Novi Sad J. Math.
Vol. 41, No. 1, 2011, 43-52

ON A SEMI-LINEAR ELLIPTIC EQUATION WITH COEFFICIENTS WHICH ARE GENERALIZED FUNCTIONS

Kamal N. Soltanov[1]

Abstract. This article investigates a certain class of the semilinear elliptic equations in which the nonlinear part has a term, the coefficient, that is a generalized function. We considered the problem which is, for instance, a semiclassical NLS type of problem, and prove a theorem on its solvability.

AMS Mathematics Subject Classification (2010): 35J60, 35B33, 35Q55, 46F10, 46T20, 33E30

Key words and phrases: Semilinear elliptic equation, generalized function coefficients, semiclassical NLS type equation, existence theorem

We consider the following problem

$$-\Delta u+f\left(x,u\right)=h\left(x\right),\quad x\in\Omega\subset R^{n}, \tag{1}$$

$$u\left|_{\,\partial\Omega}\right.=0,\quad n\geq 1 \tag{2}$$

where $h\left(x\right)$ is a generalized function ($h\in W_{2}^{-1}\left(\Omega\right)$), Ω is a bounded domain with sufficiently smooth boundary $\partial\Omega$. And we investigate the existence of solutions for the $f\left(x,u\right)$ that is represented as $f\left(x,u\right)=q\left(x\right)\left|u\left(x\right)\right|^{p-2}u\left(x\right)+f_{0}\left(x,u\left(x\right)\right)$, where $f_{0}:\Omega\times R^{1}\longrightarrow R^{1}$ is a Caratheodory function[2] and $q\left(x\right)$ is a generalized function, $p\geq 2$. It is known that in this case the equation (1) is an equation of the semiclassical Nonlinear Schrodinger type (i.e. NLS) (see, [1, 2, 3, 6, 10] and references therein). Considerable attention has been paid in recent years to the problem (1) for small $\varepsilon>0$ as the Laplacian coefficient since the solutions are known as in the semiclassical states, which can be used to describe the transition from Quantum to Classical Mechanics (see, [3, 5, 7, 11, 12, 14, 16, 17, 23, 24, 25] and references therein).

The equations of such type were studied in many articles under different conditions on the function f (see, for example, [4, 8, 9, 13, 18, 19, 20, 22] and references therein). In these articles the equation (1) was considered with various functions $f(x,u)$ that are mainly Caratheodory functions with some additional properties. Although such cases when $f(x,u)$ possesses a singularity

[1]Department of Mathematics, Faculty of Sciences, Hacettepe University, Beytepe, Ankara, TR-06532, Turkey, e-mail: soltanov@hacettepe.edu.tr

[2]Let $f:\Omega\times R^{m}\longrightarrow R$ be a given function, where Ω is a nonempty measurable set in R^{n} and $n,m\geq 1$. Then f is Caratheodory function if the following holds: $x\longrightarrow f\left(x,\eta\right)$ is measurable on Ω for all $\eta\in R^{m}$, and $\eta\longrightarrow f\left(x,\eta\right)$ is continuous on R^{m} for almost all $x\in\Omega$.

with respect to the variable x of certain type were also investigated (as equations Emden-Fowler, Yamabe, NLS, etc.), but in all of these articles the coefficient $q(x)$ is a function in the usual sense (of a Lebesgue space). Here an existence theorem for the problem (1) - (2) is proved in the model case when $f(x,u)$ only has the above expression (section 4). In section 2 we have explained how to understand the equation (1) with use of representation of certain generalized functions and properties of some special class of functions. In section 3 we have presented some general results from [21, 22], on which the proof of the solvability of the theorem is based.

1. Statement of the Main Solvability Result

Let the operator $f(x,u)$ have the form

$$f(x,u) = q(x)\,|u|^{p-2}\,u + f_0(x,u) \tag{3}$$

in the generalized sense, where $q \in W^{-1}_{p_0}(\Omega)$, $p_0 \geq 2$ (it should be noted that either $p_0 \equiv p_0(p)$ or $p \equiv p(p_0)$) and $u(x)$ is an element of the space of sufficiently smooth functions that will be determined below (see, Section 2). Consequently, the function $q(x)$ is a generalized function, which has singularity of the order 1.

Assume (i) $f_0(x,\tau)$ is a Caratheodory function on $\Omega \times R^1$ and there exist numbers $\widetilde{p}, p_1 \geq 1$, $c > 0$ such that

$$|f_0(x,\eta) - f_0(x,\xi)| \leq c\left(|\eta|^{\widetilde{p}-1} + |\xi|^{\widetilde{p}-1}\right)\,|\eta - \xi|\,, \tag{4}$$

holds for a.e. $x \in \Omega$ and any $\eta, \xi \in R^1$, moreover $f_0(x,0) \in L_{p_1}(\Omega)$, $p_1 \geq \frac{n+2}{n-2}$, where $\widetilde{p} < \frac{n+2}{n-2}$ if $n \geq 3$, $\widetilde{p} \in [1,\infty)$ if $n = 1,2$;

(ii) there exist numbers $2 \geq \theta \geq 0$, $k_0(\theta) \geq 0$, $c_0 \geq 0$, $p_2 \geq 1$ and $k_1 \in R^1$ such that $1 \leq p_2 \leq \frac{2n}{n-2}$, if $n \geq 3$, $1 \leq p_2 < \infty$, if $n = 1,2$ and

$$\langle f_0(x,u), u\rangle \geq -k_0(\theta)\,\|u\|^{\theta}_{p_2} - c_0\int_{\Omega} q(x)\,|u(x)|^p\,dx + k_1 \tag{5}$$

holds for any $u \in \overset{0}{W}{}^{1}_{2}(\Omega)$, where $\langle\cdot,\cdot\rangle$ (here and in the sequel) denote the dual form for the pair (X, X^*) of the Banach space X and its dual space X^*, in this case we have $\left(\overset{0}{W}{}^{1}_{2}(\Omega), W^{-1}_{2}(\Omega)\right)$, $k_0(\theta) \geq 0$ is arbitrary if $0 \leq \theta < 2$, and $1 > C(2,p_2)^{-2} \cdot k_0(2) \geq 0$ if $\theta = 2$; $1 \geq c_0 \geq 0$ [3].

[3] here $C(2;p_2)$ is the constant of the known inequality of Embedding Theorems for Sobolev spaces

$$\|\nabla u\|_2 \geq C(2;p_2)\,\|u\|_{p_2}\,,\ \forall u \in \overset{0}{W}{}^{1}_{2}(\Omega)\,.$$

Definition 1.1. A function $u \in \overset{0}{W}{}_2^1(\Omega)$ is called a solution of the problem (1) - (2) if the following equation is fulfilled

$$\int_\Omega [-\Delta u + f(x,u)] \ \varphi \, dx = \int_\Omega h \ \varphi \, dx \tag{6}$$

for any $\varphi \in \overset{0}{W}{}_2^1(\Omega)$.

It should be noted that the sense in which equation (6) is to be understood will be explained below (section 2). We have proved the following result for the considered problem.

Theorem 1.2. *Let the function f have the representation (3) in the generalized sense, where $q \in W_{p_0}^{-1}(\Omega)$ is a nonnegative distribution (generalized function,[4]), $p_0 = \frac{2n}{2(n-1)-p(n-2)}$, $\frac{2(n-1)}{n-2} > p > 2$ if $n \geq 3$; $p_0, p > 2$ are arbitrary if $n = 2$, and $p_0, p \geq 2$ are arbitrary if $n = 1$ (in particular, if $n = 3$ then $2 < p < 4$ and $p_0 = \frac{6}{4-p}$) and $f_0 : \Omega \times \ R^1 \longrightarrow R^1$ is a Caratheodory function such that conditions (i), (ii) are fulfilled. Then for any $h \in W_2^{-1}(\Omega)$ the problem (1) - (2) is solvable in $\overset{0}{W}{}_2^1(\Omega)$.*

For the investigation of the considered problem we used some general solvability theorems, which are conducted in section 3. We begin with explanation of equation (6).

2. The Solution Concept and Function Spaces

So we will consider the case when the function $f(x,u)$ has the form (3), where the functions q and u are the same as above. Consequently, the function $q(x)$ is a generalized function, which has singularity of order 1. Therefore we must understand the equation (1) in the sense of the generalized function space, i.e.

$$\int_\Omega [-\Delta u + f(x,u)] \ \varphi(x) \, dx \equiv$$

$$\int_\Omega \left[-\Delta u(x) + q(x) |u(x)|^{p-2} u(x) + f_0(x, u(x))\right] \ \varphi(x) \, dx = \int_\Omega h(x) \varphi(x) \, dx$$

for any $\varphi \in D(\Omega)$, where $D(\Omega)$ is $C_0^\infty(\Omega)$ and $\operatorname{supp}\varphi \subset \Omega$ with the corresponding topology.

In the beginning we need to define the expression $q \ |u|^{p-2} u$. It is known that (see, for example, [15]) in the case when $q \in W_{p_0}^{-1}(\Omega)$ we can represent it in the form $q(x) \equiv \sum_{i=0}^{n} \frac{\partial}{\partial x_i} q_i(x)$, $\frac{\partial}{\partial x_0} \equiv I$, $q_i \in L_{p_0}(\Omega)$, $i = 0, \overline{1,n}$ in the

[4] see, Definition 2 of the section 2

sense of generalized function space. From here it follows that if a solution of the considered problem belongs to the space which contains $\overset{0}{W}{}^{1}_{\widetilde{p}_1}(\Omega)$ for some number $\widetilde{p}_1 \geq 1$, then we can understand the term $q\ |u|^{p-2}u$ in the following sense

$$\left\langle q\ |u|^{p-2}u, \varphi \right\rangle \equiv \int\limits_{\Omega} q(x)\, |u(x)|^{p-2}\, u(x)\, \varphi(x)\, dx \tag{7}$$

for any $\varphi \in D(\Omega)$. Therefore we must find the needed number $\widetilde{p}_1 \geq 1$. Namely, we have to find the relation between the numbers p_0 and $\widetilde{p}_1$. So, taking into account that for a function $u \in \overset{0}{W}{}^{1}_{2}(\Omega)$, i.e. $\widetilde{p}_1 = 2$ (as $h \in W_2^{-1}(\Omega)$ by the assumption) we have $u \in L_{\widetilde{p}_1^*}(\Omega)$, where $\widetilde{p}_1^* = 2^* = \frac{2n}{n-2}$ for $n \geq 3$ by virtue of the embedding theorem, from (7) we get

$$\begin{aligned}
&\left\langle q\ |u|^{p-2}u, \varphi \right\rangle \equiv \int\limits_{\Omega} q(x)\, |u(x)|^{p-2}\, u(x)\, \varphi(x)\, dx \\
&= \int\limits_{\Omega} \sum_{i=0}^{n} \frac{\partial}{\partial x_i} q_i(x)\, |u(x)|^{p-2}\, u(x)\, \varphi(x)\, dx = -\int\limits_{\Omega} \sum_{i=1}^{n} q_i\, |u|^{p-2}\, u \frac{\partial \varphi}{\partial x_i} dx \\
&- (p-1) \int\limits_{\Omega} \sum_{i=1}^{n} q_i\, |u|^{p-2}\, \frac{\partial u}{\partial x_i} \varphi dx + \int\limits_{\Omega} q_0\, |u|^{p-2}\, u \varphi dx \\
&= I_1 + I_2 + \int\limits_{\Omega} q_0\, |u|^{p-2}\, u \varphi dx
\end{aligned} \tag{8}$$

by virtue of the generalized function theory.

Here and in what follows we assume $n \geq 3$. Because if $n = 1, 2$ then we can choose arbitrary $p \geq 2$, as will be observed below. Let us take into account that $\varphi \in D(\Omega)$ and $n \geq 3$, then in order for the expression in the left part of (8) to have the meaning, it is enough for us to take $1 \leq p - 1 \leq \frac{2n(p_0-1)}{p_0(n-2)}$ for the integral I_1 and $0 \leq p - 2 \leq \frac{n(p_0-2)}{p_0(n-2)}$ for the integral I_2. Therefore, if $2 \leq p \leq \frac{3np_0 - 2(n+2p_0)}{p_0(n-2)}$ then the left part of (8) is defined. Now, let $\varphi \in \overset{0}{W}{}^{1}_{2}(\Omega)$. Then it is sufficient to study one of the I_1 and I_2. Let us consider I_1, from which we obtain that $2 \leq p \leq \frac{2np_0 - 2(n+p_0)}{p_0(n-2)}$, moreover we can choose $p \geq 2$ only if $p_0 > n$. On the other hand, if we take into account the given p, we obtain $p_0 = \frac{2n}{2(n-1) - p(n-2)}$, and consequently, in order for $p_0 < \infty$, we must choose $2(n-1) > p(n-2)$ or $p < \frac{2(n-1)}{n-2}$. In the case when $n = 3$ then $p < 4$ and $p_0 = \frac{6}{4-p}$.

Thus we determined under what conditions the left part of (8) is defined. Hence, this implies the correctness of the statement

Proposition 2.1. *Assume $\widetilde{f}$ be an operator defined by expression $\widetilde{f}(u) \equiv q\ |u|^{p-2}u$, where $q \in W_{p_0}^{-1}(\Omega)$, and $u \in \overset{0}{W}{}^{1}_{2}(\Omega)$. If $2 \leq p < \frac{2(n-1)}{n-2}$ and*

$p_0 = \frac{2n}{2(n-1)-p(n-2)}$ *if* $n \geq 3$ *(in particular, if* $n = 3$ *then* $2 \leq p < 4$ *and* $p_0 = \frac{6}{4-p}$*) then* $\widetilde{f} : \overset{0}{W}{}_2^1(\Omega) \longrightarrow W_2^{-1}(\Omega)$ *is a bounded operator.*

And also, the following statements are true[5].

Lemma 2.2. *Let* $u \in \overset{0}{W}{}_2^1(\Omega)$ *and the number* p *satisfy the inequation* $2 < p < \frac{2(n-1)}{n-2}$, $n \geq 3$. *Then the function* $v(x) \equiv \eta(u(x)) \equiv |u(x)|^p$ *belongs to* $\overset{0}{W}{}_\beta^1(\Omega)$ *for any* $\beta \in [1, p_0']$, *where* $p_0 = \frac{2n}{2(n-1)-p(n-2)}$ *and* $p_0' = \frac{p_0}{p_0-1} = \frac{2n}{p(n-2)+2}$. *(It is obvious:* $u \in \overset{0}{W}{}_2^1(\Omega) \Longrightarrow v \equiv |u|^p \in \overset{0}{W}{}_\beta^1(\Omega)$ *for any* $\beta \in [1,2)$ *if* $n = 2$, *and for any* $\beta \in [1,2]$ *if* $n = 1$.*)*

Corollary 2.3. *Let* $u, w \in \overset{0}{W}{}_2^1(\Omega)$ *and the number* p *be such that* $2 < p < \frac{2(n-1)}{n-2}$, $n \geq 3$. *Then the function* $v(x) \equiv |u(x)|^{p-2} u(x) w(x)$ *belongs to* $\overset{0}{W}{}_\beta^1(\Omega)$ *(i.e.* $v \in \overset{0}{W}{}_\beta^1(\Omega)$*) for any* $\beta \in [1, p_0']$, *where* $p_0 = \frac{2n}{2(n-1)-p(n-2)}$ *and* $p_0' = \frac{p_0}{p_0-1}$.

Now we introduce a concept of the nonnegative generalized function

Definition 2.4. A generalized function $q(x)$ is called a non-negative distribution ("$q \geq 0$") iff $\langle q, \ \varphi \rangle \geq 0$ holds for any non-negative test function $\varphi \in D(\Omega)$.

3. General Solvability Results

Let X, Y be reflexive Banach spaces and X^*, Y^* their dual spaces, moreover Y is a reflexive Banach space with strictly convex norm together with Y^* (see, for example, references of [21]). So we present variant of the main result of [21] (the more general cases can be found in [22]). Consider the following conditions:

(a) X, Y be Banach spaces such as above and $f : D(f) \subseteq X \longrightarrow Y$ be a continuous mapping, moreover the closed ball $B_{r_0}^X(0) \subset X$ belongs to $D(f)$ ($B_{r_0}^X(0) \subseteq D(f)$);

The following conditions are fulfilled on the closed ball $B_{r_0}^X(0) \subseteq D(f)$:

(b) f is a bounded mapping, i.e. $\|f(x)\|_Y \leq \mu(\|x\|_X)$ holds for $\forall x \in B_{r_0}^X(0)$ where $\mu : R_+^1 \longrightarrow R_+^1$ is a continuous function;

(c) there is a mapping $g : D(g) \subseteq X \longrightarrow Y^*$, and a continuous function $\nu : R_+^1 \longrightarrow R^1$ nondecreasing for $\tau \geq \tau_0$ such that $D(f) \subseteq D(g)$, and for any $S_r^X(0) \subset B_{r_0}^X(0)$, $0 < r \leq r_0$, closure of $g\left(S_r^X(0)\right) \equiv S_r^{Y^*}(0)$, $S_r^X(0) \subseteq g^{-1}\left(S_r^{Y^*}(0)\right)$

$$\langle f(x), g(x) \rangle \geq \nu(\|x\|_X) \|x\|_X, \ \text{a.e. } x \in B_{r_0}^X(0) \quad \& \ \nu(r_0) \geq \delta_0 > 0 \tag{3.1}$$

holds, here $\delta_0 > 0$, $\tau_0 \geq 0$ are constants;

[5] For additional explanation of these results see, for example, Soltanov K. N. - J. Nonlinear Analysis : T.M. & APPL. (2006), 65, 2103-2134

(d) almost each $\widetilde{x} \in intB_{r_0}^X(0)$ possesses a neighborhood $V_\varepsilon(\widetilde{x})$, $\varepsilon \geq \varepsilon_0 > 0$ such that the inequation

$$\|f(x_2) - f(x_1)\|_Y \geq \Phi(\|x_2 - x_1\|_X, \widetilde{x}, \varepsilon) + \psi(\|x_1 - x_2\|_Z, \widetilde{x}, \varepsilon) \tag{3.2}$$

holds for any $x_1, x_2 \in V_\varepsilon(\widetilde{x}) \cap B_{r_0}^X(0)$, where $\Phi(\tau, \widetilde{x}, \varepsilon) \geq 0$ is a continuous function of τ and $\Phi(\tau, \widetilde{x}, \varepsilon) = 0 \Leftrightarrow \tau = 0$ (in particular, may be $\widetilde{x} = 0$, $\varepsilon = \varepsilon_0 = r_0$ and $V_\varepsilon(\widetilde{x}) = V_{r_0}(0) \equiv B_{r_0}^X(0)$, consequently $\Phi(\tau, \widetilde{x}, \varepsilon) \equiv \Phi(\tau, 0, r_0)$ on $B_{r_0}^X(0)$), Z is a Banach space and the inclusion $X \subset Z$ is compact, and $\psi(\cdot, \widetilde{x}, \varepsilon) : R_+^1 \longrightarrow R^1$ is a continuous function at τ and $\psi(0, \widetilde{x}, \varepsilon) = 0$.

Theorem 3.1. *Let the conditions (a), (b), (c), (d) be fulfilled. Then the image $f\left(B_{r_0}^X(0)\right)$ of the ball $B_{r_0}^X(0)$ is a bodily subset (i.e. with nonempty interior) of Y, moreover $f\left(B_{r_0}^X(0)\right)$ contains a bodily subset M that has the form*

$$M \equiv \left\{y \in Y \mid \langle y, g(x)\rangle \leq \langle f(x), g(x)\rangle,\ \forall x \in S_{r_0}^X(0)\right\}.$$

Now we present a solvability theorem for the nonlinear equation in Banach spaces, which is proved using Theorem 3.1. Let $F_0 : D(F) \subseteq X \longrightarrow Y$ and $F_1 : D(F_1) \subseteq X \longrightarrow Y$ be some nonlinear mappings such that $D(F_0) \cap D(F_1) = G \subseteq X$ and $G \neq \varnothing$. Consider the following equation

$$F(x) \equiv F_0(x) + F_1(x) = y, \quad y \in Y \tag{3.3}$$

where y is an arbitrary element of Y.

Let $B_r^X(0) \subseteq D(F_0) \cap D(F_1) \subseteq X$ be a closed ball, $r > 0$ be a number. Consider the following conditions:

1) $F_0 : B_r^X(0) \longrightarrow Y$ is a bounded continuous operator together with its inverse operator F_0^{-1}, (as $F_0^{-1} : D\left(F_0^{-1}\right) \subseteq Y \longrightarrow X$);

2) $F_1 : B_r^X(0) \longrightarrow Y$ is a nonlinear continuous operator;

3) There are continuous functions $\mu_i : R_+^1 \longrightarrow R_+^1$, $i = 1, 2$ and $\nu : R_+^1 \longrightarrow R^1$ such that the inequations

$$\|F_0(x)\|_Y \leq \mu_1(\|x\|_X) \ \&\ \|F_1(x)\|_Y \leq \mu_2(\|x\|_X),$$

$$\langle F_0(x) + F_1(x), g(x)\rangle \geq c\langle F_0(x), g(x)\rangle \geq \nu(\|x\|_X)\|x\|_X$$

hold for any $x \in B_r^X(0)$, moreover $\nu(r) \geq \delta_0$ holds for some number $\delta_0 > 0$, where the mapping $g : B_r^X(0) \subseteq D(g) \subseteq X \longrightarrow Y^*$ fulfills the conditions of Theorem 3.1, $c > 0$ is some number.

4) Almost each $\widetilde{x} \in intB_r^X(0)$ possesses a neighborhood $B_\varepsilon^X(\widetilde{x})$, $\varepsilon \geq \varepsilon_0 > 0$, such that the inequation

$$\|F(x_1) - F(x_2)\|_Y \geq c_1\|F_0(x_1) - F_0(x_2)\|_Y \geq$$

$$k_0(\|x_1 - x_2\|_X, \widetilde{x}, \varepsilon) - k_1(\|x_1 - x_2\|_Z, \widetilde{x}, \varepsilon), \quad X \Subset Z$$

holds for any $x_1, x_2 \in B_\varepsilon^X(\widetilde{x})$ and some number $\varepsilon_0 > 0$, where $k_i(\tau, \widetilde{x}, \varepsilon) \geq 0$, $i = 0, 1$ are continuous functions of τ for any given $\widetilde{x}$, and such that $k_0(\tau, \widetilde{x}, \varepsilon) = 0 \Longleftrightarrow \tau = 0$, $k_1(0, \widetilde{x}, \varepsilon) = 0$, and $X \Subset Z$ (i.e. $X \subset Z$ is compact).

Then the following statement is true, which follows from Theorem 3.1 .

Theorem 3.2. *Let the conditions 1, 2, 3, 4 be fulfilled. Then equation (3.3) has a solution in the ball* $B_r^X(0)$ *for any* $y \in Y$ *satisfying the inequation*

$$\langle y, g(x)\rangle \le \nu(\|x\|_X)\|x\|_X, \quad \forall x \in S_r^X(0).$$

4. Proof of Theorem 1.2

To apply Theorem 3.2 to the considered problem (1) - (2) we define the corresponding spaces and mappings in the following form

$$X \equiv \overset{0}{W}{}_2^1(\Omega), \ Y \equiv W_2^{-1}(\Omega) \equiv X^*, \ F_0 \equiv -\Delta, \ F_1(u) \equiv f(x,u), \ g \equiv id \equiv I$$

and we assume (3) in the corresponding sense, the number $p > 2$ and the function $f_0(x,\xi)$ satisfying all conditions of Theorem 1.2.

The defined spaces and mappings F_0, g satisfy the conditions of Theorem 3.2 on the ball $B_r^{\overset{0}{W}{}_2^1}(0)$ for each number $r > 0$. Indeed, it is enough to show that the inequations of Theorem 3.2 are fulfilled. As known $\|\Delta u\|_{W_2^{-1}} \equiv \|u\|_{\overset{0}{W}{}_2^1}$ holds for any $u \in \overset{0}{W}{}_2^1(\Omega)$. Let $u \in \overset{0}{W}{}_2^1(\Omega)$. Then we have

$$\|f(x,u)\|_{W_2^{-1}} \le \left\|q(x)|u|^{p-2}u + f_0(x,u)\right\|_{W_2^{-1}} \le$$

$$\left\|q(x)|u|^{p-2}u\right\|_{W_2^{-1}} + \|f_0(x,u)\|_{W_2^{-1}} \tag{9}$$

From Proposition 2.1 and Lemma 2.2 it follows that the first term in the right part is bounded , i.e.

$$\begin{aligned}
&\left\|q(x)|u|^{p-2}u\right\|_{W_2^{-1}} \\
&= \sup \frac{\left\langle q(x)|u|^{p-2}u, w\right\rangle}{\|w\|_{\overset{0}{W}{}_2^1}} \\
&\le \frac{\|q(x)\|_{W_{p_0}^{-1}}\left\||u|^{p-2}u\cdot w\right\|_{W_{p_0'}^{-1}}}{\|w\|_{\overset{0}{W}{}_2^1}} \\
&\le \|q(x)\|_{W_{p_0}^{-1}}\frac{1}{\|w\|_{\overset{0}{W}{}_2^1}}\left\{[u]^{p-1}_{\overset{0}{S}_{1,(p-1)p_0',p_0'}} + \|u\|^{p-1}_{\frac{2n}{n-2}}\right\}\|w\|_{\overset{0}{W}{}_2^1} \\
&\le C_0\|q(x)\|_{W_{p_0}^{-1}}\|u\|^{p-1}_{\overset{0}{W}{}_2^1} \\
\Longrightarrow &\left\|q(x)|u|^{p-2}u\right\|_{W_2^{-1}} \le C_0\|q(x)\|_{W_{p_0}^{-1}}\|u\|^{p-1}_{\overset{0}{W}{}_2^1}.
\end{aligned}$$

If we expand inequation (9), using condition (i) we get

$$\begin{aligned}\|f(x,u)\|_{W_2^{-1}} &\leq C_0 \|q(x)\|_{W_{p_0}^{-1}} \|u\|^{p-1}_{\overset{0}{W}{}^1_2} + \left\| c\,|u|^{\widetilde{p}} \right\|_{W_2^{-1}} + \|f_0(x,0)\|_{W_2^{-1}} \\ &\leq C_0 \|q(x)\|_{W_{p_0}^{-1}} \|u\|^{p-1}_{\overset{0}{W}{}^1_2} + c\,\|u\|^{\widetilde{p}}_{\widetilde{p}} + \|f_0(x,0)\|_{p_1} \\ &\equiv \mu\left(\|u\|_{\overset{0}{W}{}^1_2}\right).\end{aligned}$$

Hence we obtain that $F : \overset{0}{W}{}^1_2(\Omega) \longrightarrow W_2^{-1}(\Omega)$ is a bounded operator.

Now we estimate the dual form $\langle F(u), u\rangle$ for any $u \in \overset{0}{W}{}^1_2(\Omega)$, for which we have

$$\begin{aligned}\langle F(u), u\rangle &\equiv \left\langle -\Delta u + q(x)|u|^{p-2}u + f_0(x,u), u \right\rangle \\ &= \|\nabla u\|_2^2 + \left\langle q(x)|u|^{p-2}u, u\right\rangle + \langle f_0(x,u), u\rangle \\ &\geq \|\nabla u\|_2^2 + (1-c_0)\int_\Omega q(x)|u|^p\,dx - k_0(\theta)\|u\|^\theta_{p_2} + k_1.\end{aligned}$$

Taking into account the expression of the dual form and conditions of Theorem 1.2 and using Definition 1, and the Young inequality we obtain

$$\langle F(u), u\rangle \geq \delta\|\nabla u\|_2^2 + k_2 = \left(\delta\|\nabla u\|_2 + k_2\|\nabla u\|_2^{-1}\right)\|\nabla u\|_2$$

i.e. $\nu(\tau) \equiv \delta\tau + k_2\tau^{-1}$, if $\theta < 2$, $\delta = 1 - k_0(\theta)\varepsilon_0$, $k_2 = k_1 - C(\varepsilon_0, \theta)$, where $\varepsilon_0 > 0$ is a sufficiently small number, $C(\varepsilon_0,\theta)$ corresponds to ε_0; and if $\theta = 2$ then $\delta = 1 - C(2,p_2)^{-2}\cdot k_0(2)$, $k_2 = k_1$.

Further we show that condition 4 is fulfilled in a ball $B_r^{\overset{0}{W}{}^1_2}(0)$ for some $r > 0$. So, if we assume $u, v \in \overset{0}{W}{}^1_2(\Omega)$ then we get

$$\begin{aligned}&\langle F(u) - F(v), u - v\rangle \\ &\quad = \|\nabla(u-v)\|_2^2 + \left\langle q(x)\left(|u|^{p-2}u - |v|^{p-2}v\right), u - v\right\rangle \\ &\qquad + \langle f_0(x,u) - f_0(x,v), u - v\rangle \\ &\quad \geq \|\nabla(u-v)\|_2^2 - \|f_0(x,u) - f_0(x,v)\|_{W_2^{-1}}\|u-v\|_{\overset{0}{W}{}^1_2} \\ &\quad \geq \|\nabla(u-v)\|_2^2 - c\left(\|u\|^{\widetilde{p}-1}_{2^*} + \|v\|^{\widetilde{p}-1}_{2^*}\right)\|u-v\|_{\widetilde{p}_1}\|u-v\|_{\overset{0}{W}{}^1_2},\end{aligned}$$

where $1 < \widetilde{p}_1 < 2^* \equiv \frac{2n}{n-2}$ if $n \geq 3$, and $\widetilde{p}_1 \in (1,\infty)$ is arbitrary if $n = 1, 2$. Consequently, we have

$$\|F(u) - F(v)\|_{W_2^{-1}} \geq C\|u-v\|_{\overset{0}{W}{}^1_2} - c\left(\|u\|^{\widetilde{p}-1}_{\overset{0}{W}{}^1_2} + \|v\|^{\widetilde{p}-1}_{\overset{0}{W}{}^1_2}\right)\|u-v\|_{\widetilde{p}_1}$$

holds for any $u, v \in \overset{0}{W}{}_2^1(\Omega)$. From here it follows that condition 4 of Theorem 3.2 is fullfilled. Thus we proved that the conditions of Theorem 1.2 imply that all conditions of Theorem 3.2 are fulfilled on the ball $B_r^{\overset{0}{W}{}_2^1}(0)$ for any $r > 0$. Therefore problem (1) - (2) is solvable for any $h \in W_2^{-1}(\Omega)$ by virtue of Theorem 3.2.

Acknowledgement

The author is grateful to Referee for useful sugestions that improved this article.

References

[1] Alves, M., Sepulveda, M., Vera, O., Smoothing properties for the higher-order nonlinear Schrodinger equation with constant coefficients. Journal Nonlinear Analysis : T.M. & A. 71 (2009).

[2] Ambrosetti, A., Badiale, M., x Cingolani, M., Semiclassical states of nonlinear Schrödinger equations. Arch. Ration. Mech. Anal. 140 (2001), 285–300.

[3] Ambrosett, A., Malchiodi, A., Ruiz, D., Bound states of nonlinear Schrödinger equations with potentials vanishing at infinity. J. Anal. Math. 98 (2006), 317–348.

[4] Bidaut-Véron, M.-F., Pohozaev, S.I., Nonexistence results and estimates for some nonlinear elliptic problems. J. Anal. Math. 84 (2001), 1–49.

[5] Brezis, H., Ponce, A.C., Reduced measures on the boundary. J. Funct. Anal. 229 (2005).

[6] Cingolani, S., Semiclassical stationary states of nonlinear Schrödinger equations with an external magnetic field. J. Differential Equations 188 (2003), 52–79.

[7] Floer, A., Weinstein, A., Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. J. Funct. Anal. 69 (1986), 397–408.

[8] Gidas, B., Ni, W.M., Nirenberg, L., Symmetry of positive solutions of nonlinear elliptic equations in R^N. Adv. Math. Suppl. Stud. A 7 (1981), 369–402.

[9] Gilbarg, D., Trudinger, N.S., Elliptic Partial Differential Equations of Second Order. (Second ed.), Grundlehren Math. Wiss., vol. 224, Berlin, New York: Springer, 1983.

[10] Gross, M., On the number of single-peak solutions of the nonlinear Schrödinger equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002), 261–280.

[11] Gui, C., Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method. Comm. Partial Differential Equations 21 (1996), 787–820.

[12] Huicheng Yin, Pingzheng Zhang, Bound states of nonlinear Schrödinger equations with potentials tending to zero at infinity. J. Diffi Eq. 247 (2009), 618–647.

[13] Le Bris, C., Lions, P.-L., From atoms to crystals: a mathematical journey. Bull. Amer. Math. Soc. (N.S.) 42 (2005), 291–363.

[14] Lenells, J., Fokas, A.S., On a novel integrable generalization of the nonlinear Schrodinger equation. J. Nonlinearity 22 (2009), 11–27.

[15] Lions, J.-L., Magenes, E., Nonhomogeneous boundary value problems and applications, N.-Y: Springer-Verlag, 1972.

[16] Noussair, E.S., Swanson, C.A., Oscillation theory for semilinear Schrödinger equations and inequalities. Proc. Roy. Soc. Edinburgh Sect. A 75 (1975), 67–81.

[17] Rabinowitz, P.H., On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992), 27–42.

[18] Rabinowitz, P.H., Stredulinsky, E., Mixed States for an Allen-Cahn Type Equation. Communications on Pure and Applied Mathematics 54 (2003), 1078–1134.

[19] Soltanov, K.N., Some Boundary Problem for Emden-Fowler type Equation. Proc. of FSDONA - 2004, Publ. Math. Inst. Acad. Sci. Czech Rep., Praha, (2005), 311-318.

[20] Soltanov, K.N., On noncoercive semilinear equations. Journal- NA: Hybrid Systems 2 (2008), 344–358.

[21] Soltanov, K.N., On semi-continuous mappings, equations and inclusions in the Banach space. Hacettepe J. Math. & Statist. 37, 2008.

[22] Soltanov, K.N., Perturbation of the mapping and solvability theorems in the Banach space. J. Nonlinear Analysis : T.M. & APPL. 72, 2010.

[23] Stuart, C.A., Lectures on the Orbital Stability of Standing Waves and Application to the Nonlinear Schrodinger Equation. Milan J. of Math. 76, 2008.

[24] Wang, X., Zeng, B., On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions. SIAM J. Math. Anal. 28 (1997), 633–655.

[25] Wang, J., Xu, J.X., Zhang, F.B., Existence and multiplicity of semiclassical solutions for a Schrodinger equation. J. Math. Anal. and Appl. 357, 2009.

Received by the editors April 22, 2010