

SYNTHETIC GEOMETRY AND GENERALISED FUNCTIONS¹

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Abstract. We review some aspects of the geometry of length spaces and metric spaces, in particular Alexandrov spaces with curvature bounded below and/or above. We then point out some possible directions of research to explore connections between the synthetic approach to Riemannian geometry and some aspects of the approach to non-smooth differential geometry through generalised functions.

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1. Introduction

In classical Riemannian geometry, one often studies Riemannian metrics that satisfy a curvature bound of some type. Standard examples would be where one has a lower and/or upper bound on the sectional curvature of a metric or a lower bound on the Ricci tensor. Theorems that arise in such contexts include, for example, Myers's theorem, the Cartan–Hadamard theorem, the Bishop–Gromov relative volume comparison theorem and the Toponogov comparison theorem (see, e.g., [5]). The proofs of these results generally rely on the use of the exponential map, and require that the metric has at least C^2 regularity. One can often argue, however, that these results hold for metrics that are $C^{1,1}$, where one still has existence and uniqueness of solutions of the geodesic equations. If one attempts to lower the regularity of the metric further, for example to a metric that is $C^{1,\alpha}$ for some $\alpha < 1$, then one encounters metrics for which the geodesic equations do not have unique solutions. Since this implies that the exponential map is no longer a homeomorphism onto a neighbourhood of a point, this poses a genuine obstacle to generalising the standard proofs to low-regularity metrics.

Approaches to semi-Riemannian manifolds of low-regularity have been developed within the context of algebras of generalised functions (see, e.g., [13, 14] and [20] for a review). Here a low-regularity semi-Riemannian metric is embedded into the space of generalised metrics (essentially an equivalence class of nets of smooth metrics) with the properties of the low-regularity metric being encoded in the asymptotics of the net. Such methods have been employed to give

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detailed descriptions of the curvature of, for example, conical metrics and, more recently, metrics the curvature of which is well-defined as a distribution [21].

There is, alternatively, a well-developed approach to low-regularity Riemannian manifolds (and more general structures) by *synthetic geometry*. In particular, in the theory of Alexandrov spaces, one works with length spaces and metric spaces. These spaces need not be manifolds, and as such, concepts such as a Riemannian metric are not classically well-defined. On the other hand, one has well-defined notions of (minimising) geodesics and of *curvature bounds*.

The motivating idea here is that it may be fruitful to investigate elements of the theory of synthetic geometry (in particular, Alexandrov spaces, and recent constructions of Sturm [22, 23] and Lott–Villani [16] defining Ricci-curvature bounds for metric-measure spaces) within the framework of generalised functions. Conversely, given that synthetic geometry is a mature field, investigation of the results in this field may lead to some clarification of, and new directions in, the generalised functions approach to low-regularity geometry.

The main aim of this article is to introduce the basic concepts of synthetic geometry to the generalised functions community. Therefore, a large part is devoted to an introduction to and review of known material on the geometry of metric spaces and, in particular, Alexandrov spaces. Note that none of this work is original to the author, and I apologise, in advance, to anyone who believes that I have overlooked their work. We then point out some initial directions of research, some of which are currently being investigated, that may connect this approach with the theory of generalised functions.

2. The geometry of length spaces and metric spaces

The material in this section is standard. The monographs [2, 3] and the papers [4, 1] are recommended for a more detailed introduction to the theory of Alexandrov spaces. See also [10] for articles on related topics and [9] for information on the geometry of metric spaces.

Let X be a topological space. A *path* in X will be a continuous map $\gamma: I \rightarrow X$, where I is a connected subset of \mathbb{R} . The set of paths will be denoted $\mathcal{P}(X)$.

Definition 2.1. Let X be a topological space. A *length structure* on X is a pair (\mathcal{A}, L) where $\mathcal{A} \subseteq \mathcal{P}(X)$ and $L: \mathcal{A} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$. \mathcal{A} is the class of *admissible paths*, and is required to satisfy the following conditions:

A1). If $\gamma: [a, b] \rightarrow X$ is an admissible path and $[c, d] \subseteq [a, b]$, then the restriction $\gamma|_{[c, d]}$ is also admissible.

A2). If $\gamma_1: [a, c] \rightarrow X$ and $\gamma_2: [c, b] \rightarrow X$ are admissible paths such that $\gamma_1(c) = \gamma_2(c)$, then the concatenation $\gamma_1 \cdot \gamma_2: [a, b] \rightarrow X$ is an admissible path.

A3). \mathcal{A} is closed under linear reparametrisations: given an admissible path $\gamma: [a, b] \rightarrow X$ and a linear homeomorphism $\varphi: [c, d] \rightarrow [a, b]$; $t \mapsto \alpha t + \beta$, the composition $\gamma \circ \varphi(t) = \gamma(\varphi(t))$ is also an admissible path.

Given $\gamma \in \mathcal{A}$, $L(\gamma)$ is called the *length* of γ . The map L is required to have the following properties:

L1). $L(\gamma|_{[a,b]}) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]})$ for any $c \in [a, b]$.

L2). Given a path $\gamma: [a, b] \rightarrow X$ of finite length, the map $[a, b] \rightarrow \mathbb{R}; t \mapsto L(\gamma|_{[a,t]})$ is continuous.

L3). $L(\gamma \circ \varphi) = L(\gamma)$ for any linear homeomorphism φ as in Condition A3).

L4). Given $x \in X$ and any neighbourhood U of x , then the length of paths connecting x to the complement of U should be strictly positive:

$$\inf \left\{ L(\gamma) \mid \gamma \in \mathcal{A}, \gamma(a) = x, \gamma(b) \in X \setminus U \right\} > 0.$$

Definition 2.2. Let X be a topological space with length structure (\mathcal{A}, L) . Let $d_L: X \times X \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ be the metric on X defined by

$$d_L(x, y) := \inf \left\{ L[c] \mid c: [a, b] \rightarrow X \text{ an admissible curve with } c(a) = x, c(b) = y \right\}$$

for $x, y \in X$. A metric that can be obtained in this way from a length structure is called an *intrinsic metric*. A metric space (X, d) , where the metric d is intrinsic, is called a *length space*. Given points x, y in a metric space (X, d) , we will often denote the metric distance between them, $d(x, y)$, by $|xy|$.

Given $x, y \in X$, an admissible curve $c: [a, b] \rightarrow X$ is a *minimising geodesic* (or *shortest path* [3]) from x to y if $c(a) = x$, $c(b) = y$ and $L[c] = |xy|$. A length space is *complete* if, given any $x, y \in X$, there exists a minimising geodesic from x to y . Such a minimising geodesic will be denoted by $[xy]$. More generally, a *geodesic* is defined as a locally-minimising admissible curve.

Remark 2.3. If (X, d) is a locally compact length space and complete (as a metric space), then for any $x, y \in X$ such that $|xy| < \infty$, there exists a minimising geodesic between x and y [3, Theorem 2.5.23]. Moreover, the Hopf–Rinow theorem is valid on locally compact length spaces [3, Theorem 2.5.28].

Example 2.4. The main example of a length space of interest to us is when X is a C^1 manifold with a continuous Riemannian metric \mathbf{g} . We take \mathcal{A} to consist of all piece-wise C^1 paths on X and, given an admissible path $\gamma: [a, b] \rightarrow X$, we define the Riemannian length by

$$L_{\mathbf{g}}[\gamma] := \int_a^b \sqrt{\mathbf{g}(\dot{\gamma}, \dot{\gamma})} ds.$$

The distance function defined by $L_{\mathbf{g}}$, which we denote by $d_{\mathbf{g}}$ in this case, is the Riemannian distance function on X . If the metric \mathbf{g} is C^2 , then geodesics are determined by solutions of the classical geodesic equations. We may, however, still define (minimising) geodesics within the length-space approach even if the metric is, for example, only continuous, even though the Levi-Civita connection is not (classically) well-defined.

Comparison triangles

In the theory of Alexandrov spaces, curvature bounds are defined by comparing properties of geodesic triangles with the properties of corresponding geodesic triangles in model spaces. The model spaces are the two-dimensional, simply-connected Riemannian manifolds of constant curvature $K \in \mathbb{R}$, which we denote by (M_K, \mathbf{g}_K) . As such, we have

$K > 0$: M_K is the standard two-sphere with the round metric of curvature K ;
 $K = 0$: M_0 is \mathbb{R}^2 , with the flat metric;
 $K < 0$: M_K is the hyperbolic plane, with the metric of constant curvature K .

Let (X, d) be a metric space. Let $p, q, r \in X$ be the vertices of a geodesic triangle, Δpqr , with sides of length $|pq|$, $|qr|$, $|rp|$. A *comparison triangle* in M_K is a geodesic triangle $\Delta \widehat{p}\widehat{q}\widehat{r}$ in the constant curvature space (M_K, \mathbf{g}_K) with sides of the same length as those in the triangle Δpqr , i.e.

$$|\widehat{p}\widehat{q}| = |pq|, \quad |\widehat{q}\widehat{r}| = |qr|, \quad |\widehat{r}\widehat{p}| = |rp|.$$

Given a triangle Δpqr in X , it can be shown that, for $K \leq 0$, there exists a geodesic triangle in the space M_K with the required side-lengths. This triangle is unique up to an isometry of M_K . For $K > 0$, comparison triangles exist, and are unique up to an isometry of M_K , if the side-lengths obey the condition $|pq| + |qr| + |rp| < 2\pi/\sqrt{K}$.

Definition 2.5. An *Alexandrov space with curvature bounded below by K* is a connected, locally compact length space of finite (Hausdorff) dimension such that, given any $x \in X$, there exists a neighbourhood, U , of x such that for all points $p, q, r \in U$ the geodesic triangle Δpqr has a corresponding comparison triangle $\Delta \widehat{p}\widehat{q}\widehat{r}$ in (M_K, \mathbf{g}_K) , and we have

$$d(p, z) \geq \widehat{d}_K(\widehat{p}, \widehat{z})$$

for all $z \in [qr]$ with \widehat{z} the point corresponding to z in the comparison triangle. (See Figure 2.1.) Similarly, if for all such geodesic triangles we have

$$d(p, z) \leq \widehat{d}_K(\widehat{p}, \widehat{z})$$

for all $z \in [qr]$, then (X, d) is an *Alexandrov space with curvature bounded above by K* .

Remark 2.6. The above definition of an Alexandrov space with curvature bounded above/below is the most direct within our framework. In the case of curvature $\geq K$, it admits a suitable globalised version (cf., e.g., [4, §3]). There are several equivalent definitions of Alexandrov spaces with curvature bounds³.

³See, for example, [4, §2] for equivalent definitions for curvature bounded below.

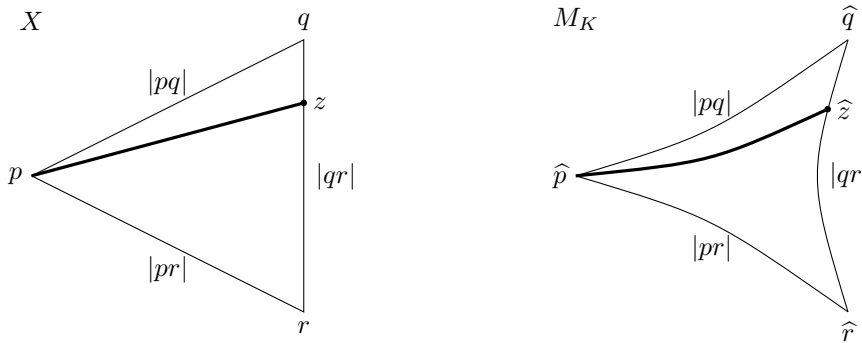


Figure 2.1: A geodesic triangle in (X, d) and comparison triangle in (M_K, \mathbf{g}_K) .

Examples

Example 2.7. Let (M, \mathbf{g}) be a Riemannian manifold without boundary, where the metric \mathbf{g} is of differentiability class C^2 or above. The Riemannian distance function $d_{\mathbf{g}}$ defined in Example 2.4 furnishes $(M, d_{\mathbf{g}})$ with the structure of a metric space. It follows from the Toponogov comparison theorem (see, e.g., [17]) that $(M, d_{\mathbf{g}})$ is an Alexandrov space with curvature bounded below (above) by $K \in \mathbb{R}$ if and only if the metric \mathbf{g} has sectional curvature bounded below (above) by K .

Example 2.8. By a theorem of Alexandrov, a convex polytope in \mathbb{R}^n is an Alexandrov space with non-negative curvature (see Figure 2.2(a)).

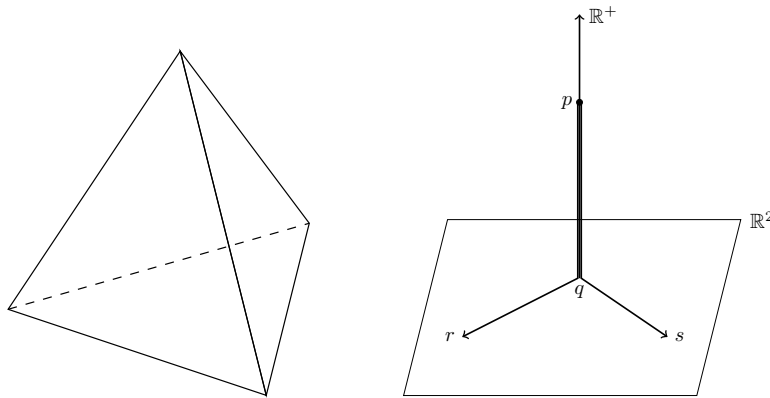


Figure 2.2: (a). An Alexandrov space with non-negative curvature. (b). An Alexandrov space with non-positive curvature.

Example 2.9. The two-dimensional plane with a line attached at a point is an Alexandrov space with non-positive curvature (see Figure 2.2(b)). Note that this space is not a manifold.

Remark 2.10. From Example 2.7, we see that Alexandrov spaces with curvature bounded below/above may be viewed as a generalisation of Riemannian manifolds with a corresponding lower/upper bound on their sectional curvature. From Example 2.9, we note that Alexandrov spaces need not be manifolds and, as such, it may not be possible to define classical differential geometrical structures such as tensor fields. As such, we may not be able to define, for example, a Riemannian metric and its corresponding curvature tensor, but we can still define the notion of a *curvature bound*.

Remark 2.11. In an Alexandrov space with curvature bounded below, geodesics do not branch [4, pp. 6]. It follows, from the branching of geodesics at the point q in Figure 2.2(b), that this space has curvature bounded above, but unbounded below. Recall that examples of Riemannian metrics where the geodesic equations have non-unique solutions have been constructed by Hartman [11]. A simple calculation of the curvature in these examples shows that the sectional curvature diverges to $-\infty$ on the set where geodesics bifurcate.

Properties of Alexandrov spaces

Alexandrov spaces with curvature bounded both above and below (that satisfy an additional completeness condition) have the following properties (see, e.g., [1]):

- 1). X is a C^3 manifold;
- 2). d is induced by a Riemannian metric, which is $C^{1,\alpha}$ for all $\alpha \in [0, 1)$. (Note, not $C^{1,1}$, so the curvature tensor is not locally bounded in general.)
- 3). The Riemannian metric can be approximated by smooth Riemannian metrics.

More generally, Alexandrov spaces with curvature bounded below have the following properties:

- 1). The Hausdorff dimension of an Alexandrov space with curvature bounded below is an integer [4, §6].
- 2). There exists a set $S_X \subset X$ of Hausdorff dimension less than or equal to $n-1$ such that $X \setminus S_X$ carries a C^0 -Riemannian structure that induces the metric d [18];
- 3). Alexandrov spaces of Hausdorff dimension 1 or 2 are topological manifolds (see, e.g., [4, Chapter 10]). However, there exist examples in dimensions $n \geq 3$ that are not topological manifolds (see, e.g., [7]).
- 4). The class of Alexandrov spaces with curvature uniformly bounded below is closed with respect to Gromov–Hausdorff convergence.
- 5). Generically, (X, d) *cannot* be approximated by smooth Riemannian manifolds (at least with respect to the Gromov–Hausdorff topology) of the same

dimension as X and with curvature bounded below. In particular, let M be a compact Alexandrov space of dimension n with curvature bounded below by K . Perelman’s stability theorem [19] then states that M^n has a neighbourhood with respect to the Gromov–Hausdorff topology such that any complete Alexandrov space of dimension n and curvature $\geq K$ in this neighbourhood, is homeomorphic to M^n .

3. Generalised functions

There is now a well-developed approach to non-smooth semi-Riemannian geometry within the context of generalised functions. In particular, an approach to non-smooth differential geometry within the context of “special” Colombeau algebras has been developed in [13, 14]⁴.

The natural question that arises is whether there is any relationship between non-smooth differential geometry as developed in the generalised functions approach, and the description in terms of Alexandrov spaces (and more general structures) that are studied in synthetic geometry. In particular, it may be lucrative to investigate more analytical aspects of, for example, Alexandrov spaces with the general philosophy of generalised functions in mind and, conversely, the geometrical insight gained from the clear geometrical motivations in synthetic geometry, combined with the large body of existing results in the field, when translated into the language of generalised functions, may allow us to gain a deeper understanding of the generalised functions approach to non-smooth geometries.

In particular, in proofs of many classical results, a strong role is played by convexity or concavity properties of the distance function on a manifold, when compared to the corresponding distance function on a constant curvature space. If one considers structures of low regularity, one may adopt such convexity/concavity conditions as the definition of particular types of curvature bound. Within a generalised functions approach, it is more natural to consider non-smooth objects as defining nets of smooth objects, with the non-smooth behaviour being encoded in the asymptotic behaviour of the net. One of our main proposals is to investigate the connection between curvature bounds defined in terms of convexity/concavity properties of geodesics, and asymptotic properties of nets of smoothed metrics that appear in the generalised functions approach.

Geroch–Traschen metrics

A natural place to begin an investigation would be to study Riemannian metrics on closed (i.e. compact without boundary) manifolds, the curvature of which is well-defined as a distribution. A class of such metrics, the *Geroch–Traschen class* [8], have recently been studied in [21] within the framework of generalised functions⁵. It was shown that such metrics lead to a well-defined, non-degenerate, generalised metric, as long as two additional conditions are satisfied:

⁴For a review of applications of generalised functions in non-smooth geometry, see [20]

⁵See [15] for an alternative approach.

Condition 1: The determinant of the generalised metric is strictly positive in the generalised sense. In particular, there exists a representative \mathbf{g}_ε of the metric and an $m \geq 0$ such that $\det \mathbf{g}_\varepsilon \geq \varepsilon^m$ as $\varepsilon \rightarrow 0$.

Condition 2: The eigenvalues of the metric \mathbf{g}_ε , λ_ε^i , $i = 1, \dots, n$ do not converge to zero too quickly as $\varepsilon \rightarrow 0$. (See the discussion after Definition 4.5 in [21].)

Condition 1 may be viewed as a localised version of the condition that the volume of M with respect to \mathbf{g}_ε does not converge to zero too quickly. Moreover, if $\det \mathbf{g}_\varepsilon$ is not converging to zero too quickly and no eigenvalue of \mathbf{g}_ε is converging to zero quickly, then no eigenvalue of \mathbf{g}_ε can diverge too quickly. If an eigenvalue were to diverge to $+\infty$, this would imply that distances are growing unboundedly, and hence that the diameter of M would be diverging. As such, Conditions 1 and 2 are, heuristically speaking, localised, generalised versions of the classical conditions that a Riemannian manifold should obey the condition

$$(3.1) \quad \text{Vol}(M) \geq V, \quad \text{diam}(M) \leq d$$

for some positive constants V, d . On a closed manifold, conditions (3.1), along with a bound on sectional curvature

$$(3.2) \quad |K_M| \leq K,$$

are precisely the conditions of the Cheeger finiteness theorem [6]. Moreover, Cheeger has shown that, given constants V, d, K and a positive integer n , then there exists a constant $c_n(V, d, K) > 0$ such that any n -dimensional, compact Riemannian manifold satisfying conditions (3.1) and (3.2) has injectivity radius bounded below by $c_n(V, d, K)$. Such injectivity radius estimates are of great importance when one wishes to establish that a sequence of Riemannian manifolds does not “collapse”. It is therefore interesting to note that there are similarities between the criteria for a Geroch–Traschen metric to give a well-defined generalised metric, and the non-collapse criteria in classical differential geometry. It would be of interest to investigate the analysis of [21] with the proof of the Cheeger finiteness theorem (and the Gromov compactness theorem) in mind.

More generally, one can extend the investigation of the Geroch–Traschen class of metrics from the viewpoint of Alexandrov spaces. It is known [8] that the $(0, 4)$ form of the curvature tensor, \mathbf{R} , of a Geroch–Traschen metric is well-defined. One should therefore investigate the consequences for such a metric if the curvature, viewed as a distribution, satisfies the weak lower curvature bound⁶:

$$\langle \mathbf{R}, \mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{X} \otimes \mathbf{Y} \rangle \geq K \langle \mathbf{g} \wedge \mathbf{g}, \mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{X} \otimes \mathbf{Y} \rangle.$$

for some K , and for all compactly supported smooth vector fields \mathbf{X}, \mathbf{Y} on M . The expectation, from the Alexandrov space point of view, would be that the geodesics in such a Geroch–Traschen metric may not branch, despite the fact

⁶This appears to be the closest analogue of a sectional curvature bound that one can expect in the distributional case.

that the metric may be well below the regularity ($C^{1,1}$) required for classical existence and uniqueness of geodesics. Given the results of [18], it is also conceivable that such a metric may have better regularity properties than a general metric with distributional curvature. If one imposes that a Geroch–Traschen metric has curvature bounded both below and above, one would expect additional extra regularity as for Alexandrov spaces with curvature bounded both above and below [1].

More broadly, within the context of synthetic geometry, lower bounds on Ricci curvature have been discussed in the context of metric measure spaces [22, 23, 16]. It would be of interest to investigate the Geroch–Traschen class of metrics within this framework. Whether the methods of [22, 23, 16] may be adapted to Lorentzian geometry and applied to the problem of lowering the regularity conditions required in the singularity theorems [12] is currently under investigation.

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