Novi Sad J. Math.
Vol. 41, No. 2, 2011, 17-27

CONVERGENCE THEOREMS OF MULTI-STEP ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN CONVEX METRIC SPACES

Gurucharan Singh Saluja[1]

Abstract. We give some necessary and sufficient conditions for multi-step iterative process with errors for asymptotically quasi-nonexpansive type mappings converging to a common fixed point in convex metric spaces. The results presented in this paper extend the corresponding results of Chang et al. [4], Kim et al. [7] and many others. Also, the corresponding results in [1, 2, 3, 5, 9, 10, 11, 12, 15, 16] are special cases of our results.

AMS Mathematics Subject Classification (2010): 47H09, 47H10, 49M05

Key words and phrases: Asymptotically nonexpansive mapping, asymptotically nonexpansive type mapping, asymptotically quasi-nonexpansive type mapping, convex metric space, common fixed point, multi-step iterative process with errors

1. Introduction and Preliminaries

Throughout this paper, we assume that E is a metric space, $F(T)$ and $D(T)$ are the sets of fixed points and domain of T respectively, and $\mathbb{N}$ is the set of all positive integers.

Definition 1.1 ([4]). Let $T\colon D(T) \subset E \to E$ be a mapping.

(1) The mapping T is said to be L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(Tx, Ty) \leq L\, d(x, y), \quad \forall x, y \in D(T). \tag{1.1}$$

(2) The mapping T is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in D(T). \tag{1.2}$$

(3) The mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p), \quad \forall x \in D(T),\ \forall p \in F(T). \tag{1.3}$$

[1]Department of Mathematics & Information Technology, Govt. Nagarjuna P.G. College of Science, Raipur (Chhattisgarh), India. e-mail: saluja_1963@rediffmail.com

(4) The mapping T is said to be asymptotically nonexpansive [6], if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y), \quad \forall x, y \in D(T), \ \forall n \in \mathbb{N}. \tag{1.4}$$

(5) The mapping T is said to be of asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, p) \leq k_n d(x, p), \quad \forall x \in D(T), \ \forall p \in F(T), \ \forall n \in \mathbb{N}. \tag{1.5}$$

(6) T is said to be of asymptotically nonexpansive type [8], if

$$\limsup_{n\to\infty} \left\{ \sup_{x,y\in D(T)} \{d(T^n x, T^n y) - d(x, y)\} \right\} \leq 0. \tag{1.6}$$

(7) T is said to be asymptotically quasi-nonexpansive type, if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \left\{ \sup_{x\in D(T), p\in F(T)} \{d(T^n x, p) - d(x, p)\} \right\} \leq 0. \tag{1.7}$$

Remark 1.1. It is easy to see that if $F(T)$ is nonempty, then the nonexpansive mapping, quasi-nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mapping all are the special cases of asymptotically quasi-nonexpansive type mapping.

In recent years, the problem concerning convergence of iterative sequences (and sequences with errors) for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings converging to some fixed points in Hilbert spaces or Banach spaces have been considered by many authors.

In 1973, Petryshyn and Williamson [12] obtained a necessary and sufficient condition for Picard iterative sequences and Mann iterative sequences to converge to a fixed point for quasi-nonexpansive mappings. In 1994, Tan and Xu [15] also proved some convergence theorems of Ishikawa iterative sequences satisfying Opial's condition or having Frečhet differential norm. In 1997, Ghosh and Debnath [5] extended the result of Petryshyn and Williamson [12] and gave a necessary and sufficient condition for Ishikawa iterative sequences to converge to a fixed point of quasi-nonexpansive mappings. In 2001, in [1, 2, 3] the author proved some other kinds of necessary and sufficient conditions for Ishikawa iterative sequences with errors for asymptotically nonexpansive mappings to converge to a fixed point. Also in 2001 and 2002, Liu [9, 10, 11] obtained some necessary and sufficient conditions for Ishikawa iterative sequences or Ishikawa iterative sequences with errors to converge to a fixed point for asymptotically quasi-nonexpansive mappings.

Recently, in 2004 Chang et al. [4] generalized and improved the result of Liu [11] in a convex metric space and proved the following:

Theorem CKJ. Let (E, d, W) be a complete convex metric space and $T\colon E \to E$ be an asymptotically quasi-nonexpansive type mapping satisfying the following conditions: there exist constants $L > 0$ and $\alpha > 0$ such that

$$d(Tx, p) \leq L\ d(x, p)^{\alpha}, \quad \forall x \in E, \quad \forall p \in F(T).$$

For any given $x_0 \in E$, let $\{x_n\}$ be the iterative sequence with errors defined by:

$$\begin{aligned} x_{n+1} &= W(x_n, T^n y_n, u_n; a_n, b_n, c_n), \\ y_n &= W(x_n, T^n x_n, v_n; a'_n, b'_n, c'_n), \quad n \geq 0, \end{aligned} \tag{1.8}$$

where $\{a_n\}$, $\{a'_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{c_n\}$ and $\{c'_n\}$ are six sequences in $[0, 1]$ satisfying $a_n+b_n+c_n = a'_n+b'_n+c'_n = 1, \forall n \geq 0$ and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in E. If the sequences $\{b_n\}$ and $\{c_n\}$ appeared in (1.8) satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} b_n < \infty$,

(ii) $\sum_{n=1}^{\infty} c_n < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T in E if and only if

$$\liminf_{n\to\infty} D(x_n, F(T)) = 0,$$

where $D(y, S)$ denotes the distance from y to the set S, i.e.

$$D(y, S) = \inf_{s\in S} d(y, s).$$

The purpose of this paper is to study the convergence problem of multi-step iterative sequences with errors for a finite family of asymptotically quasi-nonexpansive type mappings in convex metric spaces and give some necessary and sufficient conditions to converge to common fixed points for the above mappings. The results presented in the paper extend and improve the corresponding results of Chang [1]-[3], Chang et al. [4], Ghosh and Debnath [5], Kim et al. [7], Liu [9, 10, 11], Petryshyn and Williamson [12] and Tan and Xu [15]. Our results also contain the corresponding results of [1]-[5], [9]-[12], [16] as special cases.

For the sake of convenience, we first recall some definitions and notation.

Definition 1.2. Let (E, d) be a metric space and $I = [0, 1]$. A mapping $W\colon E^3 \times I^3 \to E$ is said to be a convex structure on E if it satisfies the following condition:

$$d(u, W(x, y, z; \alpha, \beta, \gamma)) \leq \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z),$$

for any $u, x, y, z \in E$ and for any $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$.

If (E, d) is a metric space with a convex structure W, then (E, d) is called a *convex metric space* and is denoted by (E, d, W).

Remark 1.2. It is easy to prove that every linear normed space is a convex metric space with a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in I$ with $\alpha+\beta+\gamma = 1$. However, there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi [13]).

Definition 1.3. Let (E, d, W) be a convex metric space, $T_1, T_2, \dots, T_N \colon E \to E$ be N mappings and let $x_1 \in E$ be a given point. Then the sequence $\{x_n\}$ defined by:

$$\begin{aligned}
x_{n+1} &= x_n^{(N)} \\
&= W(x_n, T_N^n x_n^{(N-1)}, u_n^{(N)}; \alpha_n^{(N)}, \beta_n^{(N)}, \gamma_n^{(N)}), \\
x_n^{(N-1)} &= W(x_n, T_{N-1}^n x_n^{(N-2)}, u_n^{(N-1)}; \alpha_n^{(N-1)}, \beta_n^{(N-1)}, \gamma_n^{(N-1)}), \\
&\vdots \\
&\vdots \\
x_n^{(3)} &= W(x_n, T_3^n x_n^{(2)}, u_n^{(3)}; \alpha_n^{(3)}, \beta_n^{(3)}, \gamma_n^{(3)}), \\
x_n^{(2)} &= W(x_n, T_2^n x_n^{(1)}, u_n^{(2)}; \alpha_n^{(2)}, \beta_n^{(2)}, \gamma_n^{(2)}), \\
x_n^{(1)} &= W(x_n, T_1^n x_n, u_n^{(1)}; \alpha_n^{(1)}, \beta_n^{(1)}, \gamma_n^{(1)}), \qquad (1.9)
\end{aligned}$$

is called the multi-step iterative sequence with errors for N mappings $T_1, T_2, \dots, T_N$, where $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ are sequences in $[0, 1]$ satisfying $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $\forall i = 1, 2, \dots, N$ and $\forall n \in \mathbb{N}$ and $\{u_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ is a bounded sequence in E.

In (1.9), if $N = 2$, $T_1 = T_2 = T$, $x_n^{(1)} = y_n$, $u_n^{(2)} = u_n$, $\alpha_n^{(2)} = a_n$, $\beta_n^{(2)} = b_n$, $\gamma_n^{(2)} = c_n$, $u_n^{(1)} = v_n$, $\alpha_n^{(1)} = a'_n$, $\beta_n^{(1)} = b'_n$ and $\gamma_n^{(1)} = c'_n$, then the scheme (1.9) reduces to the two-step iterative scheme with errors for a mapping defined by Chang et al. [4].

In order to prove our main theorem of this paper, we need the following lemma:

Lemma 1.1 ([14]). *Let $\{a_n\}$, $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \qquad n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$. Then

(a) $\lim_{n\to\infty} a_n$ *exists.*

(b) If $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

2. Main Results

In order to prove our main result, we will first prove the following important lemma.

Lemma 2.1. *Let (E, d, W) be a convex metric space, $T_1, T_2, \dots, T_N : E \to E$ be N asymptotically quasi-nonexpansive type mappings. Suppose $\mathcal{F} = \bigcap_{i=1}^N F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the multi-step iterative sequences with errors defined by (1.9), where $\{u_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ is a bounded sequence in E. Then*

(a) $$d(x_{n+1}, p) \le d(x_n, p) + \theta_n, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0,$$

where $\theta_n = N\beta_n^{(N)}\varepsilon + M\sum_{k=1}^N \gamma_n^{(k)}$.

(b) $$d(x_{n+m}, p) \le d(x_n, p) + \sum_{k=n}^{n+m-1} \theta_k, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0, \quad \forall m \ge 1.$$

Proof. (a) Let $p \in \mathcal{F}$ and since $\{u_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ is a bounded sequence in E, so we put

$$M = \max\{\sup_{n\ge1} d(u_n^{(i)}, p) : i = 1, 2, \dots, N\}. \tag{2.1}$$

Since T_i for all $i = 1, 2, \dots, N$ is asymptotically quasi-nonexpansive type, it follows that

$$\limsup_{n\to\infty}\left\{\sup_{x\in E, p\in\mathcal{F}}\{d(T_i^n x, p) - d(x, p)\}\right\} \le 0.$$

This implies that for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for $n \ge n_0$ we have

$$\sup_{x\in E, p\in\mathcal{F}}\{d(T_i^n x, p) - d(x, p)\} < \varepsilon. \tag{2.2}$$

Since $\{x_n\}, \{x_n^{(1)}\}, \dots, \{x_n^{(N-1)}\} \subset E$, we have

$$\begin{aligned}
d(T_1^n x_n, p) - d(x_n, p) &< \varepsilon, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0\\
d(T_2^n x_n^{(1)}, p) - d(x_n^{(1)}, p) &< \varepsilon, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0\\
d(T_3^n x_n^{(2)}, p) - d(x_n^{(2)}, p) &< \varepsilon, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0\\
&\vdots\\
d(T_N^n x_n^{(N-1)}, p) - d(x_n^{(N-1)}, p) &< \varepsilon, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0.
\end{aligned} \tag{2.3}$$

Thus for each $n \geq 1$ and for any $p \in \mathcal{F}$, using (1.9), (2.1) and (2.3), we note that

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T_N^n x_n^{(N-1)}, u_n^{(N)}; \alpha_n^{(N)}, \beta_n^{(N)}, \gamma_n^{(N)}), p) \\ &\leq \alpha_n^{(N)} d(x_n, p) + \beta_n^{(N)} d(T_N^n x_n^{(N-1)}, p) + \gamma_n^{(N)} d(u_n^{(N)}, p) \\ &\leq \alpha_n^{(N)} d(x_n, p) + \beta_n^{(N)} [d(x_n^{(N-1)}, p) + \varepsilon] + \gamma_n^{(N)} M \\ &\leq \alpha_n^{(N)} d(x_n, p) + \beta_n^{(N)} d(x_n^{(N-1)}, p) + \beta_n^{(N)} \varepsilon + \gamma_n^{(N)} M \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(x_n^{(N-1)}, p) &= d(W(x_n, T_{N-1}^n x_n^{(N-2)}, u_n^{(N-1)}; \alpha_n^{(N-1)}, \beta_n^{(N-1)}, \gamma_n^{(N-1)}), p) \\ &\leq \alpha_n^{(N-1)} d(x_n, p) + \beta_n^{(N-1)} d(T_{N-1}^n x_n^{(N-2)}, p) \\ &\quad + \gamma_n^{(N-1)} d(u_n^{(N-1)}, p) \\ &\leq \alpha_n^{(N-1)} d(x_n, p) + \beta_n^{(N-1)} [d(x_n^{(N-2)}, p) + \varepsilon] + \gamma_n^{(N-1)} M \\ &\leq \alpha_n^{(N-1)} d(x_n, p) + \beta_n^{(N-1)} d(x_n^{(N-2)}, p) + \beta_n^{(N-1)} \varepsilon \\ &\quad + \gamma_n^{(N-1)} M. \end{aligned} \tag{2.5}$$

Continuing on this process, we get

$$\begin{aligned} d(x_n^{(N-2)}, p) &\leq \alpha_n^{(N-2)} d(x_n, p) + \beta_n^{(N-2)} d(x_n^{(N-3)}, p) + \beta_n^{(N-2)} \varepsilon \\ &\quad + \gamma_n^{(N-2)} M \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} d(x_n^{(1)}, p) &= d(W(x_n, T_1^n x_n, u_n^{(1)}; \alpha_n^{(1)}, \beta_n^{(1)}, \gamma_n^{(1)}), p) \\ &\leq \alpha_n^{(1)} d(x_n, p) + \beta_n^{(1)} d(T_1^n x_n, p) + \gamma_n^{(1)} d(u_n^{(1)}, p) \\ &\leq \alpha_n^{(1)} d(x_n, p) + \beta_n^{(1)} [d(x_n, p) + \varepsilon] + \gamma_n^{(1)} M \\ &\leq [\alpha_n^{(1)} + \beta_n^{(1)}] d(x_n, p) + \beta_n^{(1)} \varepsilon + \gamma_n^{(1)} M \\ &= [1 - \gamma_n^{(1)}] d(x_n, p) + \beta_n^{(1)} \varepsilon + \gamma_n^{(1)} M \\ &\leq d(x_n, p) + \beta_n^{(1)} \varepsilon + \gamma_n^{(1)} M. \end{aligned} \tag{2.7}$$

Using equations (2.4) - (2.7), we get

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_n, p) + N\beta_n^{(N)} \varepsilon + [\gamma_n^{(1)} + \gamma_n^{(2)} + \cdots + \gamma_n^{(N)}] M \\ &\leq d(x_n, p) + N\beta_n^{(N)} \varepsilon + M \sum_{k=1}^{N} \gamma_n^{(k)} \\ &\leq d(x_n, p) + \theta_n \end{aligned} \tag{2.8}$$

where $\theta_n = N\beta_n^{(N)} \varepsilon + M \sum_{k=1}^{N} \gamma_n^{(k)}$. This completes the proof of (a).

(b) It follows from conclusion (a) that for any $m \geq 1$, we have

$$\begin{aligned} d(x_{n+m}, p) &\leq d(x_{n+m-1}, p) + \theta_{n+m-1} \\ &\leq d(x_{n+m-2}, p) + \theta_{n+m-2} + \theta_{n+m-1} \\ &\leq \dots \\ &\leq \dots \\ &\leq d(x_n, p) + [\theta_{n+m-1} + \theta_{n+m-2} + \cdots + \theta_n] \\ &\leq d(x_n, p) + \sum_{k=n}^{n+m-1} \theta_k, \quad \forall n \geq n_0, \quad \forall p \in \mathcal{F}. \end{aligned} \tag{2.9}$$

This completes the proof of (b). □

Theorem 2.1. *Let (E, d, W) be a complete convex metric space, $T_1, T_2, \dots, T_N \colon E \to E$ be N asymptotically quasi-nonexpansive type mappings. Suppose $\mathcal{F} = \bigcap_{i=1}^N F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the multi-step iterative sequences with errors defined by (1.9). If $\sum_{n=1}^\infty \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \dots, N$. Then the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$ if and only if*

$$\liminf_{n\to\infty} D(x_n, \mathcal{F}) = 0.$$

where $D(y, S)$ denotes the distance from y to the set S, i.e.

$$D(y, S) = \inf_{s \in S} d(y, s).$$

Proof. The necessity is obvious. Now, we will only prove the sufficient condition. Suppose that the condition $\liminf_{n\to\infty} D(x_n, \mathcal{F}) = 0$ is satisfied. Then from Lemma 2.1 (a), we have

$$d(x_{n+1}, p) \leq d(x_n, p) + \theta_n, \quad \forall p \in \mathcal{F}, \quad \forall n \geq 1, \tag{2.10}$$

where $\theta_n = N\beta_n^{(N)}\varepsilon + M \sum_{k=1}^N \gamma_n^{(k)}$. Since $\sum_{n=1}^\infty \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \dots, N$, it follows that $\sum_{n=1}^\infty \theta_n < \infty$. From (2.10), we can obtain that

$$D(x_{n+1}, \mathcal{F}) \leq D(x_n, \mathcal{F}) + \theta_n. \tag{2.11}$$

Since $\liminf_{n\to\infty} D(x_n, \mathcal{F}) = 0$, by Lemma 1.1, we have

$$\lim_{n\to\infty} D(x_n, \mathcal{F}) = 0. \tag{2.12}$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence in E. Since $\sum_{n=1}^\infty \theta_n < \infty$ and by the condition (2.12), for any given $\varepsilon > 0$, there exists a positive integer

$n_1 \geq n_0$ (where n_0 is the positive integer appeared in Lemma 2.1) such that for any $n \geq n_1$, we have

$$\sum_{n=n_1}^{\infty} \theta_n \quad < \quad \varepsilon, \tag{2.13}$$

and

$$D(x_n, \mathcal{F}) \quad < \quad \varepsilon. \tag{2.14}$$

By the definition of infimum, it follows from (2.14) that for any given $n \geq n_1$ there exists an $p(n) \in \mathcal{F}$ such that

$$d(x_n, p(n)) < 2\varepsilon. \tag{2.15}$$

On the other hand, it follows from Lemma 2.1 that for the given $\varepsilon > 0$ and for any $n \geq n_1 \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p(n)) + d(x_n, p(n)) \\ &\leq d(x_n, p(n)) + \sum_{k=n}^{n+m-1} \theta_k + d(x_n, p(n)) \\ &= 2d(x_n, p(n)) + \sum_{k=n}^{n+m-1} \theta_k, \qquad m \geq 1. \end{aligned}$$

Therefore, from (2.13), (2.15) and the above inequality, we have

$$d(x_{n+m}, x_n) \quad < \quad 4\varepsilon + \varepsilon = 5\varepsilon, \qquad m \geq 1. \tag{2.16}$$

This implies that $\{x_n\}$ is a Cauchy sequence in E. Since E is complete, there exists $p^* \in E$ such that $x_n \to p^*$ as $n \to \infty$.

Now we have to prove that p^* is a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$, that is, $p^* \in \mathcal{F}$.

By contradiction, we assume that p^* is not in $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$. Since the set of fixed points of asymptotically quasi-nonexpansive type mapping is closed, so is $\mathcal{F}$, therefore $D(p^*, \mathcal{F}) > 0$. So, for all $p \in \mathcal{F}$, we have

$$d(p^*, p) \quad \leq \quad d(p^*, x_n) + d(x_n, p). \tag{2.17}$$

By the arbitrariness of $p \in \mathcal{F}$, we know that

$$D(p^*, \mathcal{F}) \quad \leq \quad d(p^*, x_n) + D(x_n, \mathcal{F}). \tag{2.18}$$

By $\lim_{n\to\infty} D(x_n, \mathcal{F}) = 0$, above inequality and $x_n \to p^*$ as $n \to \infty$, we have

$$D(p^*, \mathcal{F}) \quad = \quad 0, \tag{2.19}$$

which contradicts $D(p^*, \mathcal{F}) > 0$. Thus the iterative sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$. This completes the proof. □

Theorem 2.2. *Let (E, d, W) be a complete convex metric space, $T_1, T_2, \dots, T_N \colon E \to E$ be N asymptotically nonexpansive mappings. Suppose $\mathcal{F} = \bigcap_{i=1}^N F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the multi-step iterative sequences with errors defined by (1.9). If $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \dots, N$. Then the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$ if and only if*

$$\liminf_{n\to\infty} D(x_n, \mathcal{F}) \quad = \quad 0.$$

Proof. Since T_i for all $i = 1, 2, \dots, N$ is asymptotically nonexpansive with $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, we know that there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d(T_i^n x, p) \leq k_n d(x, p), \quad \forall p \in \mathcal{F}, \quad \forall x \in E, \quad n \geq 1.$$

This implies that

$$d(T_i^n x, p) - k_n d(x, p) \leq 0, \quad \forall p \in \mathcal{F}, \quad \forall x \in E, \quad n \geq 1.$$

Therefore we have

$$\limsup_{n\to\infty} \left\{ \sup_{x\in E, p\in\mathcal{F}} \{d(T_i^n x, p) - d(x, p)\} \right\} \leq 0.$$

This implies that $T_i \colon E \to E$ is asymptotically quasi-nonexpansive type mapping for all $i = 1, 2, \dots, N$. The conclusion of Theorem 2.2 can be obtained from Theorem 2.1 immediately. □

By using the same method as in Theorem 2.1, we can easily prove the following theorem.

Theorem 2.3. *Let (E, d, W) be a complete convex metric space, $T_1, T_2, \dots, T_N \colon E \to E$ be N quasi-nonexpansive mappings. Suppose $\mathcal{F} = \bigcap_{i=1}^N F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the multi-step iterative sequences with errors defined by (1.9). If $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \dots, N$. Then, the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$ if and only if*

$$\liminf_{n\to\infty} D(x_n, \mathcal{F}) \quad = \quad 0.$$

From Theorem 2.1, we can also obtain the following:

Theorem 2.4. *Let E be a Banach space and $T_1, T_2, \dots, T_N \colon E \to E$ be N asymptotically quasi-nonexpansive type mappings. Suppose $\mathcal{F} = \bigcap_{i=1}^N F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the multi-step iterative sequences with errors defined by (1.9). If $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \dots, N$. Then, the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$ if and only if*

$$\liminf_{n\to\infty} D(x_n, \mathcal{F}) \quad = \quad 0.$$

Proof. Since E is a Banach space, it is a complete convex metric space with a convex structure $W(x, y, z; \alpha, \beta, \gamma) := \alpha x + \beta y + \gamma z$, for all $x, y, z \in E$ and for all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$. Therefore, the conclusion of Theorem 2.4 can be obtained from Theorem 2.1 immediately. □

Remark 2.1. We would like to point out that Theorems 2.1, 2.2 and 2.3 of this paper generalize and improve the corresponding results of Chang [1]-[3], Ghosh and Debnath [5], Liu [9] - [11], Petryshyn and Williamson [12] and Tan and Xu [15] to the case of a finite family of more general class nonexpansive, quasi-nonexpansive, asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings and multi-step iteration considered here. Our results also contain the corresponding results of [1]-[5], [9]-[12], [16] as their special cases. Especially, Theorem 2.4 generalizes and improves the result of Liu [11] in the following aspects:

(i) The condition that "E is a compact subset of a uniformly convex Banach space" is removed. We only assume that E is a general Banach space.

(ii) The asymptotically quasi-nonexpansive mapping in [11] is extended to asymptotically quasi-nonexpansive type mapping.

(iii) The condition "(L, α)-uniformly Lipschitz" in [11] is removed.

(iv) The two-step iteration scheme for one mapping is extended to multi-step iteration scheme for N mappings.

Remark 2.2. Our results also extend the corresponding results of Kim et al. [7] to the case of a more general class of asymptotically quasi-nonexpansive mappings and multi-step iteration scheme considered here.

Remark 2.3. Our results also extend the corresponding results of Chang et al. [4] to the case of multi-step iteration scheme for a finite family of mappings.

Remark 2.4. Especially, Theorem 2.1 improves and generalizes Theorem 2.1 of Chang et al. [4] in the following aspects:

(i) The condition "there exist constants $L > 0$ and $\alpha > 0$ such that

$$d(Tx, p) \leq Ld(x, p)^{\alpha}, \quad \forall p \in F(T), \quad \forall x \in E\text{"},$$

is removed.

(ii) The two-step iteration scheme for one mapping is extended to multi-step iteration scheme for a finite family of mappings.

References

[1] Chang S.S., Some results for asymptotically pseudo-contractive mapping and asymptotically nonexpansive mappings. Proc. Amer. Math. Soc. 129 No.3 (2001), 845-853.

[2] Chang S.S., Iterative approximation problem of fixed points for asymptotically nonexpansive mappings in Banach spaces. Acta Math. Appl. 24 No.2 (2001), 236-241.

[3] Chang S.S., On the approximating problem of fixed points for asymptotically nonexpansive mappings. Indian J. Pure and Appl. 32 No.9 (2001), 1-11.

[4] Chang S.S., Kim J.K., Jin D.S., Iterative sequences with errors for asymptotically quasi-nonexpansive type mappings in convex metric spaces. Archives of Inequality and Applications 2 (2004), 365-374.

[5] Ghosh M.K., Debnath L., Convergence of Ishikawa iterates of quasi-nonexpansive mappings. J. Math. Anal. Appl. 207 (1997), 96-103.

[6] Goebel K., Kirk W.A., A fixed point theorem for asymptotically nonexpansive mappings. Proc. Amer. Math. Soc. 35 (1972), 171-174.

[7] Kim J.K., Kim K.H., Kim K.S., Three-step iterative sequences with errors for asymptotically quasi-nonexpansive mappings in convex metric spaces. Nonlinear Anal. Convex Anal. RIMS Vol. 1365 (2004), 156-165.

[8] Kirk W.A., Fixed point theorems for non-lipschitzian mappings of asymptotically nonexpansive type. Israel J. Math. 17 (1974), 339-346.

[9] Liu Q.H., Iterative sequences for asymptotically quasi-nonexpansive mappings. J. Math. Anal. Appl. 259 (2001), 1-7.

[10] Liu Q.H., Iterative sequences for asymptotically quasi-nonexpansive mappings with error member. J. Math. Anal. Appl. 259 (2001), 18-24.

[11] Liu Q.H., Iterative sequences for asymptotically quasi-nonexpansive mappings with error member of uniformly convex Banach spaces. J. Math. Anal. Appl. 266 (2002), 468-471.

[12] Petryshyn W.V., Williamson T.E., Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings. J. Math. Anal. Appl. 43 (1973), 459-497.

[13] Takahashi W., A convexity in metric space and nonexpansive mappings I. Kodai Math. Sem. Rep. 22 (1970), 142-149.

[14] Tan K.K., Xu H.K., Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. 178 (1993), 301-308.

[15] Tan K.K., Xu H.K., Fixed point iteration processes for asymptotically nonexpansive mappings. Proc. Amer. Math. Soc. 122 (1994), 733-739.

[16] Xu B.L., Noor M.A., Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 267 No.2 (2002), 444-453.

Received by the editors October 4, 2008