} \in L^2(\mathbb{R})\}$. For $B(\cdot) \in \mathcal{L}\big(H; \mathcal{L}(H)\big)$, defined by $[B(u)v](s) := u(s)\int_{\mathbb{R}} \psi(s-\tau)v(\tau)\,d\tau$, where $\psi \in C^\infty(\mathbb{R})$ has compact support, assumption **B** is true.

Applying the Hermite transform to both sides of (8) we come to the next problem:

$$\frac{\partial}{\partial t}\hat{X}(t,z) = A\hat{X}(t,z) + B\big(\hat{X}(t,z)\big)\hat{\mathbb{W}}(t,z), \qquad t \geq 0, \quad \hat{X}(0,z) = \hat{\Phi}(z), \quad z \in \overline{\mathbb{K}}_p. \tag{10}$$

where $\hat{X}(t,z) = \mathcal{H}[X(t)](z)$, $\hat{\mathbb{W}}(t,z) = \mathcal{H}[\mathbb{W}(t)](z)$, $\hat{\Phi}(z) = \mathcal{H}\Phi(z)$ and $\overline{\mathbb{K}}_p$ with $p \in \mathbb{N}\setminus\{1\}$ is such that $\hat{\Phi}(z)$ and $\hat{\mathbb{W}}(t,z)\,, t \in \mathbb{R}$, are absolutely convergent and $\hat{\Phi}(z) \in \mathrm{dom}A$ for all $z \in \overline{\mathbb{K}}_p$. Let $\hat{B}(t,z) = B(\cdot)\hat{\mathbb{W}}(t,z)$, then we have $\hat{B}(t,z) \in \mathcal{L}(H)$ for all $t \in \mathbb{R}$ and $z \in \overline{\mathbb{K}}_p$ and we can write (10) in the next way:

$$\frac{\partial}{\partial t}\hat{X}(t,z) = A\hat{X}(t,z) + \hat{B}(t,z)\hat{X}(t,z), \qquad t \geq 0, \quad \hat{X}(0,z) = \hat{\Phi}(z), \quad z \in \overline{\mathbb{K}}_p. \tag{11}$$

Let A be the generator of a C_0-semigroup $\{S(t)\,,\, t\geq 0\}$. Define the sequence of operators $\{T_k(t,z)\}\,,\, t\geq 0\,, z\in\overline{\mathbb{K}}_p$ by

$$T_0(t,z)=S(t),$$

$$T_k(t,z)x=\int_0^t S(t-s)\hat{B}(s,z)T_{k-1}(s,z)x\,ds\,,\ x\in H\quad k=1,2,\dots\,. \tag{12}$$

Lemma 3.2. *For any $t\geq 0$, $z\in\overline{\mathbb{K}}_p$ and $k\in\mathbb{N}$*

$$\|T_k(t,z)\|_{\mathcal{L}(H_{\mathbb{C}})}\leq M^{k+1}\|B\|^kC^ke^{at}\frac{t^k}{k!}\,, \tag{13}$$

where $M>0$ and $a\in\mathbb{R}$ are such that $\|S(t)\|\leq Me^{at}$ for $t\geq 0$, $\|B\|=\|B\|_{\mathcal{L}(H,\mathcal{L}(H))}$, $C=2^{-p}\sup\limits_{n\in\mathbb{N}}\big(n^{\frac{1}{12}}\sup\limits_{t\in\mathbb{R}}|\xi_n(t)|\big)\sum\limits_{n\in\mathbb{N}}n^{-p-\frac{1}{12}}$.

Proof. Suppose (13) holds for some $k\in\mathbb{N}$, then for $x\in H_{\mathbb{C}}$ we have

$$\|T_{k+1}(t,z)x\|_{H_{\mathbb{C}}}=\left\|\int_0^t S(t-s)\sum_{n\in\mathbb{N}}\xi_{i(n)}(s)B\big(T_k(s,z)x\big)e_{j(n)}z^{\epsilon_n}\,ds\right\|\leq$$

$$\leq\sum_{n\in\mathbb{N}}\int_0^t|\xi_{i(n)}(s)|\,\big\|S(t-s)B\big(T_k(s,z)x\big)e_{j(n)}\big\|\,ds\,|z^{\epsilon_n}|\leq$$

$$\leq M\|B\|\int_0^t e^{a(t-s)}\|T_k(s,z)x\|\,ds\sum_{n\in\mathbb{N}}(2n)^{-p}\sup_{s\in\mathbb{R}}|\xi_{i(n)}(s)|\leq$$

$$\leq M^{k+2}\|B\|^{k+1}C^{k+1}e^{at}\int_0^t\frac{s^k}{k!}\,ds=M^{k+2}\|B\|^{k+1}C^{k+1}e^{at}\frac{t^{k+1}}{(k+1)!}\,.$$

Since (13) is true for $k=0$, by induction it is true for all $k\in\mathbb{N}$. □

Define

$$\hat{X}(t,z)=\sum_{k=0}^{\infty}T_k(t,z)\,. \tag{14}$$

It follows from Lemma 3.2 that the series (14) is absolutely convergent in $\mathcal{L}(H_{\mathbb{C}})$ for all $t\geq 0$ and $z\in\overline{\mathbb{K}}_p$. Thus $\hat{X}(t,z)\in\mathcal{L}(H_{\mathbb{C}})$, with $\|\hat{X}(t,z)\|_{\mathcal{L}(H_{\mathbb{C}})}\leq Me^{(a+MC\|B\|)t}$. For $\Phi\in(\mathrm{dom}A)$ by the properties of C_0-semigroups, we obtain $T_0(t,z)\hat{\Phi}(z)\in\mathrm{dom}A$ for all $t\geq 0$ and $z\in\overline{\mathbb{K}}_p$. Condition (9) implies that for all $t\geq 0$ and $z\in\overline{\mathbb{K}}_p$ we have $\hat{B}(s,z)(\mathrm{dom}A)\subseteq\mathrm{dom}A$. Therefore, by induction we obtain that $T_k(t,z)\hat{\Phi}(z)\in\mathrm{dom}A$ for $\Phi\in(\mathrm{dom}A)\,,k\in\mathbb{N}$, $t\geq 0$ and $z\in\overline{\mathbb{K}}_p$. Condition (9) also implies that $\hat{B}(s,z)T_k(s,z)\hat{\Phi}(z)\in\mathrm{dom}A$ for $\Phi\in(\mathrm{dom}A)\,,k\in\mathbb{N}\cup\{0\}$ and

$$\frac{\partial}{\partial t}S(t-s)\hat{B}(s,z)T_k(s,z)\hat{\Phi}(z)=AS(t-s)\hat{B}(s,z)T_k(s,z)\hat{\Phi}(z)\,,\ t\geq 0\,,z\in\overline{\mathbb{K}}_p\,.$$

Thus for $\Phi \in (\mathrm{dom} A)$ we have

$$\frac{\partial}{\partial t} T_0(t,z)\hat{\Phi}(z) = AT_0(t,z)\hat{\Phi}(z)\,, \tag{15}$$

$$\frac{\partial}{\partial t} T_k(t,z)\hat{\Phi}(z) = \int_0^t AS(t-s)\hat{B}(s,z)T_{k-1}(s,z)\hat{\Phi}(z)\,ds + \\ +\hat{B}(t,z)T_{k-1}(t,z)\hat{\Phi}(z)\,. \tag{16}$$

Since A is closed we can rewrite (16) as

$$\frac{\partial}{\partial t} T_k(t,z)\hat{\Phi}(z) = AT_k(t,z)\hat{\Phi}(z) + \hat{B}(t,z)T_{k-1}(t,z)\hat{\Phi}(z)\,. \tag{17}$$

We will use the next estimate for $\hat{W}(t,z)$, $t \geq 0$, $z \in \mathbb{K}_p$:

$$\|\hat{W}(t,z)\|_{H_{\mathbb{C}}} \leq \sum_{n\in\mathbb{N}} |\xi_{i(n)}(t)z_n| \leq \sum_{i,j\in\mathbb{N}} \frac{\sup_{i\in\mathbb{N}}\left(i^{\frac{1}{12}}\sup_{t\in\mathbb{R}}|\xi_i(s)|\right)}{i^{\frac{1}{12}}(2ij)^p} =: M_w\,. \tag{18}$$

Note also that from Assumption **B**, by the uniform boundedness principle, it follows that there exists $K > 0$ such that for all $x \in H$

$$\big\|[A, B(\cdot)x]\big\|_{\mathcal{L}(H)} \leq K\|x\|\,. \tag{19}$$

Lemma 3.3. *For any $\Phi \in (\mathrm{dom} A)$, $k \in \mathbb{N}$, $t \geq 0$ and $z \in \mathbb{K}_p$*

(20)

$$\Big\|AT_k(t,z)\hat{\Phi}(z)\Big\| \leq M^{k+1}\|B\|^{k-1}C^{k-1}e^{at}\frac{t^k}{k!}\Big(C\|B\|\,\|A\hat{\Phi}(z)\| + M_wK\|\hat{\Phi}(z)\|\Big)\,,$$

where M, a, C are as in Lemma 3.2, M_w is defined by (18) and K is from (19).

Proof. For $\Phi \in (\mathrm{dom} A)$, $k \in \mathbb{N}$ we have

$$AT_k(t,z)\hat{\Phi}(z) = \int_0^t AS(t-s)\hat{B}(s,z)T_{k-1}(s,z)\hat{\Phi}(z)\,ds =$$

$$\int_0^t S(t-s)\big(\hat{B}(s,z)A + [A,\hat{B}(s,z)]\big)T_{k-1}(s,z)\hat{\Phi}(z)\,ds =$$

$$= T_k(t,z)A\hat{\Phi}(z) + \int_0^t AS(t-s)[A,\hat{B}(s,z)]T_{k-1}(s,z)\hat{\Phi}(z)\,ds\,.$$

Using this representation and the estimate (13) we obtain (20). □

From Lemmas 3.2 and 3.3 follows convergence of the series $\sum_{k=0}^{\infty} AT_k(t,z)\hat{\Phi}(z)$ and $\sum_{k=0}^{\infty} \hat{B}(t,z)T_k(t,z)\hat{\Phi}(z)$ in $H_{\mathbb{C}}$ for all $t \geq 0$ and $z \in \overline{\mathbb{K}}_p$. Taking the sum of equalities (15) and (17) with $k = 1, 2, \ldots$, we obtain that $\hat{X}(t,z)$ solves the problem (11). By Proposition 2.10, there exists $X(t) \in (\mathcal{S})_{-1}(H)$ such that $\hat{X}(t,z) = \mathcal{H}[X(t)](z)$ and it is a solution of the problem (8).

Thus we obtain the following result:

Theorem 3.4. *Let A be the generator of a C_0-semigroup, $B(\cdot) \in \mathcal{L}\big(\mathcal{H}, \mathcal{L}(\mathcal{H})\big)$ and Assumption **B** be true. Then for all $\Phi \in (\mathrm{dom} A)$ the Cauchy problem (8) has a strong solution in $(\mathcal{S})_{-1}(H)$.*

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