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ON SOME CONNECTIONS ON LOCALLY PRODUCT RIEMANNIAN MANIFOLDS - PART II

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Abstract. In the first part of this paper, we considered some kinds of connections (the members of "projective class") in almost product spaces. The main purpose of considering such connections was the extension of the Riemannian connection from a submanifold of a Riemannian manifold to the whole space. We used the methodology of almost product spaces for it. For two of them, there exist invariant tensors. For the holomorphically projective connection, this invariant tensor is well-known. Here we calculate the invariant tensor for a product semi-symmetric connection and give a proof that there does not exist a curvature or Ricci type invariant for the mirror connection.

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1. About locally product spaces

As we have said in the first part of this paper, M_n is an $n-$dimensional locally product space of $p-$ and $q-$dimensional spaces M_p and M_q $(p+q=n)$ if and only if M_n is covered by such a system of coordinate neighborhoods $\{(U,x^h)\}$ that at any intersection of two coordinate neighborhoods (U,x^h) and $(U',x^{h'})$ we have

$$x^{a'} = x^{a'}(x^a) \text{ and } x^{x'} = x^{x'}(x^x) \tag{1.1}$$

with

$$|\partial_a x^{a'}| \neq 0 \text{ and } |\partial_x x^{x'}| \neq 0,$$

where ∂_h denotes $\partial/\partial x^h$, the indices a,b,c,d run over the range $1,2,...,p$, the indices x,y,z,w run over the range $p+1,...,p+q=n$, and the indices h,i,j,k,l run over the range $1,2,...,n$. Such a coordinate system will be called a **separating coordinate system** in M_n. The **locally product structure tensor** F_i^h is defined by

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$$(F_i^h) = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_y^x \end{pmatrix} \tag{1.2}$$

in each separating coordinate neighborhood and it satisfies $F_j^i F_i^h = \delta_j^h$. If M_n is a Riemannian manifold, its metric tensor is of the form

$$(g_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{xy} \end{pmatrix} \tag{1.3}$$

and component-subspaces are mutually orthogonal. Then there exists the covariant structure tensor

$$F_{ji} = F_j^t g_{ti} \tag{1.4}$$

and there holds

$$F_j^t F_k^s g_{ts} = g_{jk}. \tag{1.5}$$

By this fact, it is obvious that the covariant structure tensor is symmetric and in fact

$$F_{ji} = \begin{bmatrix} g_{cb} & 0 \\ 0 & -g_{xy} \end{bmatrix} \tag{1.6}$$

If g_{cb} depend only on x^a and g_{xy} depend only on x^z, we call the space a *locally decomposable Riemannian space.* For such a space, we have

$$\left\{ {x \atop cb} \right\} = 0, \quad \left\{ {x \atop zb} \right\} = 0, \quad \left\{ {a \atop cy} \right\} = 0, \quad \left\{ {a \atop zy} \right\} = 0.$$

It is a very well known fact ([10]) that the necessary and sufficient condition for a locally product Riemannian space to be a locally decomposable Riemannian space is that

$$\overset{\circ}{\nabla}_j \, F_i^h = 0, \tag{1.7}$$

and there holds an equivalent condition for the covariant structure tensor. Here $\overset{\circ}{\nabla}$ denotes the covariant differentiation with respect to the Levi-Civita connection on M_n.

For the purposes of the present considerations, we induce the tensors

$$O_{kl}^{ij} = \frac{1}{2}(\delta_k^i \delta_l^j + F_k^i F_l^j), \ {}^*O_{kl}^{ij} = \frac{1}{2}(\delta_k^i \delta_l^j - F_k^i F_l^j), \tag{1.8}$$

which are called **Obata operators.**

Locally product spaces have been investigated very widely in the last century. Here we use results of the papers [2, 3, 4, 6, 7]. Special kinds of connections on Riemannian spaces with or without structures have been investigated in [1, 5, 8, 9, 10]. Here we use the method which has been promoted in the book of K. Yano ([11]).

2. Invariants of some connections on Riemannian, locally product and locally decomposable spaces

As it is very well known ([1, 2, 10]), a projective connection and a holomorphically projective connection on a space with affine connection or on a space which is locally a product of spaces of affine connection have curvature-like tensors that are the same for all connections of such a class, i.e. invariant on their generators. Here we shall briefly present a method for the calculation of these invariants.

On a space of affine connection (as well as in a Riemannian space), the connection with the coefficients

$$\Lambda^i_{jk} = \Gamma^i_{jk} + p_j \delta^i_k + p_k \delta^i_j, \tag{2.1}$$

are coefficients of a connection projective to the affine the connection with coefficients Γ. If the connection with coefficients Γ is symmetric, then the connection with coefficients Λ is also symmetric. The affine connection with coefficients Γ (which also can be a Riemannian connection, but for a different metric tensor) and the connection with coefficients (2.1) have their autoparallel lines in common. For their curvature tensors, there holds the relation

$$R^i_{jkl} = K^i_{jkl} + \delta^i_k p_{lj} - \delta^i_l p_{kj} + \delta^i_j (p_{lk} - p_{kl}), \tag{2.2}$$

where R^i_{jkl} denotes the coefficient of the curvature tensor of the connection given by (2.1), K^i_{jkl} is the coefficient of the curvature tensor of the connection with the coefficients Γ, and by p_{kj} are denoted components of the tensor $\nabla_k p_j - p_k p_j$ (∇_k denotes the covariant derivative towards the affine connection with the coefficients Γ), which is not symmetric in the general case. Contracting the relation (2.2) in the indices i and l, we obtain

$$n p_{kj} = K_{jk} - R_{jk} + p_{jk} \tag{2.3}$$

and, after some calculation,

$$p_{kj} = \frac{1}{n^2-1}[nK_{jk} + K_{kj} - R_{kj} - nR_{jk}].$$

Then there holds

$$\begin{aligned} & R^i_{jkl} + \frac{1}{n^2-1}[\delta^i_k (R_{lj} + nR_{jl}) - \delta^i_l (R_{kj} + nR_{jk})] + \\ & + \frac{1}{n+1}\delta^i_j (R_{kl} - R_{lk}) = \\ = \; & K^i_{jkl} + \frac{1}{n^2-1}[\delta^i_k (K_{lj} + nK_{jl}) - \delta^i_l (K_{kj} + nK_{jk})] + \\ & + \frac{1}{n+1}\delta^i_j (K_{kl} - K_{lk}). \end{aligned} \tag{2.4}$$

The tensor of the right-hand side of (2.4) is called *the projective curvature tensor* for the affine connection with coefficients Γ. It is the same for all affine connections which are projectively related to the same symmetric connection. It does not depend on the covector field p_j, which is its generator.

Holomorphically projective related connections on a locally product Riemannian space are $F-$connections, which have holomorphically planar curves in common with considered symmetric affine connection (possible, with the Riemannian connection, too). Holomorphically planar curves ($\frac{d^2x^h}{dt^2}+\Gamma^h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}=\alpha(t)\frac{dx^h}{dt}+\beta(t)F^h_r\frac{dx^r}{dt}$) have been considered in [1, 2] and the curvature-like invariant tensor of a whole family of holomorhically projective connections has been given in [1, 2], in two different ways of calculation and, we get two differently looking invariants. Here we are going to present the third (original) way of making this invariant, but shortly and roughly, because we shall need this method for some considerations of two other connections. A holomorphically planar curve is a curve whose holomorphic section stays parallel to itself during a parallel displacement along such a curve.

A symmetric affine connection with coefficients Λ^i_{jk} is said to be holomorphically projective related to a symmetric affine connection with coefficients Γ^i_{jk} if and only if their coefficients satisfy the relation

$$\Lambda^i_{jk}=\Gamma^i_{jk}+p_j\delta^i_k+p_k\delta^i_j+q_jF^i_k+q_kF^i_j, \tag{2.5}$$

where p_i are components of a covector field and $q_j=p_aF^a_j$ and by the last fact the connection with coefficients Λ^i_{jk} is an $F-$connection if and and only if the connection with coefficients Γ^i_{jk} is an $F-$connection. We shall suppose that the last fact is satisfied.

It is easy to calculate that for the curvature tensors of these two connections there holds

$$\begin{aligned} R^h_{ijk} &= K^h_{ijk}+\delta^h_jp_{ki}-\delta^h_kp_{ji}+F^h_jF^s_ip_{ks}-F^h_kF^s_ip_{js}\\ &\quad+\delta^h_i(p_{kj}-p_{jk})+F^h_i(F^t_jp_{kt}-F^t_kp_{jt}) \end{aligned} \tag{2.6}$$

where R denotes the curvature tensor of connection with the coefficients Λ, K denotes the curvature tensor of the connection with the coefficients Γ and

$$p_{ji}=\nabla_jp_i-p_jp_i-q_jq_i \tag{2.7}$$

and ∇ is the covariant derivative operator with respect to the connection with coefficients Γ. Contracting the indices h and k, we obtain the relation between the Ricci tensors

$$\begin{aligned} R_{ij} &= K_{ij}+(p_{ij}+p_{ji})-(n+2)p_{ji}+\\ &\quad+F^a_jF^b_i(p_{ab}+p_{ba})-\varphi_1F^s_ip_{js}. \end{aligned} \tag{2.8}$$

Here, φ_1 denotes $F^a_a = p - q$.

After long calculation with multiple using of the Obata operator induced by formula (1.8), we obtain

$$\begin{aligned} p_{jt} &= \frac{1}{(n+2)^2-\varphi_1^2}[\varphi_1(R_{cj}-K_{cj})F^c_t-(n+2)(R_{tj}-K_{tj})- \\ &\quad -2\varphi_1(p_{ab}+p_{ba})O^{ab}_{jc}F^c_t+2(n+2)(p_{ab}+p_{ba})O^{ab}_{jt}] \end{aligned}$$

and finally

$$\begin{aligned} p_{jt} = {} & \frac{1}{(n+2)^2-\varphi_1^2}\{\varphi_1(R_{cj}-K_{cj})F^c_t- \\ & (n+2)(R_{tj}-K_{tj})-\frac{2\varphi_1}{(n-2)^2-\varphi_1^2}[\varphi_1(R_{ab}+R_{ba})O^{ba}_{jt} \\ & -(n-2)(R_{ab}+R_{ba})O^{ba}_{jc}F^c_t-\varphi_1(K_{ab}+K_{ba})O^{ba}_{jt} \\ & +(n-2)(K_{ab}+K_{ba})O^{ba}_{jc}F^c_t] \\ & +\frac{2(n+2)}{(n-2)^2-\varphi_1^2}[\varphi_1(R_{ab}+R_{ba})O^{ba}_{jc}F^c_t \\ & -(n-2)(R_{ab}+R_{ba})O^{ba}_{jt}-\varphi_1(K_{ab}+K_{ba})O^{ba}_{jc}F^c_t \\ & +(n-2)(K_{ab}+K_{ba})O^{ba}_{jt}]\}. \end{aligned} \tag{2.9}$$

The tensor p_{jt} does not depend on the generator of holomorphically projective connection with coefficients Λ. If we induce the following abbreviations

$$\begin{aligned} & \alpha_1 = \frac{\varphi_1}{(n+2)^2-\varphi_1^2}; \beta_1 = \frac{n+2}{(n+2)^2-\varphi_1^2}; \\ & \alpha_2 = 2\frac{\varphi_1^2+(n^2-4)}{[(n+2)^2-\varphi_1^2][(n-2)^2-\varphi_1^2]}; \\ & \beta_2 = \frac{4\varphi_1 n}{[(n+2)^2-\varphi_1^2][(n-2)^2-\varphi_1^2]}; \\ & \pi_{jt} = (K_{ab}+K_{ba})O^{ba}_{jt}; \\ & \bar{\pi}_{jt} = (R_{ab}+R_{ba})O^{ba}_{jt}, \end{aligned} \tag{2.10}$$

putting these abbreviations into the expression (2.9) and then putting this result into (2.6), we obtain that the tensor

$$
\begin{aligned}
(2.11) \qquad & K^h_{ijk} - \alpha_1[K_{ck}\delta^h_j F^c_i - K_{cj}\delta^h_k F^c_i + F^h_j K_{ik} - F^h_k K_{ij} \\
& +\delta^h_i(K_{ck}F^c_j - K_{cj}F^c_k) + F^h_i(K_{jk} - K_{kj})] \\
& +\beta_1[\delta^h_j K_{ik} - \delta^h_k K_{ij} + K_{sk}F^s_i F^h_j - K_{sj}F^s_i F^h_k + \\
& +\delta^h_i(K_{jk} - K_{kj}) + F^h_i(F^t_j K_{tk} - F^t_k K_{tj})] + \\
& +\alpha_2[\pi_{ki}\delta^h_j - \pi_{ji}\delta^h_k + \pi_{ks}F^h_j F^s_i - \pi_{js}F^h_k F^s_i + \\
& +\delta^h_i(\pi_{kj} - \pi_{jk}) + F^h_i(F^t_j\pi_{kt} - F^t_k\pi_{jt})] - \\
& -\beta_2[\delta^h_j\pi_{kc}F^c_i - \delta^h_k\pi_{jc}F^c_i + F^h_j\pi_{ki} - F^h_k\pi_{ji} + \\
& +\delta^h_i(\pi_{kc}F^c_j - \pi_{jc}F^c_k) + F^h_i(\pi_{kj} - \pi_{jk})].
\end{aligned}
$$

is equal to the tensor of the same structure, but depending on $R^h_{ijk}, R_{jk}, \bar{\pi}_{jk}$ instead of $K^h_{ijk}, K_{jk}, \pi_{jk}$.

This tensor is called **holomorphically projective curvature tensor** of the connection with coefficients Γ ((2.5)). This method of calculation is different from the methods given in [1] and [2], and the final form is also different.

3. Invariant curvature-like tensor of product semi-symmetric metric connections

On a locally decomposable Riemannian space there exists a **product semi-symmetric** metric connection with components ([7])

$$
(3.1) \qquad \Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_j\delta^i_k - p^i g_{jk} + q_j F^i_k - q^i F_{jk}
$$

Such a connection is always metric and the structure tensor is parallel with respect to it if and only if there holds $q_k = p_a F^a_k$. We shall suppose that it is satisfied and that the connection (3.1) is an $F-$connection. The connection generators p and q have the same length. Autoparallel lines of a product semi-symmetric metric connection are holomorphically planer curves on a decomposable Riemannian space if and only if the vector p is collinear with the vector tangent to holomorphically planer curve and of constant length. So, the product semi-symmetric metric connection is a member of "projective class" of connections.

For the curvature tensor components of the connection (3.1) there holds

$$
\begin{aligned}
(3.2) \qquad R^i_{jkl} \quad = \quad & K^i_{jkl} + \delta^i_k p_{lj} - \delta^i_l p_{kj} + g_{jl}p^i_k - g_{kj}p^i_l \\
& +F^i_k q_{lj} - F^i_l q_{kj} + F_{jl}q^i_k - F_{jk}q^i_l,
\end{aligned}
$$

where $p^i_l = p_{lj}g^{ji}$, $q^i_l = q_{lj}g^{ji}$ and

$$
\begin{aligned}
p_{kj} &= \overset{0}{\nabla}_k\, p_j - p_k p_j - q_k q_j + \frac{1}{2} p_s p^s g_{kj} + \frac{1}{2} p_s q^s F_{kj}, \\
q_{kj} &= \overset{0}{\nabla}_k\, q_j - p_k q_j - q_k p_j + \frac{1}{2} p_s p^s F_{kj} + \frac{1}{2} p_s q^s g_{kj},
\end{aligned}
$$

as the lengths of vectors p and q are equal. K^i_{jkl} denote curvature tensor components of the Levi-Civita connection and $\overset{0}{\nabla}$ the covariant differentiation operator with respect to it. We can notice that $p_{ka} F^a_j = q_{kj}$. After contraction (3.1) in i and l, we obtain

$$
\begin{aligned}
R_{jk} &= K_{jk} + (2-n) p_{kj} + p_{kj} - g_{kj} p_{ls} g^{ls} + F^l_k F^s_j p_{ls} - \\
&\quad - \varphi_1 F^a_j p_{ka} - F_{jk} p_{ls} F^{ls}.
\end{aligned} \tag{3.3}
$$

In the last formula, we used the relation between p_{kj} and q_{kj}. It is necessary to express p_{kj} through Ricci tensors, curvature scalars, metric and structure tensors. After multiplying the equation (3.3) by $F^j_p F^k_q$ and substracting the result from (3.3), we obtain

$$
(2-n) p_{ls}\ {}^*O^{ls}_{kj} = R_{sl}\ {}^*O^{ls}_{kj} - K_{sl}\ {}^*O^{ls}_{kj} + \varphi_1 p_{ls}\ {}^*O^{ls}_{kt} F^t_j. \tag{3.4}
$$

and from this equation there holds immediatelly

$$
(2-n) p_{ls}\ {}^*O^{ls}_{kt} F^t_j = R_{sl}\ {}^*O^{ls}_{kt} F^t_j - K_{sl}\ {}^*O^{ls}_{kt} F^t_j + \varphi_1 p_{ls}\ {}^*O^{ls}_{kj}. \tag{3.5}
$$

After multiplying (3.4) by $2-n$ and (3.5) by φ_1 and making the sum of results, we obtain

$$
[(2-n)^2 - \varphi_1^2] p_{ls}\ {}^*O^{ls}_{kj} = (2-n-\varphi_1)[R_{sl}\ {}^*O^{ls}_{kj} - K_{sl}\ {}^*O^{ls}_{kj}]
$$

and, hence,

$$
p_{ls}\ {}^*O^{ls}_{kj} = \frac{1}{2-n+\varphi_1}[R_{sl}\ {}^*O^{ls}_{kj} - K_{sl}\ {}^*O^{ls}_{kj}]. \tag{3.6}
$$

On the other hand, the equation (3.3) can be rewritten as

$$
\begin{aligned}
R_{jk} - K_{jk} &= (4-n) p_{kj} - p_{ls}\ {}^*O^{ls}_{kj} - \varphi_1 F^s_j p_{ks} - \\
&\quad - (g_{kj} g^{ls} + F_{kj} F^{ls}) p_{ls}.
\end{aligned} \tag{3.7}
$$

From this relation, we have

$$(3.8)\qquad \begin{aligned}(R_{ck}-K_{ck})F^c_j &= (4-n)p_{kc}F^c_j - p_{ls}\ {}^*O^{ls}_{kc}F^c_j\\ &-\varphi_1 p_{kj} - (F_{kj}g^{ls}+g_{kj}F^{ls})p_{ls}.\end{aligned}$$

Now we multiply (3.8) by φ_1 and (3.7) by $4-n$. After addition of results, we have

$$(3.9)\qquad \begin{gathered}[(n-4)^2-\varphi_1^2]p_{kj} = \varphi_1[(R_{ck}-K_{ck})F^c_j-\\ p_{ls}\ {}^*O^{ls}_{kc}F^c_j + (F_{kj}g^{ls}+g_{kj}F^{ls})p_{ls}] - (n-4)[R_{jk}-K_{jk}-\\ -p_{ls}\ {}^*O^{ls}_{kj} + (g_{kj}g^{ls}+F_{kj}F^{ls})p_{ls}].\end{gathered}$$

From (3.3), we obtain by transvecting g^{jk}

$$R-K = 2(2-n)p_{ls}g^{ls} - 2\varphi_1 p_{ls}F^{ls},$$

and, consequently

$$p_{ls}F^{ls} = \frac{(2-n)p_{ls}g^{ls}}{\varphi_1} - \frac{R-K}{2\varphi_1}.$$

In order to eliminate the scalar function $p_{ls}F^{ls}$, we can easily prove that there hold

$${}^*O^{ls}_{kj}g^{kj}=0,\ {}^*O^{ls}_{kc}F^c_j g^{jk}=0.$$

Then, we can transvect (3.9) by g^{jk} and finally get

$$(3.10)\qquad \begin{aligned}p_{ls}g^{ls} &= \frac{\varphi_1}{2[(n-2)^2-\varphi_1^2]}(R_{ck}-K_{ck})F^{ck} -\\ &-\frac{(n-2)}{2[(n-2)^2-\varphi_1^2]}(R-K)\end{aligned}$$

$$(3.11)\qquad p_{ls}F^{ls} = \frac{1}{2\varphi_1[(n-2)^2-\varphi_1^2]}[\varphi_1^2(R-K)-(n-2)(R_{ck}-K_{ck})F^{ck}]$$

and then both scalar functions are eliminated. Now we can find the final form of p_{kj}, but we shall involve some numerical abbreviations first:

$$(3.12)\qquad \begin{aligned}\alpha_1 &= \frac{\varphi_1+(n-2)(n-4)}{2\varphi_1[(n-2)^2-\varphi_1^2]}, \alpha_2 = \frac{n-2+(n-4)\varphi_1}{2[(n-2)^2-\varphi_1^2]},\\ \beta_1 &= (n-4)^2-\varphi_1^2, \beta_2 = 2-n+\varphi_1;\end{aligned}$$

Then

$$p_{kj}=\frac{\varphi_1}{\beta_1}(R_{ck}-K_{ck})F_j^c-\frac{\varphi_1}{\beta_2}(R_{sl}\ {}^*O_{kc}^{ls}-K_{sl}\ {}^*O_{kc}^{ls})F_j^c- \\ -\frac{n-4}{\beta_1}(R_{jk}-K_{jk})-\frac{n-4}{\beta_1\beta_2}(R_{sl}\ {}^*O_{kj}^{ls}-K_{sl}\ {}^*O_{kj}^{ls})+ \\ +\frac{1}{\beta_1}(\alpha_1(R_{ab}-K_{ab})F^{ab}-\alpha_2(R-K))F_{kj}- \\ -\frac{1}{\beta_1}(\alpha_2(R_{ab}-K_{ab})-\alpha_1(R-K))g_{kj}. \tag{3.13}$$

So we have proved that there holds

Theorem 1. *On a locally product Riemannian space, if there is given a product semi-symmetric metric connection* (3.1) *with curvature tensor components* R_{ijkl} *and Ricci tensor component* R_{jk}*, the tensor*

$$\begin{aligned}\mathcal{A}(R)= \quad & R_{ijkl}-\frac{\varphi_1}{\beta_1}[R_{cl}F_j^cg_{ik}-R_{ck}F_j^cg_{il}+R_{ck}F_i^cg_{lj}-R_{cl}F_i^cg_{kj}\\ & +R_{jl}F_{ik}-R_{jk}F_{li}+F_{jl}R_{ik}-F_{jk}R_{il}]\\ & +\frac{\varphi_1}{\beta_1}[R_{ab}\ {}^*O_{lc}^{ba}F_j^cg_{ik}-R_{ab}\ {}^*O_{kc}^{ba}F_j^cg_{il}+R_{ab}\ {}^*O_{kc}^{ba}F_i^cg_{lj}\\ & -R_{ab}\ {}^*O_{lc}^{ba}F_i^cg_{kj}+R_{ab}\ {}^*O_{lj}^{ba}F_{ik}-R_{ab}\ {}^*O_{kj}^{ba}F_{il}+\\ & +R_{ab}\ {}^*O_{ki}^{ba}F_{jl}-R_{ab}\ {}^*O_{li}^{ba}F_{jk}]\\ & +\frac{n-4}{\beta_1}(g_{ik}R_{lj}-g_{il}R_{kj}+g_{lj}R_{ki}-g_{ki}R_{li}\\ & +F_{ik}R_{cl}F_j^c-F_{li}R_{ck}F_j^c+F_{jl}R_{ck}F_i^c-F_{jk}R_{cl}F_i^c)\\ & +\frac{n-4}{\beta_1\beta_2}[R_{ab}\ {}^*O_{lj}^{ba}g_{ik}-R_{ab}\ {}^*O_{kj}^{ba}g_{il}+R_{ab}\ {}^*O_{ki}^{ba}g_{lj}-\\ & -R_{ab}\ {}^*O_{li}^{ba}g_{kj}+R_{ab}\ {}^*O_{lc}^{ba}F_j^cF_{ik}-R_{ab}\ {}^*O_{kc}^{ba}F_j^cF_{il}+\\ & +R_{ab}\ {}^*O_{kc}^{ba}F_i^cF_{jl}-R_{ab}\ {}^*O_{lc}^{ba}F_i^cF_{jk}]-\\ & -\frac{4}{\beta_1}(\alpha_1R_{ab}F^{ab}-\alpha_2R)[2(g_{lj}F_{ik}-g_{il}F_{kj}+g_{ik}F_{lj}-g_{kj}F_{li})-\\ & -2(g_{ik}g_{lj}-g_{il}g_{kj}+F_{ik}F_{lj}-F_{li}F_{kj})]\end{aligned} \tag{3.14}$$

does not depend on the generators p *and* q *of the product semi-symmetric metric connection.*

The tensor in (3.14) is equal to a tensor of the same structure, but depending on K_{ijkl}, K_{jk} and K instead of R_{ijkl}, R_{jk} and K. It is invariant on the generator of such a connection and, is the same for the whole family of product semi-symmetric metric connection.

4. A relation between holomorhically projective and mirror connection on a locally decomposable Riemannian space

The **mirror connection** on a locally decomposable Riemannian space acts along a curve which is holomorphically projective to the Levi-Civita connection such that the image of the vector tangent to that curve by the structure tensor stays parallel to itself. Its components are given by

$$\Gamma^i_{jk} = \{^{\,i}_{jk}\} + q_j F^i_k + q_k F^i_j, \tag{4.1}$$

where $\{^{\,i}_{jk}\}$ are components of the Levi-Civita connection, F^i_j are components of the structure tensor and q_j are components of a vector field.

The mirror connection is neither an $F-$connection nor metric connection. Components of its curvature tensor are given by

$$\begin{aligned} R^i_{jkl} &= K^i_{jkl} - F^i_l q_{kj} + F^i_k q_{lj} + F^i_j (q_{lk} - q_{kl}) \\ &\quad + \delta^i_k q_l q_j - \delta^i_l q_k q_j, \end{aligned} \tag{4.2}$$

where by q_{kj} are denoted components of the tensor $\overset{0}{\nabla}_k\, q_j - p_k q_j - q_k p_j$ and p_j are components of the vector $q_a F^a_j$.

We can notice that for every mirror connection on a locally decomposable Riemannian space there exists a uniquely determined connection which is holomorphically projective to the Levi-Civita connection, which is an $F-$connection, and which is projective to the mirror connection ((2.5))

$$\Lambda^i_{jk} = \{^{\,i}_{jk}\} + q_j F^i_k + q_k F^i_j + p_j \delta^i_k + p_k \delta^i_j; \tag{4.3}$$

$$\Lambda^i_{jk} = \Gamma^i_{jk} + p_j \delta^i_k + p_k \delta^i_j, \tag{4.4}$$

where $p_j = q_a F^a_j$. According to (2.2), the components of curvature tensors of these two connections satisfy the relation

$$\overline{R}^i_{jkl} = R^i_{jkl} + \delta^i_k P_{lj} - \delta^i_l P_{kj} + \delta^i_j (P_{lk} - P_{kl}), \tag{4.5}$$

where P_{kj} denotes the tensor $\nabla_k p_j - p_k p_j$ and $\overline{R}^i_{jkl}$ and R^i_{jkl} are curvature tensors components of connections given by (4.3) and (4.1) recpectively; the operator ∇_k denotes the covariant differentiation with respect to the connection with coefficients Γ^i_{jk}. As these coefficients are given by formula (4.1), for the tensor P_{kj} there holds $P_{kj} =\overset{0}{\nabla}_k p_j - 2q_k q_j - p_k p_j$ (where $\overset{0}{\nabla}_k$ denotes the covariant differentiation operator towards the Levi-Civita connection). If we denote by p_{kj} the tensor $q_{ka}F^a_j$, in the case of mirror connection, there will hold $P_{kj} =\overset{0}{\nabla}_k p_j - q_k q_j - p_k p_j$ and, consequently, $p_{kj} = P_{kj} + q_k q_j$.

By suitable transformations of the formula (4.5) (contraction, symmetrization, alternation), we obtain that there holds

$$P_{kj} = \frac{1}{n^2-1}[R_{kj} + nR_{jk} - (\overline{R}_{kj} + n\overline{R}_{jk})],$$

where R_{jk} denotes components of the Ricci tensor of mirror connection and $\overline{R}_{jk}$ denotes components of the Ricci tensor of the holomorphically projective $F-$connection. Then, for the mirror connection there hold

$$\begin{aligned} p_{kj} &= \frac{1}{n^2-1}[R_{kj} + nR_{jk} - (\overline{R}_{kj} + n\overline{R}_{jk})] + q_k q_j, \\ q_{kj} &= \frac{1}{n^2-1}[R_{ka} + nR_{ak} - (\overline{R}_{ka} + n\overline{R}_{ak})]F^a_j + q_k p_j. \end{aligned}$$

The coefficients of a holomorphically projective $F-$connection are given by the relation (4.3). Then for the components of curvature tensor of such a connection, according to Section 2, are given by

$$\begin{aligned} \overline{R}^i_{ijkl} \quad = \quad & K^i_{jkl} + \delta^i_k \overline{p}_{lj} - \delta^i_l \overline{p}_{kj} + F^i_k F^s_j \overline{p}_{ls} - F^i_l F^s_j \overline{p}_{ks} + \\ & + \delta^i_j(\overline{p}_{lk} - \overline{p}_{kl}) + F^i_j(F^t_k \overline{p}_{lt} - F^t_l \overline{p}_{kt}), \end{aligned}$$

where $\overline{p}_{ji}$ denotes the tensor $\overset{0}{\nabla}_j p_i - p_j p_i - q_j q_i$; we can see that $\overline{p}_{ji} = p_{ji}$.

If we follow the construction from Section 2, we obtain the formula (2.9). In the exactly same way, we can eliminate the tensor p_{kj}. Using the abbreviations given by (2.10), we can obtain a simplier expression for p_{kj}. Putting it into (4.2), we obtain

$$
\begin{aligned}
(4.6)\qquad & R^i_{jkl} - F^i_k F^a_j[\alpha_1 \overline{R}_{cl} F^c_a - \beta_1 \overline{R}_{al} - \alpha_2\ \bar{\pi}_{la} + \beta_2\ \bar{\pi}_{lc}\ F^c_a] \\
& + F^i_l F^a_j[\alpha_1 \overline{R}_{ck} F^c_a - \beta_1 \overline{R}_{ak} - \alpha_2\ \bar{\pi}_{ka} + \beta_2\ \bar{\pi}_{kc}\ F^c_a] \\
& - F^i_j F^a_k[\alpha_1 \overline{R}_{cl} F^c_a - \beta_1 \overline{R}_{al} - \alpha_2\ \bar{\pi}_{la} + \beta_2\ \bar{\pi}_{lc}\ F^c_a] \\
& + F^i_j F^a_k[\alpha_1 \overline{R}_{ck} F^c_a - \beta_1 \overline{R}_{ak} - \alpha_2\ \bar{\pi}_{ka} + \beta_2\ \bar{\pi}_{kc}\ F^c_a] \\
= \ & K^i_{jkl} - F^i_k F^a_j[\alpha_1 K_{cl} F^c_a - \beta_1 K_{al} - \alpha_2 \pi_{la} + \beta_2 \pi_{lc} F^c_a] \\
& + F^i_l F^a_j[\alpha_1 K_{ck} F^c_a - \beta_1 K_{ak} - \alpha_2 \pi_{ka} + \beta_2 \pi_{kc} F^c_a] \\
& - F^i_j F^a_k[\alpha_1 K_{cl} F^c_a - \beta_1 K_{al} - \alpha_2 \pi_{la} + \beta_2 \pi_{lc} F^c_a] \\
& + F^i_j F^a_k[\alpha_1 K_{ck} F^c_a - \beta_1 K_{ak} - \alpha_2 \pi_{ka} + \beta_2 \pi_{kc} F^c_a] \\
& + \delta^i_k q_j q_l - \delta^i_l q_j q_k.
\end{aligned}
$$

On the left-hand side of (4.6), there appear curvature tensor components (R^i_{jkl}) of the mirror connection given by (4.1), but all Ricci tensor components ($\overline{R}$) are those of holomophically projective connection (given by (4.3)) which is projectively related to the mirror connection and uniquely determined. Also, the generator of the mirror (and holomorphically projective) connection on the right-hand side of (4.6) is not fully eliminated. As the tensor on the left-hand side is completely different from the holomorphically-projective curvature tensor, we call it a **semi-invariant tensor of mirror connection of a locally decomposable Riemannian space.** It is a consequence of projectivity of these two connections.

5. On the existence of a possible tensor invariant of a mirror connection of a locally decomposable Riemannian space

For the mirror connection on a locally decomposable space with the coefficients given by the formula (4.1), the curvature tensor is expressed by the formula (4.2), where by q_{kj} are denoted components of the tensor $\overset{0}{\nabla}_k\ q_j - p_k q_j - q_k p_j$ and p_j are components of the vector $q_a F^a_j$. We are going to show here that, besides the semi-invariant which has been shown in the previous section, there is no invariant tensors of curvature type or Ricci type for the family of mirror connections. As the Levi-Civita connection is an $F-$connection, we shall focus on the tensor $p_{kj} = \overset{0}{\nabla}_k\ p_j - p_k p_j - q_k q_j$ ($p_{kj} = q_{ka} F^a_j$). If we contract the relation (4.2) in the indices $i, l,$ we obtain

$$
(5.1)\qquad R_{jk} = K_{jk} - \varphi_1 p_{ka} F^a_j + F^b_k F^a_j (p_{ab} + p_{ba}) - p_{kj} + (1-n) q_k q_j.
$$

Now we put that there holds

$$
(5.2)\qquad p_{kj} = S_{kj} + \alpha_1 p_k p_j + \alpha_2 q_k q_j + \beta_1 p_k q_j + \beta_2 q_k p_j.
$$

Our goal is to eliminate the product of generator components at the end of (5.1). From (5.1) and (5.2) we obtain

$$
\begin{aligned}
R_{jk} = {} & K_{jk} - \varphi_1 S_{ka} F^a_j + F^a_j F^b_k (S_{ab} + S_{ba}) - S_{kj} \\
& - \varphi_1 \alpha_1 p_k q_j - \varphi_1 \alpha_2 q_k p_j - \varphi_1 \beta_1 p_k p_j - \varphi_1 \beta_2 q_k q_j \\
& + \alpha_1 q_k q_j + \alpha_2 p_k p_j + \beta_1 q_j p_k + \beta_2 p_j q_k \\
& + \alpha_1 q_k q_j + \alpha_2 p_k p_j + \beta_1 p_j q_k + \beta_2 q_j p_k \\
& - \alpha_1 p_k p_j - \alpha_2 q_k q_j - \beta_1 p_k q_j - \beta_2 q_k p_j + (1-n) q_k q_j .
\end{aligned} \tag{5.3}
$$

Now we are going to make an equation system, such that the last four lines of (5.3) vanish. If it would be possible to solve such a system, (5.3) will reduce to its first line.

The equation system looks this way

$$
\begin{aligned}
-\varphi_1 \alpha_1 + \beta_2 &= 0 \\
-\varphi_1 \alpha_2 + \beta_1 &= 0 \\
-\varphi_1 \beta_1 + 2\alpha_2 - \alpha_1 &= 0 \\
-\varphi_1 \beta_2 + 2\alpha_1 - \alpha_2 - (n-1) &= 0
\end{aligned} \tag{5.4}
$$

From the first two equations, we obtain $\alpha_1 = \frac{\beta_2}{\varphi_1}; \alpha_2 = \frac{\beta_1}{\varphi_1}$. Putting these results into the third equation, we obtain $\beta_2 = (2 - \varphi_1^2)\beta_1$. If we put

$$2 - \varphi_1^2 = \theta, \tag{5.5}$$

the system will have the solution

$$\beta_1 = \frac{\varphi_1(n-1)}{\theta^2 - 1}; \beta_2 = \frac{\varphi_1 \theta (n-1)}{\theta^2 - 1}; \alpha_1 = \frac{\theta(n-1)}{\theta^2 - 1}; \alpha_2 = \frac{n-1}{\theta^2 - 1}. \tag{5.6}$$

As $\theta^2 - 1 = (\theta - 1)(\theta + 1) = (1 - \varphi_1^2)(3 - \varphi_1^2)$. It will be possible to solve the system if and only if $\varphi_1 \neq \pm 1$ (as $\varphi_1 = p - q$ is an integer) and then we have

$$R_{jk} = K_{jk} - \varphi_1 S_{ka} F^a_j + F^a_j F^b_k (S_{ab} + S_{ba}) - S_{kj}, \tag{5.7}$$

where S_{kj} is given by the equation (5.2). If we put $T_{jk} = R_{jk} - K_{jk}$, there will hold

$$T_{jk} = -\varphi_1 S_{ka} F^a_j + F^a_j F^b_k (S_{ab} + S_{ba}) - S_{kj}. \tag{5.8}$$

Now we shall calculate the components of the tensor S_{kj} on the component-subspaces, as well as on the mixed ones, using separating coordinate system. If

$M_n = M_p \times M_q$ and if on M_p, $F^i_j = \delta^\alpha_\beta$, then on M_q there holds $F^i_j = -\delta^\varkappa_\lambda$. The indices from the beginning of alphabet $(\alpha, \beta, \gamma, ...)$ are related to the subspace M_p and the indices from the middle of alphabet $(\varkappa, \lambda, \mu, ...)$ are related to the subspace M_q. According to the relation (5.8), there will hold

$$\begin{aligned} T_{\alpha\beta} &= -\varphi_1 S_{\beta\alpha} + S_{\alpha\beta}, \\ T_{\beta\alpha} &= -\varphi_1 S_{\alpha\beta} + S_{\beta\alpha,} \end{aligned}$$

$$S_{\alpha\beta} = \frac{1}{1-\varphi_1^2}[T_{\alpha\beta} + \varphi_1 T_{\beta\alpha}], \tag{5.9}$$

$$\begin{aligned} T_{\varkappa\alpha} &= (\varphi_1 - 2) S_{\alpha\varkappa} - S_{\varkappa\alpha}, \\ T_{\alpha\varkappa} &= -(\varphi_1 + 2) S_{\varkappa\alpha} - S_{\alpha\varkappa}, \end{aligned}$$

$$S_{\alpha\varkappa} = -\frac{1}{1 + (\varphi_1+2)(\varphi_1-2)}[T_{\alpha\varkappa} - (\varphi_1 + 2) T_{\varkappa\alpha}], \tag{5.10}$$

$$S_{\varkappa\alpha} = -\frac{1}{1 + (\varphi_1+2)(\varphi_1-2)}[T_{\varkappa\alpha} + (\varphi_1 - 2) T_{\alpha\varkappa}], \tag{5.11}$$

$$\begin{aligned} T_{\varkappa\lambda} &= \varphi_1 S_{\lambda\varkappa} + S_{\varkappa\lambda,} \\ T_{\lambda\varkappa} &= \varphi_1 S_{\varkappa\lambda} + S_{\lambda\varkappa}, \end{aligned}$$

$$S_{\lambda\varkappa} = \frac{1}{1-\varphi_1^2}[T_{\lambda\varkappa} - \varphi_1 T_{\varkappa\lambda}]. \tag{5.12}$$

According to relations (5.9)-(5.12), it is natural to expect that there holds

$$\begin{aligned} S_{kj} = {} & AT_{kj} + BT_{jk} + F^a_k(CT_{aj} + DT_{ja}) + \\ & F^a_j(ET_{ka} + GT_{ak}) + F^a_j F^b_k(HT_{ab} + KT_{ba}), \end{aligned} \tag{5.13}$$

for some constants A, B, C, D, E, G, H, K. Now we can express components of the tensor S given by the equation (5.13) by the use of the separating coordinate system. Comparing the results with equations (5.9)-(5.12), we obtain a system of eight linear equations with eight variables A, B, C, D, E, G, H, K. If it is possible to solve such a system uniquely, then S_{ij} will really look as it is given by (5.13).

The system is looking this way

$$
\begin{aligned}
A+C+E+K &= \frac{1}{1-\varphi_1^2};\\
A+C-E-K &= -\frac{1}{1+(\varphi_1+2)(\varphi_1-2)};\\
A-C+E-K &= -\frac{1}{1+(\varphi_1+2)(\varphi_1-2)};\\
A-C-E+K &= \frac{1}{1-\varphi_1^2};\\
B+D+G+H &= \frac{\varphi_1}{1-\varphi_1^2};\\
B+D-G-H &= -\frac{\varphi_1-2}{1+(\varphi_1+2)(\varphi_1-2)};\\
B-D+G-H &= \frac{\varphi_1+2}{1+(\varphi_1+2)(\varphi_1-2)};\\
B-D-G+H &= \frac{-\varphi_1}{1-\varphi_1^2}.
\end{aligned}
\tag{5.14}
$$

These are in fact two systems consisting of four linear equations of four variables each. They have a unique solution if $\varphi_1 \neq \pm 1$.

The solution of 8×8 system is

$$
\begin{aligned}
A &= \frac{1}{2(1-\varphi_1^2)} - \frac{1}{2[1+(\varphi_1+2)(\varphi_1-2)]};\\
K &= \frac{1}{2(1-\varphi_1^2)} + \frac{1}{2[1+(\varphi_1+2)(\varphi_1-2)]};\\
C &= E=0;\\
B &= -\frac{1}{1+(\varphi_1+2)(\varphi_1-2)}; H=-B;\\
G &= \varphi_1 A; D=\varphi_1 K.
\end{aligned}
\tag{5.15}
$$

Now the equation (5.13) has the form

$$
S_{kj} = AT_{kj} + BT_{jk} + \varphi_1 K F_k^a T_{ja} + \varphi_1 A F_j^a T_{ak} + F_j^a F_k^b (KT_{ba} - BT_{ab}). \tag{5.16}
$$

Our goal is to express the tensor T in terms of metric tensor, structure tensor and constants. Now we apply the above equation to the equation (5.8). We obtain

$$
\begin{aligned}
0 = {} & (K-2B-\varphi_1^2 A-1)T_{jk} + (K-A-B)T_{kj}\\
& -\varphi_1(B+A-K)F_j^a T_{ak} + \varphi_1(B+A-K)F_k^b T_{jb}\\
& +(A-\varphi_1^2 K+2B)F_j^a F_k^b T_{ab} + (B+A-K)F_j^a F_k^b T_{ba}
\end{aligned}
\tag{5.17}
$$

After calculating the coefficients in the above equation using the solution of the linear equations system, we obtain that the expression on the right-hand side of the equation (5.17) vanishes identically. So we cannot express the tensor T as we have intended. Such a method cannot give us any way to calculate any invariant for a mirror connection, neither of curvature type, nor of Ricci type.

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