

NOTES ON GENERALIZED PSEUDO-DIFFERENTIAL OPERATORS

Shantanu Dave¹

Abstract. This article provides a survey of abstract pseudo-differential operators based on [3]. In particular we shall comment on the relationship between the algebraic and analytic concepts of order and under what conditions they agree with each other. We shall also note some regularity features that can be provided in abstract generality and correspond to geometrical properties.

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1. Introduction

The algebra of classical pseudo-differential operators on a closed manifold M and its symbol calculus has been vastly useful in obtaining topological information about M , for example by means of index of some elliptic operator D on it [1]. This inspired some very abstract generalizations of both the concept of elliptic operators and the algebras of pseudo-differential operators which can be applied to the study of geometrical spaces other than smooth manifolds. In doing so, some deeper understanding of concepts, such as regularity, has been obtained. Most of these abstractions have been carried out in an operator-theoretic setup. Of particular interest, for our applications to certain singular spaces, are the abstract pseudo-differential operators of [3].

Our aim here is to furnish an introduction to this abstract pseudo-differential calculus without the prerequisite of knowledge in noncommutative geometry.

2. Notation

In this article all algebras are over the complex numbers \mathbb{C} . A filtration on an algebra \mathcal{D} is an increasing sequence of subspaces, $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \dots$ such that

$$\mathcal{D} = \bigcup_k \mathcal{D}_k \quad \text{and} \quad \mathcal{D}_k \cdot \mathcal{D}_l \subseteq \mathcal{D}_{k+l}.$$

We then say that \mathcal{D} is a filtered algebra. Also, if an element X is contained in a subspace \mathcal{D}_k we say that $order(X) \leq k$.

¹Department of Mathematics, University of Vienna Nordbergstr. 15, 1090 Wien, Austria, e-mail: shantanu.dave@univie.ac.at

For example, the algebra of differential operators $\mathcal{D}(M)$ on any manifold M can be given a filtration by setting $\mathcal{D}_k(M)$ to be the space of all operators of order less than or equal to k .

We note that a symbol map on a filtered algebra \mathcal{D} is a sequence of maps $\sigma_k : \mathcal{D}_k \rightarrow \mathcal{D}_k/\mathcal{D}_{k-1}$.

We say that an unbounded self-adjoint operator D on a Hilbert space \mathbb{H} is positive if its spectrum $sp(D) \subseteq [0, \infty)$. We shall use the convention to pick the sign of the Laplace operator Δ_M on a complete Riemannian manifold M so that it is a positive operator on $L^2(M)$. Thus for instance $\Delta_{\mathbb{R}^n} = -\sum \frac{\partial^2}{\partial x_j^2}$.

3. Abstract differential operators

To motivate the notion of abstract differential operators we look at a familiar case.

First suppose that M is a closed Riemannian manifold and Δ the scalar Laplace operator. Then the algebra $\mathcal{D}(M)$ of differential operator on M and its filtration with respect to the order of operators is completely determined by two pieces of data: a) the algebra $\mathcal{C}^\infty(M)$ of smooth functions on M considered as operators by multiplication on $L^2(M)$ and, b) the Laplace operator Δ , as follows.

Lemma 3.1. *The algebra $\mathcal{D}(M)$ of differential operators on M is the smallest algebra of operators on $L^2(M)$ that contains $\mathcal{C}^\infty(M)$ and is closed under commutation with Δ , that is*

$$\mathcal{D}(M) \ni T \rightarrow [\Delta, T] \in \mathcal{D}(M).$$

Furthermore, the filtration on $\mathcal{D}(M)$ is obtained by observing that the operators of order k , the space $\mathcal{D}_k(M)$, is generated by elements obtained by at most k iterated commutators of a smooth function and Δ .

Proof. The algebra of differential operators is generated by the vector fields and the smooth functions. The vector fields are linear combinations of gradient vector fields over $\mathcal{C}^\infty(M)$ and one directly checks that,

$$(1) \quad \text{grad } f = \frac{1}{2}(\Delta f - [\Delta, f]).$$

This leads to the proof of both the assertions in the lemma. \square

More generally, recall that the differential operators $\mathcal{D}(M : S)$ on sections of some Hermitian vector bundle S on M , can be generated by the space $\text{End}(S)$ of bundle endomorphisms and all covariant derivations ∇_X with respect to some connection ∇ on S . Here again, if H is a Laplace type operator. The algebra $\mathcal{D}(M : S)$ is completely determined by the order zero operators $\mathcal{D}_0(M : S) = \text{End}(M : S)$ and the operator H due to the formula for a Weitzenböck connection given by

$$\nabla_{f_0 \text{ grad } f_1} = \frac{1}{2} f_0 ([f_1, H] + \Delta f_1).$$

But a similar procedure works only for flat connections and then also recovers the grading only on a subalgebra of $\mathcal{D}(M : S)$ which is generated by covariant derivatives with respect to the above connection and multiplication by smooth functions.

In [3], it is observed that given such a description of the algebra of differential operators using an operator Δ one can obtain an algebra of classical pseudodifferential operators. Our purpose is to explain this construction and some of its applications.

We shall now define an abstract notion of differential operators based on [3, 7]. First, let Δ be a positive self-adjoint operator on a Hilbert space \mathbb{H} . Let \mathbb{H}^∞ be the intersection $\bigcap_n \text{Domain}(\Delta^n)$. We shall assign to Δ an order $r \geq 0$ and for each positive real m define the ‘Sobolev space’ H^m as the completion of $\text{Domain}(\Delta^{\frac{m}{r}})$ with respect to the graph norm

$$\|u\|_m^2 := \|u\|^2 + \|\Delta^{\frac{m}{r}} u\|^2.$$

Indeed in case of the concrete Laplace operator Δ this procedure defines the usual Sobolev norms due to the elliptic estimates, often called Gårding’s inequalities.

Back in our abstract setup one can define:

Definition 3.2. A filtered algebra \mathcal{D} of operators on \mathbb{H}^∞ with filtration

$$\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \dots \subseteq \mathcal{D},$$

is an algebra of differential operators for the operator Δ if the following hold.

1. If T belongs to \mathcal{D} then so does $[\Delta, T]$ and

$$\text{order}([\Delta, T]) \leq \text{order}(T) + r - 1.$$

2. If $\text{order}(T) = k$ then T extends to a bounded linear operator from H^{s+k} to H^s for all s . This condition can be rephrased as to demand that the algebraic order corresponding to the filtration must be equal to the analytic order corresponding to the mapping properties of the operators.

The condition 2 is a reformulation of the elliptic estimate of the Laplace operator in the abstract settings. In general, the algebra \mathcal{D} is not completely determined by commutators with the subalgebra of order zero operators \mathcal{D}_0 , but this would be the case of interest.

As is obvious from the definition, one can always set Δ to be a positive elliptic (scalar valued) differential operator on some manifold M . If M is closed then, due to the elliptic estimates, the notion of Sobolev space remains unchanged. In particular we recover the usual filtration with some possible rescaling. But it is a different matter when the operator Δ is only a hypoelliptic differential operator as seen in the following simple geometrical example.

Example 1. Consider the sphere $S^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. Define a vector field

$$V := \frac{d}{dt}(e^{it}z_1, e^{it}z_2) \Big|_{t=0}.$$

Let Δ_{S^3} be the standard Laplacian on the sphere S^3 . Let us set $\Delta_{Hopf} := \Delta_{S^3} + V^2 + V^4$. This new operator is clearly not elliptic. To see another form of it let us choose an orthonormal basis to the space $\{span(V(z))\}^\perp$ defining vector fields X and Y of unit norm, then

$$\Delta_{Hopf} = -X^2 - Y^2 + V^4.$$

Thus our new operator defines some weighted Sobolev spaces. Given u supported in a small coordinate chart (x_1, x_2, v) the Sobolev norm can be expressed by (an equivalent) norm:

$$\|u\|_m^2 = \int |\hat{u}(\xi)|^2 (1 + \xi_1^2 + \xi_2^2 + \xi_3^4)^{\frac{m}{4}} d\xi,$$

where ξ_1, ξ_2, ξ_3 are dual variables to coordinates x_1, x_2, v is the small neighbourhood.

Let us consider the algebra of differential operators on S^3 . Since the vector fields X, Y, V along with $\mathcal{C}^\infty(S^3)$ generate all differential operators, we can get a new filtration on $\mathcal{D}(S^3)$ as follows. We can assign to the vector fields X and Y the order 2 and to the vector field V its usual order 1. Also assign the multiplication by smooth functions an order of 0. We shall denote this new filtered algebra by $\mathcal{D}_H(S^3)$. The order of the operator Δ_{Hopf} in this new filtration is still 4 but it is now elliptic with respect to the symbol calculus obtained from this filtration.

Lemma 3.3. *The algebra $\mathcal{D}_H(M)$ is an algebra of differential operators for the operator Δ_H in the sense of Higson where Δ_H is assigned a degree $r = 4$.*

Proof. Since differential operators are local it suffices to do all the computations locally. Let us choose foliation coordinates (x_1, x_2, x_3) in a neighbourhood U around a point in S^3 such that $V = \partial_{x_3}$. Let g^{ij} be the metric coefficients on the cotangent space and Γ_{ij}^k be the Christoffel symbols in the local coordinates in U . Then clearly

$$\Delta_{Hopf} = - \sum_{ij=1}^3 g^{ij} \left(\partial_i \partial_j + \sum_k \Gamma_{ij}^k \partial_k \right) + \partial_{x_3}^2 + \partial_{x_3}^4.$$

First we shall prove that for any differential operator $D \in \mathcal{D}_H(M)$ the commutator $[\Delta_H, D]$ is of order less or equal order(D) + 3. To that end one notes that the symbol of D in foliation coordinates near any point $p \in M$ is a polynomial $\sigma_D(x, \xi)$, and as is well known the bracket of two differential operators is given by the polynomial

$$\sigma_{[P, Q]} = [\partial_x \sigma_P(x, \xi) \partial_\xi \sigma_Q(x, \xi) - \partial_\xi \sigma_P(x, \xi) \partial_x \sigma_Q(x, \xi)]$$

We bear in mind that the new notion of order of the operators gives rise to a new notion of degree of polynomials namely that the variables ξ_1, ξ_2 are to be considered as polynomials of degree 2, whereas ξ_3 is to be treated as a polynomial of degree one². Hence the estimate on order of commutators follows from direct computation of Poisson brackets and checking the degree of the resulting polynomial.

To obtain the elliptic estimate that says that the analytic order is the same as the algebraic order for every differential operator it is enough to check the same only for the generators of the algebra $\mathcal{D}_H(M)$ which is straightforward as it can be checked that $H^2_{Hopf}(S^3) \subseteq H^1(S^3)$ and vector fields map $H^1(M)$ continuously to $L^2(S^3)$ \square

This construction works in general for any foliated manifold which were the source of inspiration in [3].

Thus we see that the notions of filtration on the algebra of differential operators naturally correspond to notions of regularity in the Sobolev sense.

In addition, the idea of a symbol map is akin to the filtration used. A (universal) symbol map on a filtered algebra $\mathcal{A} = \bigcup \mathcal{A}_i$ $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots$, is just a sequence of maps $\sigma_i : \mathcal{A}_i \rightarrow \mathcal{A}_i/\mathcal{A}_{i-1}$. Thus by choosing a filtration we prescribe a choice of symbol map and even the meaning of ellipticity which indicates invertibility modulo lower order operators.

4. Pseudo-differential operators

As mentioned already we shall follow the Connes-Moscovici construction of pseudo-differential operators.

So given an algebra of differential operators \mathcal{D} for a positive self-adjoint operator Δ in the sense of Definition 3.2 we shall first extend the notion of order of an operator to a bigger class of operators in an obvious way. We say that an operator T on \mathbb{H}^∞ is of order l if it extends to a bounded operator $H^{s+l} \rightarrow H^s$ for every $s \geq 0$.

Definition 4.1. We shall follow [7] in our formulation and call an operator T on \mathbb{H}^∞ a basic pseudo-differential operator of order k if for any $l \in \mathbb{Z}$ one can represent it in the form

$$T = X\Delta^{\frac{m}{r}} + R,$$

where $X \in \mathcal{D}$ and $order(X) \leq k - m$ and $order(R) \leq l$.

More generally, a pseudo-differential operator is a finite linear combination of basic pseudo-differential operators.

At the first glance it is not at all clear that the space of pseudo-differential operators so defined is even an algebra. Hence we introduce an asymptotic

²For subsequent use we say that a function $f \in C^\infty(\mathbb{R}^3/0)$ is homogeneous of degree d if

$$(2) \quad f(\lambda^2\xi_1, \lambda^2\xi_2, \lambda\xi_3) = \lambda^d f(\xi_1, \xi_2, \xi_3).$$

expansion for these abstract pseudo-differential algebras which is of fundamental importance.

Definition 4.2. We say that an operator T has an asymptotic expansion $T \approx \sum_j T_j$ if given any $l \in \mathbb{Z}$ there exists an $N = N(l)$ such that for all $n \geq N$ the order relation

$$\text{order}(T - \sum_{i \leq n} T_i) \leq l,$$

is satisfied.

The basic non-commutative asymptotic expansion says that

Proposition 4.3. Let T be a pseudo-differential operator associated to \mathcal{D} and Δ as defined above. Let $\Delta_1 = \Delta + K$ be an invertible operator such that $K : \mathbb{H} \rightarrow \mathbb{H}^\infty$ is bounded³. Then

$$[\Delta_1^{-z}, T] \approx \sum_j \binom{-z}{j} \nabla_1^j(T) \Delta_1^{-z-j}.$$

Here $\nabla_1(T) := [\Delta_1, T]$ and ∇_1^j is its j -th iterate.

This proposition, proved easily by means of holomorphic functional calculus, provides a powerful tool in understanding the abstract calculus of pseudo-differential operators. This also shows in a standard way that the pseudo-differential operators form an algebra.

Let us go back to our basic example when M is a closed manifold with Δ the Laplace operator and $\mathcal{D}(M)$ the honest to goodness algebra of differential operators on M . Recall now that Kohn-Nirenberg [8] characterize a pseudo-differential operators of order k as an operator $T : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ such that for any collection of vector fields X_1, X_2, \dots, X_k , the operator

$$[X_1, [X_2, [\dots [X_k, T] \dots]]$$

extends to a bounded operator $H^{s+k}(M) \rightarrow H^s(M)$. This immediately tells us that our abstract construction does in fact give honest pseudo-differential operators in the old framework of closed manifold M , which is good news. The asymptotic expansion tells us that in fact we are talking about classical pseudo-differential operators.

Example 2. We shall return to the example of the sphere S^3 with the operator $\Delta_H = \Delta_{H_{opf}}$ and filtered algebra $\mathcal{D}_H(S^3)$ as described in Example 2. Let us denote by $\Psi_H^\infty(S^3)$ the corresponding abstract algebra of pseudo-differential operators on $H = L^2(S^3)$. We shall provide a local symbol class to express the operators in this algebra.

Definition 4.4. Let U be a foliation coordinate. A symbol $a(x, \xi) \in S_{0,1}(T^*U)$ will be referred to as a foliation symbol of order m if the following holds.

³Such an operator Δ_1 always exists by spectral theory.

1. There exists an integer m such that for every multi-index α and β there exists a constant C such that:

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha, \beta} \left[(1 + \xi_1^2 + \xi_2^2 + \xi_3^4)^{\frac{1}{4}} \right]^{m - \|\alpha\|}.$$

We shall refer to such symbols to be of the class $S^H m$.

2. There is an asymptotic expansion for the symbol $a(x, \xi)$ given by

$$a(x, \xi) \sim a_m(x, \xi) + a_{m-1}(x, \xi) + a_{m-2}(x, \xi) + \dots,$$

where each $a_j(x, \xi)$ is homogeneous of degree j in the sense of (2) and, as usual, the asymptotic expansion implies that for any given N there is a large l_0 such that

$$a(x, \xi) - \sum_{j=0}^l a_{m-j}(x, \xi) \in S^{H N} \quad \forall l \geq l_0.$$

The following results describe the complete symbol for the algebra $\Psi_H^\infty(M)$.

Proposition 4.5. *An operator $T \in \Psi_H^m(M)$ is a pseudodifferential operator in the usual sense and its symbol is a foliation symbol of order m in the sense of Definition 4.4.*

Sketch of the proof. The statement follows directly for operators of the form $X \Delta_{Hopf}^{\frac{m}{4}}$ with X being a differential operator. To see its validity in more general case we fix an operator Δ_1 as prescribed in Proposition 4.3. For a fixed basic operator T in $\Psi_H^\infty(S^3)$ and fixed integer $l \in \mathbb{Z}$ we note that the operator R_l provided by Definition 4.1 has a distributional kernel in $D'(S^3 \times S^3)$. Using the regularity property of Δ_1^{-1} one notices that the remainder term R_l is more and more regular as $l' \rightarrow \infty$ in an appropriate Besov space. The proof thus follows by an expansion in l . \square

5. Generators for pseudo-differential operators

So far we have constructed our abstract algebra of pseudo-differential operators assuming that we are given two pieces of information, namely a filtered algebra \mathcal{D} of abstract differential operators associated with a positive self-adjoint operator Δ . In practice we saw that the subalgebra \mathcal{D}_0 of order zero differential operators is also a natural geometric entity like $\mathcal{C}^\infty(M)$ or $End(S)$ which along-side with Δ generates our differential operators $\mathcal{D}(M)$.

In general one can apply the following procedure. Fix a positive self-adjoint operator Δ and the space of order zero differential operators \mathcal{D}_0 . Then iteratively define the order k differential operators by

$$(3) \quad \mathcal{D}_k := \sum \mathcal{D}_i \cdot \mathcal{D}_{k-i} + [\Delta, \mathcal{D}_{k-1}] + \mathcal{D}_0[\Delta, \mathcal{D}_{k-1}] + \mathcal{D}_{k-1}.$$

In this situation we shall call the algebra $\mathcal{D} = \bigcup_k \mathcal{D}_k$ the filtered algebra generated by \mathcal{D}_0 and Δ .

This is all very good except for that there is no reason to believe that such an algebraically generated space of differential operators would have the crucial analytic property that each differential operator of order k maps $H^{s+k} \rightarrow H^s$ continuously. To establish such a relationship between analysis and algebraic-generation, we shall use the language of spectral triples. The setup of spectral triples envisioned by Connes provides some deep insights. A principal example is provided by a Dirac operator on a spin manifold (see [2] for more examples).

Definition 5.1. Suppose that we are given an algebra \mathcal{A} of bounded operators on \mathbb{H} , and an operator D on \mathbb{H} satisfying the following two conditions:

1. The domain of D is invariant under \mathcal{A} , that is, $a \cdot \text{Domain}(\Delta) \subseteq \text{Domain}(\Delta)$ and the commutators $[a, D]$ extend to a bounded operator on \mathbb{H} for every $a \in \mathcal{A}$.
2. For every $a \in \mathcal{A}$ the operator $a \cdot (1 + D^2)^{-1}$ is a compact operator.

Then we call the structure $(\mathcal{A}, \mathbb{H}, D)$ a spectral triple.

What interests us here is that a spectral triple provides us with a candidate algebra of differential operators.

Now we can hopefully choose the algebra $\mathcal{D}_0(\mathcal{A}) := \mathcal{A} + [D, \mathcal{A}]$ to be the algebra of order 0 differential operators and hope that by picking $\Delta = D^2$ we can generate an appropriate algebra of differential operators $\mathcal{D}(\mathcal{A})$ using the procedure described by (3).

So it turns out that the analytic condition corresponding to the elliptic estimates for Δ can be reformulated as saying that all order zero differential operators should also be order zero pseudo-differential operators!

Remember that from the Kohn-Nirenberg formulation the order zero pseudo-differential operators on M should satisfy the condition that iterated commutators with vector fields should continue to be bounded operators on $L^2(M)$. Here such a condition is formulated as saying that the space of order zero pseudo-differential operators are all those bounded operators which extend to a bounded operator after finitely many commutators with $|D|$ (the positive part of $|D|$). Thus we say that $T \in B(\mathbb{H})$ is in $Op^0(\mathcal{A})$ if any finite number of commutators,

$$[|D|, [|D|, [\dots [|D|, T], \dots]]],$$

extends to a bounded operator on \mathbb{H} . Then

Definition 5.2. A spectral triple is called regular if $\mathcal{D}_0(\mathcal{A}) \subset Op^0(\mathcal{A})$.

What this entails is the following:

Proposition 5.3. *A spectral triple defined by $\mathcal{A}, \mathbb{H}, D$ is regular if and only if the filtered algebra generated by $\mathcal{D}_0 = \mathcal{A} + [D, \mathcal{A}]$ with respect to Δ is an algebra of differential operators.*

This is essentially proved in Appendix B of [3] and is clearly mentioned in [7].

6. Regularity

So far we have seen that, given a positive self-adjoint operator Δ on a Hilbert space \mathbb{H} , we can construct an algebra of differential operators generated by a subalgebra \mathcal{D}_0 , provided it corresponds to a regular spectral triple $(\mathcal{A}, \mathbb{H}, D)$. We have also noted that, with any algebra of differential operator associated with Δ there is an algebra of classical pseudo-differential operators. Let us consider the classical case of the algebra \mathcal{A} equal $C^\infty(M)$, where M is a spin manifold. Here one can make another important observation, namely that the Fréchet topology on $C^\infty(M)$ can be obtained from the semi-norms $a \rightarrow \|\delta^n(a)\|$, where, as before, $\delta(a) = [[D], a]$. This provides the differential structure of M in the context of spectral triples. In fact Connes [2] shows that many other geometrical properties such as distance function on M can easily be recovered from the spectral triple data.

The question therefore is that, in cases where we do not have a manifold but something more singular albeit with interesting geometrical properties like a length space (see for example [5]), could we still apply a similar construction and obtain a calculus of pseudo-differential operators that encodes the regularity in the structure concerned? This question is out of the scope of our present article and we shall focus on further understanding our abstract pseudo-differential operators for now.

Our first desire is to check if in this abstract settings the spectral zeta functions are able to produce interesting global invariants. On a compact manifold, it is a non-trivial theorem due to Seeley [9] (and a different proof can be found in Guillemin [6]) that the complex powers Δ^z of an elliptic positive operator are in fact pseudo-differential operators of order $\Re(z)$ and hence when $\Re(z)$ is very large then

$$\zeta_A(z) := \text{Tr}(A\Delta^{-z})$$

is an analytic function for any (classical) pseudo-differential operator A . Furthermore, it has a meromorphic extension to the whole of the complex plane with only simple poles. The assignment $\tau(A) := \text{res}_{z=0} \zeta_A(z)$ provides a trace on the algebra of classical pseudo-differential operators and up to a constant multiple it is the unique trace on this algebra, referred to as the noncommutative residue. The importance of noncommutative residue lies in the fact that it is locally computable from the asymptotic expansion of A (even though the asymptotic expansion itself is not invariantly well-defined) and it provides global invariance as well.

The main aim of [3] is to generalize this to a very abstract setting and provide under suitable conditions a formula of a cyclic Chern-character of the class in K-homology given by $\mathcal{A}, \mathbb{H}, D/|D|$ in terms of residues of certain zeta functions.

The fact that on an n dimensional manifold M all pseudo-differential operators of order $< -n$ are of trace class, whereas the operators of order $\leq -n$ in general are in the Dixmier ideal of $B(L^2(M))$, bears on the position of poles of the zeta functions $\zeta_A(z)$. Guillemin found a new proof of the Weyl's asymptotic formula, namely that the spectral counting function $N_D(\lambda)$, defined below, for

a positive elliptic differential operator D is asymptotically given by

$$N_D(\lambda) := \#\{\lambda_i \in \text{sp}(D) \mid \lambda_i \leq \lambda\} \simeq O(\lambda^{\frac{n}{\text{order}(D)}}),$$

by showing that the zeta function $\zeta_{Id}(z) = \text{Tr}(D^{-z})$ has its highest pole at $z = \frac{n}{\text{order}(D)}$. One can state the following result for our example operator Δ_{Hopf} .

Lemma 6.1. *The zeta function $\zeta_{\Delta_{Hopf}}(z) := \text{Tr}(\Delta_{Hopf}^{-z})$ is analytic on the half-plane $\text{Re}(z) > \frac{5}{4}$ and has a meromorphic continuation with simple poles.*

Of course, the Weyl asymptotics has practical applications including estimation of rates of convergences of numerical approximations. It also provides a very clear understanding of regularity in the usual Sobolev sense. Moreover it establishes a surprising relationship between the poles of the analytically defined spectral zeta function and the geometric quantities such as dimension and volume. Such relationships have been incorporated in the abstract setup of pseudo-differential operators. In practice, it remains a difficult proposition to prove a result like the theorem of Seeley [9, 6] that is behind the analyticity of the zeta function on some half-plane of the complex domain, and even harder to establish a meromorphic continuation.

Thus we realize that for a particular application of the abstract pseudo-differential calculus to a singular space one would have to furnish statements that confirm that the axioms of regularity are satisfied. But in cases where the axioms can be ascertained, these abstract pseudo-differential operators provide very useful geometrical information.

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