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ALGEBRAS OF GENERALIZED GEVREY ULTRADISTRIBUTIONS

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Abstract. New algebras of generalized functions containing Gevrey ultradistributions are introduced. We also study the embeddings of the spaces of Gevrey ultradistributions into such algebras.

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1. Introduction

The importance of ultradistributions in the theory of partial differential equations as well as in applied problems is well known, see [8] and [11], so it is an important issue to provide differential algebras of generalized functions containing spaces of ultradistributions.

A central role in the field of differential algebras of generalized functions is played by the Colombeau algebra. This algebra contains the space of distributions as a subspace and has the algebra of smooth functions as a faithful subalgebra.

Generalized Gevrey ultradistributions of Colombeau type have been defined, but as a side-theme, in the paper [5]. The first paper aiming to construct differential algebras containing ultradistributions is [10]. Let us also mention the interesting approach of the paper [4] to algebras of generalized ultradistributions.

However, a Colombeau type theory of generalized Gevrey ultradistributions has been addressed in [1], where the core of a full theory was developed and also introduced a new way of defining differential algebras of generalized Gevrey ultradistributions that makes such a complete theory possible. In [1] we recovered a whole list of important results known for the usual Colombeau theory in the setting of generalized Gevrey ultradistributions. But, it was not clear in that paper why different Gevrey exponents occurred in the embedding of the spaces of Gevrey ultradistributions.

The aim of this paper is to introduce new algebras of generalized functions containing Gevrey ultradistributions and also to study the embeddings of the spaces of Gevrey ultradistributions into such algebras.

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2. Generalized Gevrey ultradistributions

We first introduce the algebra of moderate elements and its ideal of null elements depending on the order $\tau > 0$. The set Ω is a non'void open set of $\mathbb{R}^m$.

Definition 2.1. The space of moderate elements, denoted $\mathcal{E}_m^\tau(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{]0,1]}$ satisfying for every compact K of Ω, $\forall \alpha \in \mathbb{Z}_+^m, \exists k > 0$,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| = O\left(\exp\left(k\varepsilon^{-\frac{1}{\tau}}\right)\right), \varepsilon \to 0.$$

The space of null elements, denoted $\mathcal{N}^\tau(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{]0,1]}$ satisfying for every compact K of Ω,$\forall \alpha \in \mathbb{Z}_+^m, \forall k > 0$,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| = O\left(\exp\left(-k\varepsilon^{-\frac{1}{\tau}}\right)\right), \varepsilon \to 0.$$

The main properties of the spaces $\mathcal{E}_m^\tau(\Omega)$ and $\mathcal{N}^\tau(\Omega)$ are given in the following proposition.

Proposition 2.2. *1) The space of moderate elements $\mathcal{E}_m^\tau(\Omega)$ is an algebra stable by derivation.*
2) The space $\mathcal{N}^\tau(\Omega)$ is an ideal of $\mathcal{E}_m^\tau(\Omega)$.

Proof. See [1]. □

We have also the null characterization of the ideal $\mathcal{N}^\tau(\Omega)$.

Proposition 2.3. *Let $(u_\epsilon)_\epsilon \in \mathcal{E}_m^\tau(\Omega)$, then $(u_\epsilon)_\epsilon \in \mathcal{N}^\tau(\Omega)$ if and only if for every compact K of Ω, $\forall k > 0$,*

$$\sup_{x \in K} |f_\varepsilon(x)| = O\left(\exp\left(-k\varepsilon^{-\frac{1}{\tau}}\right)\right).$$

Proof. See [1]. □

According to the topological construction of Colombeau type algebras of generalized functions, we introduce the desired algebras.

Definition 2.4. The algebra of generalized ultradistributions of order $\tau > 0$, denoted $\mathcal{G}^\tau(\Omega)$, is the quotient algebra

$$\mathcal{G}^\tau(\Omega) = \frac{\mathcal{E}_m^\tau(\Omega)}{\mathcal{N}^\tau(\Omega)}.$$

A comparison of the structure of the algebras $\mathcal{G}^\tau(\Omega)$ and the Colombeau algebra $\mathcal{G}(\Omega) := \frac{\mathcal{E}_m(\Omega)}{\mathcal{N}(\Omega)}$, where $\mathcal{E}_m(\Omega)$ is the space of moderate elements and $\mathcal{N}(\Omega)$ its ideal of null elements, is as follow. Due to the inequalities

$$\exp\left(-\varepsilon^{-\frac{1}{\sigma}}\right) < \exp\left(-\varepsilon^{-\frac{1}{\tau}}\right) < \varepsilon, \forall \varepsilon \in]0,1], \text{ and } \sigma < \tau,$$

we have the strict inclusions $\mathcal{N}^{\sigma}(\Omega) \subset \mathcal{N}^{\tau}(\Omega) \subset \mathcal{N}(\Omega) \subset \mathcal{E}_m(\Omega) \subset \mathcal{E}_m^{\tau}(\Omega) \subset \mathcal{E}_m^{\sigma}(\Omega)$.

Let Ω' be an open subset of Ω and let $f = (f_\varepsilon)_\varepsilon + \mathcal{N}^{\tau}(\Omega) \in \mathcal{G}^{\tau}(\Omega)$, the restriction of f to Ω', denoted $f_{|\Omega'}$, is defined as

$$\left(f_{\varepsilon|\Omega'}\right)_\varepsilon + \mathcal{N}^{\tau}(\Omega') \in \mathcal{G}^{\tau}(\Omega').$$

Theorem 2.5. *The functor $\Omega \to \mathcal{G}^{\tau}(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^m$.*

Proof. See [1]. □

Now it is legitimate to define the support of $f \in \mathcal{G}^{\tau}(\Omega)$.

Definition 2.6. The support of $f \in \mathcal{G}^{\tau}(\Omega)$ is the complement of the largest open set U such that $f_{|U} \in \mathcal{N}^{\tau}(U)$.

The space of functions slowly increasing, denoted as $\mathcal{O}_M(\mathbb{K}^m)$, is the space of C^∞-functions all derivatives growing at most like some power of $|x|$, as $|x| \to +\infty$, where $\mathbb{K}^m \simeq \mathbb{R}^m$ or $\mathbb{R}^{2m}$.

Proposition 2.7. *If $v \in \mathcal{O}_M(\mathbb{K}^m)$ and $f = (f_1, f_2, ..., f_m) \in \mathcal{G}^{\tau}(\Omega)^m$, then $v \circ f := (v \circ f_\varepsilon)_\varepsilon + \mathcal{N}^{\tau}(\Omega)$ is a well defined element of $\mathcal{G}^{\tau}(\Omega)$.*

3. Generalized point values

The ring of exponential generalized complex numbers, denoted $\mathcal{C}^{\tau}$, is defined by the quotient

$$\mathcal{C}^{\tau} = \frac{\mathcal{E}_0^{\tau}}{\mathcal{N}_0^{\tau}} \quad ,$$

where

$$\mathcal{E}_0^{\tau} = \left\{(a_\varepsilon)_\varepsilon \in \mathbb{C}^{]0,1]}; \exists k > 0, \exists c > 0, \exists \varepsilon_0 \in]0,1], \text{ such that } \forall \varepsilon \leq \varepsilon_0, |a_\varepsilon| \leq c \exp\left(k\varepsilon^{-\frac{1}{\tau}}\right)\right\}$$

and

$$\mathcal{N}_0^{\tau} = \left\{(a_\varepsilon)_\varepsilon \in \mathbb{C}^{]0,1]}; \forall k > 0, \exists c > 0, \exists \varepsilon_0 \in]0,1], \text{ such that } \forall \varepsilon \leq \varepsilon_0, |a_\varepsilon| \leq c \exp\left(-k\varepsilon^{-\frac{1}{\tau}}\right)\right\}$$

The ring $\mathcal{C}^{\tau}$ motivates the following, easy to prove, result.

Proposition 3.1. *If $u \in \mathcal{G}^{\tau}(\Omega)$ and $x \in \Omega$, then the element $u(x)$ represented by $(u_\varepsilon(x))_\varepsilon$ is an element of $\mathcal{C}^{\tau}$ independent of the representative $(u_\varepsilon)_\varepsilon$ of u.*

A generalized Gevrey ultradistribution is not defined by its point values, in order to give a solution to this situation, set

$$\Omega_M^{\tau} = \left\{(x_\varepsilon)_\varepsilon \in \Omega^{]0,1]} : \exists k > 0, \exists c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0, |x_\varepsilon| \leq ce^{k\varepsilon^{-\frac{1}{\tau}}}\right\} \tag{3.1}$$

Define in Ω_M^τ the equivalence relation by

$$x_\varepsilon \sim y_\varepsilon \Longleftrightarrow \forall k > 0, \exists c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0, |x_\varepsilon - y_\varepsilon| \leq c e^{-k\varepsilon^{-\frac{1}{\tau}}} \tag{3.2}$$

Definition 3.2. The set $\widetilde{\Omega}^\tau = \Omega_M^\tau / \sim$ is called the set of exponential generalized points. The set of its compactly supported points is defined by

$$\widetilde{\Omega}_c^\tau = \left\{ \widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}^\tau : \exists K \text{ a compact set of } \Omega, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0, x_\varepsilon \in K \right\} \tag{3.3}$$

Proposition 3.3. *Let $f \in \mathcal{G}^\tau(\Omega)$ and $\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c^\tau$, then the generalized point value of f at $\widetilde{x}$, i.e. $f(\widetilde{x}) = [(f_\varepsilon(x_\varepsilon))_\varepsilon]$ is a well-defined element of the algebra of exponential generalized complex numbers $\mathcal{C}^\tau$.*

The characterization of nullity of $f \in \mathcal{G}^\tau(\Omega)$ is given by the following theorem.

Theorem 3.4. *Let $f \in \mathcal{G}^\tau(\Omega)$, then*

$$f = 0 \text{ in } \mathcal{G}^\tau(\Omega) \Longleftrightarrow f(\widetilde{x}) = 0 \text{ in } \mathcal{C}^\tau \text{ for all } \widetilde{x} \in \widetilde{\Omega}_c^\tau$$

4. Embedding of Gevrey ultradistributions

We recall some definitions and results on Gevrey ultradistributions. A function $f \in E^\sigma(\Omega)$, if $f \in C^\infty(\Omega)$ and for every compact K of Ω, $\exists c > 0, \forall \alpha \in \mathbb{Z}_+^m$,

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1} (\alpha!)^\sigma.$$

Obviously, we have $E^t(\Omega) \subset E^\sigma(\Omega)$ if $1 \leq t \leq \sigma$. It is well known that $E^1(\Omega) = A(\Omega)$ is the space of all real analytic functions in Ω. Denote by $D^\sigma(\Omega)$ the space $E^\sigma(\Omega) \cap C_0^\infty(\Omega)$, then $D^\sigma(\Omega)$ is nontrivial if and only if $\sigma > 1$. The topological dual of $D^\sigma(\Omega)$, denoted $D'_\sigma(\Omega)$, is called the space of Gevrey ultradistributions of order σ. The space $E'_\sigma(\Omega)$ is the topological dual of $E^\sigma(\Omega)$ and is identified with the space of Gevrey ultradistributions with compact support.

Definition 4.1. A differential operator of infinite order $P(D) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma$ is called a σ-ultradifferential operator, if $\forall\, h > 0, \exists c > 0$ such that $\forall \gamma \in \mathbb{Z}_+^m$,

$$|a_\gamma| \leq c \frac{h^{|\gamma|}}{(\gamma!)^\sigma}. \tag{4.1}$$

The importance of σ-ultradifferential operators lies in the following result.

Proposition 4.2. *Let $T \in E'_\sigma(\Omega)$, $\sigma > 1$ and $suppT \subset K$, then there exist a σ-ultradifferential operator $P(D) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma$, $M > 0$ and continuous functions $f_\gamma \in C_0(K)$ such that $\sup_{\gamma \in \mathbb{Z}_+^m, x \in K} |f_\gamma(x)| \leq M$ and $T = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma f_\gamma$.*

The space $\mathcal{S}^{(\rho)}(\mathbb{R}^m), \rho > 1$, see [7], is the space of functions $\varphi \in C^\infty(\mathbb{R}^m)$ such that $\forall b > 0$, we have

$$\|\varphi\|_{b,\rho} = \sup_{\alpha,\beta\in\mathbb{Z}_+^m} \int \frac{|x|^{|\beta|}}{b^{|\alpha+\beta|}\alpha!^\rho\beta!^\rho} |\partial^\alpha\varphi(x)|\, dx < \infty.$$

Lemma 4.3. *There exists $\phi \in \mathcal{S}^{(\rho)}(\mathbb{R}^m)$ satisfying*

$$\int \phi(x)\, dx = 1 \ \textit{and} \int x^\alpha \phi(x)\, dx = 0, \forall\alpha \in \mathbb{Z}_+^m \backslash \{0\}.$$

Definition 4.4. The net $\phi_\varepsilon = \varepsilon^{-m}\phi(./\varepsilon), \varepsilon \in]0,1]$, where ϕ satisfies the conditions of Lemma 4.3, is called a $\rho-$net of mollifiers.

The space $E^\sigma(\Omega)$ is embedded into $\mathcal{G}^\tau(\Omega)$ by the standard canonical injection

$$\begin{array}{llll} I: & E^\sigma(\Omega) & \to & \mathcal{G}^\tau(\Omega) \\ & f & \mapsto & [f] = cl(f_\varepsilon) \end{array},$$

where $f_\varepsilon = f$, $\forall\varepsilon \in]0,1]$.

The following proposition gives the natural embedding of Gevrey ultradistributions into $\mathcal{G}^\tau(\Omega)$.

Theorem 4.5. *The map*

$$\begin{array}{llll} J_0: & E'_{\tau+\rho}(\Omega) & \to & \mathcal{G}^\tau(\Omega) \\ & T & \mapsto & [T] = cl\left((T*\phi_\varepsilon)_{/\Omega}\right)_\varepsilon \end{array},$$

is an embedding.

Proof. Let $T \in E'_{\tau+\rho}(\Omega)$ with $suppT \subset K$, then there exists an $(\tau+\rho)$-ultradifferential operator $P(D) = \sum_{\gamma\in\mathbb{Z}_+^m} a_\gamma D^\gamma$ and continuous functions f_γ with $suppf_\gamma \subset K, \forall\gamma \in \mathbb{Z}_+^m$, and $\sup_{\gamma\in\mathbb{Z}_+^m, x\in K} |f_\gamma(x)| \leq M$, such that

$$T = \sum_{\gamma\in\mathbb{Z}_+^m} a_\gamma D^\gamma f_\gamma.$$

Let $\alpha \in \mathbb{Z}_+^m$, then

$$|\partial^\alpha(T*\phi_\varepsilon(x))| \leq \sum_{\gamma\in\mathbb{Z}_+^m} a_\gamma \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x+\varepsilon y)| \left|D^{\gamma+\alpha}\phi(y)\right| dy.$$

From (4.1) we have, $\forall h > 0, \exists c > 0$, such that

$$|\partial^\alpha(T*\phi_\varepsilon(x))| \leq \sum_{\gamma\in\mathbb{Z}_+^m} c\frac{h^{|\gamma|}}{\gamma!^{\tau+\rho}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x+\varepsilon y)| \left|D^{\gamma+\alpha}\phi(y)\right| dy.$$

And from the inequality

$$(\beta+\alpha)!^t \leq 2^{t|\beta+\alpha|}\alpha!^t\beta!^t,\ \forall t \geq 1,$$

$$\begin{aligned}\frac{1}{\alpha!^{\tau+\rho}}\left|\partial^\alpha\left(T*\phi_\varepsilon(x)\right)\right| &\leq \|\phi\|_{b,\rho} Mc\sum_{\gamma\in\mathbb{Z}_+^m} 2^{-|\gamma|}\frac{\left(2^{\tau+\rho+1}bh\right)^{|\gamma+\alpha|}}{(\gamma+\alpha)!^\tau}\frac{1}{\varepsilon^{|\gamma+\alpha|}} \\ &\leq c\exp\left(k_1\varepsilon^{-\frac{1}{\tau}}\right),\end{aligned}$$

where $k_1=\tau\left(2^{\tau+\rho+1}bh\right)^{\frac{1}{\tau}}$, and $h>\frac{1}{2}$, which means

$$\left|\partial^\alpha\left(T*\phi_\varepsilon(x)\right)\right| \leq c(\alpha)\exp\left(k_1\varepsilon^{-\frac{1}{\tau}}\right).$$

Suppose that $(T*\phi_\varepsilon)_\varepsilon \in \mathcal{N}^\tau(\Omega)$, then for every compact L of Ω, $\exists c>0$, $\forall k>0$, $\exists\varepsilon_0\in\left]0,1\right]$,

$$|T*\phi_\varepsilon(x)| \leq c\exp\left(-k\varepsilon^{-\frac{1}{\tau}}\right), \forall x\in L, \forall\varepsilon\leq\varepsilon_0.$$

Let $\chi\in D^{\tau+\rho}(\Omega)$ and $\chi=1$ in a neighborhood of K, then $\forall\psi\in E^{\tau+\rho}(\Omega)$,

$$\langle T,\psi\rangle=\langle T,\chi\psi\rangle=\lim_{\varepsilon\to 0}\int(T*\phi_\varepsilon)(x)\,\chi(x)\,\psi(x)\,dx=0,$$

i.e. $T=0$ □

In order to show the commutativity of the following diagram of embeddings

$$\begin{array}{ccc} D^\sigma(\Omega) & \rightarrow & \mathcal{G}^{\sigma+\rho-1}(\Omega) \\ & \searrow & \uparrow \\ & & E'_{\sigma+2\rho-1}(\Omega) \end{array},$$

we have to prove the following fundamental result.

Proposition 4.6. *Let $f\in D^\sigma(\Omega)$, then*

$$\left(f-(f*\phi_\varepsilon)_{/\Omega}\right)_\varepsilon \in \mathcal{N}^{\sigma+\rho-1}(\Omega).$$

Proof. Let $f\in D^\sigma(\Omega)$ and $\tau=\sigma+\rho-1$, then there exists a constant $c>0$, such that

$$|\partial^\alpha f(x)| \leq c^{|\alpha|+1}\alpha!^\sigma, \forall\alpha\in\mathbb{Z}_+^m, \forall x\in\Omega.$$

Let $\alpha\in\mathbb{Z}_+^m$, the Taylor's formula and the properties of ϕ_ε give

$$\partial^\alpha(f*\phi_\varepsilon-f)(x)=\sum_{|\beta|=N}\int\frac{(\varepsilon y)^\beta}{\beta!}\partial^{\alpha+\beta}f(\xi)\,\phi(y)\,dy,$$

where $x \leq \xi \leq x + \varepsilon y$. Consequently, for $b > 0$, we have

$$\left|\partial^{\alpha}\left(f * \phi_{\varepsilon} - f\right)(x)\right| \leq \varepsilon^{N} \sum_{|\beta|=N} \int \frac{|y|^{N}}{\beta!} \left|\partial^{\alpha+\beta} f(\xi)\right| |\phi(y)|\, dy$$

$$\leq \alpha!^{\sigma} \varepsilon^{N} \sum_{|\beta|=N} \beta!^{\tau} 2^{\sigma|\alpha+\beta|} b^{|\beta|} \int \frac{\left|\partial^{\alpha+\beta} f(\xi)\right|}{(\alpha+\beta)!^{\sigma}} \frac{|y|^{|\beta|}}{b^{|\beta|}\beta!^{\rho}} |\phi(y)|\, dy.$$

Let $k > 0$ and $T > 0$, then

$$\left|\partial^{\alpha}\left(f * \phi_{\varepsilon} - f\right)(x)\right| \leq \alpha!^{\sigma} \left(\varepsilon N^{\tau}\right)^{N} \left(k^{\tau} T\right)^{-N} \times$$

$$\times \sum_{|\beta|=N} \int 2^{\sigma|\alpha+\beta|} \left(k^{\tau} b T\right)^{|\beta|} \frac{\left|\partial^{\alpha+\beta} f(\xi)\right|}{(\alpha+\beta)!^{\sigma}} \frac{|y|^{|\beta|}}{b^{|\beta|}\beta!^{\rho}} |\phi(y)|\, dy$$

$$\leq c\alpha!^{\sigma} \left(\varepsilon N^{\tau}\right)^{N} \left(k^{\tau} T\right)^{-N} \|\phi\|_{b,\rho} \left(2^{\sigma} c\right)^{|\alpha|} \sum_{|\beta|=N} \left(2^{\sigma} k^{\tau} b T\right)^{|\beta|} c^{|\beta|},$$

hence, taking $2^{\sigma} k^{\tau} b T c \leq \frac{1}{2a}$, with $a > 1$, we obtain

$$\left|\partial^{\alpha}\left(f * \phi_{\varepsilon} - f\right)(x)\right| \leq c\alpha!^{\sigma} \left(\varepsilon N^{\tau}\right)^{N} \left(k^{\tau} T\right)^{-N} \|\phi\|_{b,\rho} \left(2^{\sigma} c\right)^{|\alpha|} a^{-N} \sum_{|\beta|=N} \left(\frac{1}{2}\right)^{|\beta|}$$

$$\leq \|\phi\|_{b,\rho} c^{|\alpha|+1} \alpha!^{\sigma} \left(\varepsilon N^{\tau}\right)^{N} \left(k^{\tau} T\right)^{-N} a^{-N}. \tag{4.2}$$

Let $\varepsilon_0 \in \left]0, 1\right]$ such that $\varepsilon_0^{\frac{1}{\tau}} \frac{\ln a}{k} < 1$ and take $T > 2^{\tau}$, then

$$\left(T^{\frac{1}{\tau}} - 1\right) > 1 > \frac{\ln a}{k} \varepsilon^{\frac{1}{\tau}}, \forall \varepsilon \leq \varepsilon_0,$$

in particular, we have

$$\left(\frac{\ln a}{k} \varepsilon^{\frac{1}{\tau}}\right)^{-1} T^{\frac{1}{\tau}} - \left(\frac{\ln a}{k} \varepsilon^{\frac{1}{\tau}}\right)^{-1} > 1.$$

Then, there exists $N = N(\varepsilon) \in \mathbb{Z}^{+}$, such that

$$\left(\frac{\ln a}{k} \varepsilon^{\frac{1}{\tau}}\right)^{-1} < N < \left(\frac{\ln a}{k} \varepsilon^{\frac{1}{\tau}}\right)^{-1} T^{\frac{1}{\tau}},$$

which gives

$$a^{-N} \leq \exp\left(-k\varepsilon^{-\frac{1}{\tau}}\right) \qquad \text{and} \qquad \frac{\varepsilon N^{\tau}}{k^{\tau} T} \leq \left(\frac{1}{\ln a}\right)^{\tau} < 1,$$

if we choose $\ln a > 1$. Finally, from (4.2), we have

$$\left|\partial^{\alpha}\left(f * \phi_{\varepsilon} - f\right)(x)\right| \leq c \exp\left(-k\varepsilon^{-\frac{1}{\tau}}\right),$$

i.e. $f * \phi_{\varepsilon} - f \in \mathcal{N}^{\sigma+\rho-1}(\Omega)$ □

As in [6] and [1], we embed $D'_{\tau+\rho}(\Omega)$ into $\mathcal{G}^{\tau}(\Omega)$ using the sheaf properties, then we have the following commutative diagram

$$\begin{array}{ccc} E^{\sigma}(\Omega) & \rightarrow & \mathcal{G}^{\sigma+\rho-1}(\Omega) \\ \downarrow & \nearrow & \\ D'_{\sigma+2\rho-1}(\Omega) & & \end{array}$$

5. Regular generalized ultradistributions

To define the algebra of regular generalized ultradistributions one needs first to define these regular moderate elements and these null elements.

Definition 5.1. The space of $\sigma-$regular elements, denoted $\mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^{]0,1]}$ satisfying, for every compact K of Ω, $\exists k > 0$, $\exists c > 0, \exists \varepsilon_0 \in]0,1], \forall \alpha \in \mathbb{Z}_+^m, \forall \varepsilon \leq \varepsilon_0$,

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq c^{|\alpha|+1}\alpha!^\sigma \exp\left(k\varepsilon^{-\frac{1}{\tau}}\right).$$

Proposition 5.2. *1) The space $\mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$ is an algebra stable under the action of σ-ultradifferential operators.*

2) The space $\mathcal{N}^{\tau,\sigma,\infty}(\Omega) := \mathcal{N}^{\tau}(\Omega) \cap \mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$ is an ideal of $\mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$.

Proof. See [1]. □

Now, we define the Gevrey regular elements of $\mathcal{G}^{\tau}(\Omega)$.

Definition 5.3. The order of τ, denoted $\mathcal{G}_\sigma^{\tau,\infty}(\Omega)$, is the quotient algebra

$$\mathcal{G}_\sigma^{\tau,\infty}(\Omega) = \frac{\mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)}{\mathcal{N}^{\tau,\sigma,\infty}(\Omega)}.$$

It is clear that $E^{\sigma}(\Omega) \hookrightarrow \mathcal{G}_\sigma^{\tau,\infty}(\Omega)$ and $\mathcal{G}_\sigma^{\tau,\infty}$ is subsheaf of $\mathcal{G}^{\tau}$. This motivates the following definition.

Definition 5.4. We define the $\mathcal{G}_\sigma^{\tau,\infty}$-singular support of a generalized ultradistribution $f \in \mathcal{G}^{\tau}(\Omega)$, denoted σ- $singsupp_g(f)$, as the complement of the largest open set Ω' such that $f \in \mathcal{G}_\sigma^{\tau,\infty}(\Omega')$.

The following result is a Paley-Wiener type characterization of $\mathcal{G}_\sigma^{\tau,\infty}(\Omega)$.

Proposition 5.5. *Let $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_C^{\tau}(\Omega)$ the set of generalized ultradistribution with compact support, then f is σ-regular if and only if $\exists k_1 > 0, \exists k_2 > 0, \exists c > 0, \exists \varepsilon_1 > 0, \forall \varepsilon \leq \varepsilon_1$, such that*

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c \exp\left(k_1\varepsilon^{-\frac{1}{\tau}} - k_2|\xi|^{\frac{1}{\sigma}}\right), \forall \xi \in \mathbb{R}^m,$$

where $\mathcal{F}(f_\varepsilon)$ denotes Fourier transform of f_ε.

The algebra $\mathcal{G}_\sigma^{\tau,\infty}(\Omega)$ plays the same role as the Oberguggenberger subalgebra of regular elements $\mathcal{G}^\infty(\Omega)$ in the Colombeau algebra $\mathcal{G}(\Omega)$.

Theorem 5.6. *We have*

$$\mathcal{G}_{\sigma}^{\sigma+\rho-1,\infty}(\Omega)\cap D'_{\sigma+2\rho-1}(\Omega)=E^{\sigma}(\Omega).$$

Proof. Let $S\in\mathcal{G}_{\sigma}^{\sigma+\rho-1,\infty}(\Omega)\cap D'_{\sigma+2\rho-1}(\Omega)$, for any fixed $x_0\in\Omega$ we take $\psi\in D^{\sigma+2\rho-1}(\Omega)$ with $\psi\equiv 1$ on the neighborhood U of x_0, then $T=\psi S\in E'_{\sigma+2\rho-1}(\Omega)$. Let ϕ_ε be a net of mollifiers with $\check{\phi}=\phi$ and let $\chi\in D^{\sigma}(\Omega)$ such that $\chi\equiv 1$ on $K=supp\psi$. As $[T]\in\mathcal{G}_{\sigma}^{\sigma+\rho-1,\infty}(\Omega)$, $\exists k_1>0,\exists k_2>0,\exists c_1>0,\exists\varepsilon_1>0,\forall\varepsilon\leq\varepsilon_1$,

$$\left|\mathcal{F}\left(\chi\left(T*\phi_\varepsilon\right)\right)(\xi)\right|\leq c_1\exp\left(k_1\varepsilon^{-\frac{1}{\sigma+\rho-1}}-k_2\left|\xi\right|^{\frac{1}{\sigma}}\right),$$

then

$$\begin{aligned}\left|\mathcal{F}\left(\chi\left(T*\phi_\varepsilon\right)\right)(\xi)-\mathcal{F}(T)(\xi)\right|&=\left|\mathcal{F}\left(\chi\left(T*\phi_\varepsilon\right)\right)(\xi)-\mathcal{F}(\chi T)(\xi)\right|\\&=\left|\left\langle T(x),\left(\chi(x)e^{-i\xi x}\right)*\phi_\varepsilon(x)-\left(\chi(x)e^{-i\xi x}\right)\right\rangle\right|.\end{aligned}$$

As $E'_{\sigma+2\rho-1}(\Omega)\subset E'_{\sigma}(\Omega)$, then $\exists L$ a compact subset of Ω such that $\forall h>0,\exists c>0$, and

$$\begin{aligned}&\left|\mathcal{F}\left(\chi\left(T*\phi_\varepsilon\right)\right)(\xi)-\mathcal{F}(T)(\xi)\right|\leq\\&\qquad\leq c\sup_{\alpha\in\mathbb{Z}_+^m,x\in L}\frac{h^{|\alpha|}}{\alpha!^{\sigma}}\left|\left(\partial_x^{\alpha}\left(\chi(x)e^{-i\xi x}*\phi_\varepsilon(x)-\chi(x)e^{-i\xi x}\right)\right)\right|.\end{aligned}$$

We have $e^{-i\xi}\chi\in D^{\sigma}(\Omega)$ and by (4.2), we obtain $\forall k_3>0,\exists c_2>0,\exists\eta>0,\forall\varepsilon\leq\eta$,

$$\sup_{\alpha\in\mathbb{Z}_+^m,x\in L}\frac{c_2^{|\alpha|}}{\alpha!^{\sigma}}\left|\partial_x^{\alpha}\left(\chi(x)e^{-i\xi x}*\phi_\varepsilon(x)-\chi(x)e^{-i\xi x}\right)\right|\leq c_2e^{-k_3\varepsilon^{-\frac{1}{\sigma+\rho-1}}},$$

so there exists $c'=c'(k_3)>0$, such that

$$\left|\mathcal{F}(T)(\xi)-\mathcal{F}\left(\chi\left(T*\phi_\varepsilon\right)\right)(\xi)\right|\leq c'e^{-k_3\varepsilon^{-\frac{1}{\sigma+\rho-1}}}.$$

Let $\varepsilon\leq\min(\eta,\varepsilon_1)$, then

$$\begin{aligned}\left|\mathcal{F}(T)(\xi)\right|&\leq\left|\mathcal{F}(T)(\xi)-\mathcal{F}\left(\chi\left(T*\phi_\varepsilon\right)\right)(\xi)\right|+\left|\mathcal{F}\left(\chi\left(T*\phi_\varepsilon\right)\right)(\xi)\right|\\&\leq c'e^{-k_3\varepsilon^{-\frac{1}{\sigma+\rho-1}}}+c_1e^{k_1\varepsilon^{-\frac{1}{\sigma+\rho-1}}-k_2|\xi|^{\frac{1}{\sigma}}}.\end{aligned}$$

Take $c=\max(c',c_1)$, $\varepsilon=\left(\dfrac{k_1}{(k_2-r)\left|\xi\right|^{\frac{1}{\sigma}}}\right)^{\sigma+\rho-1}$, $r\in\left]0,k_2\right[$ and $k_3=\dfrac{k_1r}{k_2-r}$, then $\exists\delta>0,\exists c>0$ such that

$$\left|\mathcal{F}(T)(\xi)\right|\leq ce^{-\delta|\xi|^{\frac{1}{\sigma}}},$$

which means $T=\psi S\in E^{\sigma}(\Omega)$. As $\psi\equiv 1$ on the neighborhood U of x_0, consequently $S\in E^{\sigma}(\Omega)$, which finishes the proof. □

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