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# ON SOME GENERALIZED VALUATION MONOIDS

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**Abstract.** The valuation monoids and pseudo-valuation monoids have been established through valuation domains and pseudo-valuation domains respectively. In this study we continue these lines to describe the almost valuation monoids, almost pseudo-valuation monoids and pseudo-almost valuation monoids. Further we also characterized the newly described monoids as the spirit of valuation monoids pseudo-valuation monoids.

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## 1. Introduction and Preliminaries

Let $R$ be an integral domain with quotient field $K$. A prime ideal $P$ of $R$ is called strongly prime if $xy \in P$, where $x$, $y \in K$, then $x \in P$ or $y \in P$ (alternatively $P$ is strongly prime if and only if $x^{-1}P \subset P$ whenever $x \in K \backslash R$ [10, Definition, p.2]). A domain $R$ is called a pseudo-valuation domain if every prime ideal of $R$ is a strongly prime [10, Definition, p.2]. It was shown in Hedstrom and Houston [10, Theorem 1.5(3)], an integral domain $R$ is a pseudo-valuation domain if and only if for every nonzero $x \in K$, either $x \in R$ or $ax^{-1} \in R$ for every nonunit $a \in R$. Every valuation domain is a pseudo-valuation domain [10, Proposition. 1.1] but converse is not true; for example, the valuation domain $V$ of the form $K + M$, where $K$ is a field and $M$ is the maximal ideal of $V$. If $F$ is a proper subfield of $K$, then $R = F + M$ is a pseudo-valuation domain which is not a valuation domain. Further, $R$ and $V$ have the same quotient field $L$ and $M$ is the maximal ideal of $R$ [9, Theorem A]. A quasi-local domain $(R, M)$ is a pseudo-valuation domain if and only if $x^{-1}M \subset M$ whenever $x \in K \backslash R$ [10, Theorem. 1.4]. Also, a Noetherian pseudo-valuation domain was discussed in [10]. A Noetherian domain $R$ with quotient field $K$ is a pseudo-valuation domain if and only if $x^{-1} \in R^{'}$ whenever $x \in K \backslash R$, where $R^{'}$ is the integral closure of $R$ in $K$ [10, Theorem 3.1]. $\mathbb{Z}[\sqrt{5}]_{(2,1+\sqrt{5})}$ is a Noetherian pseudo-valuation domain which is not a valuation domain and is not in the form of $D + M$ [10, Example 3.6].

It is already an established fact that there is a common structural behaviour between an integral domain $R$ and the multiplicative monoid $R^{*}(= R - \{0\})$, for

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example, an integral domain $R$ is called a valuation domain if it is a valuation monoid [11, p.167]. By a valuation monoid $H$ we mean that for all $a, b \in H$, either $a \mid_H b$ ($a$ divides $b$ in $H$) or $b \mid_H a$ ($b$ divides $a$ in $H$) (see [11, Definition 15.1]). Similarly, $H$ is called a pseudo-valuation monoid if $x \in G \backslash H$ and $a \in H \backslash H^{\times}$ (where $H^{\times}$ is a set of invertible elements of $H$) implies $x^{-1}a \in H$ [11, Definition 16.7]. An integral domain $R$ is called a pseudo valuation domain if it is pseudo-valuation monoid and vice visa. For the definitions and terminology one may consult [11].

At the present, there are numerious studies dealing with valuation domains, pseudo-valuation domains and their generalizations. For a complete survey on pseudo-valuation domain one can consult [3]. Ayman Badawi generalized pseudo-valuation domains in the perspective of arbitrary rings, for instance, a prime ideal $P$ is strongly prime if $aP$ and $bP$ are compareable for all $a, b \in R$, and $R$ is said to be a pseudo-valuation ring if each prime ideal of $R$ is strongly prime [7]. One may consult [8], [4], [5] and [6] for studying the generalization of pseudo-valuation domain in the context of an arbitrary ring. However, a reasonably different type of monoids have been explored in [11].

In this study we introduced almost valuation monoids, almost pseudo-valuation monoids and pseudo-almost valuation monoids. We used different ideal systems to characterize these monoids on same lines as adopted for valuation monoids and pseudo valuation monids. We considered a (multiplicative) monoid, a cancellative commutative semigroup having identity and with adjoined zero. We represent the semigroup operation by ordinary multiplication. As in [11], a zero element 0 with the property that $0x = 0$; yet $xy = 0$ implies $x = 0$ or $y = 0$. An excellent example of a multiplicative monoid is a multiplicative monoid of an integral domain.

## 2. Basic terminology

Here we give the already established terminology which will be helpful for understanding the work discussed in this note.

For a monoid $H$, $H^*$ represents $H \backslash \{0\}$ and $a, b \in H$ are associates if $a \mid_H b$ and $b \mid_H a$. Associates of 1 in $H$ are called units (invertible elements) and the set of units of $H$ is denoted by $H^{\times}$. Furthermore, $H$ is said to be reduced if $H^{\times} = \{1\}$. Thus $H^{\times}$ is a subgroup of $H$ and we can consider the quotient monoid $H/H^{\times}$ which is obviously reduced and it is denoted by $H_{red}$. $H$ is said to be a groupoid if $H^*$ is a group (equivalently: every nonzero element of $H$ is invertible, or $H^* = H^{\times}$). We have a quotient groupoid of a cancellative monoid $H$ in the place of quotient field as in integral domain. By a quotient groupoid of $H$ we mean a groupoid $G(H)$ such that $H \subset G(H)$ is a submonoid and $G(H) = \{c^{-1}h : h \in H \text{ and } c \in H^*\}$.

As in the case of integral domains we can also define various ideal systems on a monoid $H$. This fact has been adequately discussed in [11]. We added definitions here for better understanding. For the properties of these ideal systems one may consult [11, Chapter 2]. An ideal system $r$ of a monoid $H$ is a map on

$P(H)$, the power set of $H$, defined as $X \mapsto X_r$ such that for all $X, Y \in P(H)$ and $c \in H$ the following conditions hold:

(1) $X \cup \{0\} \subseteq X_r$,

(2) $X \subseteq Y_r$ implies $X_r \subseteq Y_r$,

(3) $cH \subseteq \{c\}_r$, and

(4) $(cX)_r = cX_r$.

An ideal $I$ is an $r-$ideal if $I = I_r$ and is $r-$finitely generated if $I = J_r$ for a finitely generated ideal $J$ of $H$. From (1) we observe that for every $r-$system we have $H_r = H$ and from (3) we conclude that every principal ideal is an $r-$ideal. If $I$ is an $r-$ideal and $X$ is any subset of $H$, then the set $(I : X) = \{x \in H \mid xX \subseteq I\}$ is an $r-$ideal and $(I : X) = (I : X_r)$. An ideal system $r$ on $H$ is said to be finitary if for each $X \in P(H)$, $X_r = F_r$, where $F$ ranges over the finite subsets of $X$. The $s-$ideal system is the map on $P(H)$ such that for $X \subset H$, we define, $X_s = \{0\}$ when $X = \varnothing$ and $XH$ when $X \neq \varnothing$. Also, a $d-$ideal system is given by $X \mapsto X_d = X$.

## 3. Almost Valuation Monoid

By [1], an integral domain $D$ is said to be an almost valuation domain if for every $0 \neq x \in K$, there is a positive integer $n$ such that either $x^n \in D$ or $x^{-n} \in D$. We first define an almost valuation monoid because it will be needed while discussing the pseudo-almost valuation monoid. After defining an almost valuation monoid we made its relation with pseudo-almost valuation monoid. By the motivation of definition of an almost pseudo-valuation domain we give the following definition.

**Definition 1.** A cancellative monoid $H$ with quotient groupoid $G(H)$ is said to be an almost valuation monoid if for any $x \in G(H)$ there exists a positive integer $n$ such that either $x^n \in H$ or $x^{-n} \in H$. Equivalently, for each pair $a, b \in H$, there is a positive integer $n = n(a, b)$ such that $a^n \mid b^n$ or $b^n \mid a^n$.

The following proposition characterizes the definition of an almost valuation monoid.

**Proposition 1.** *For a monoid $H$, the following assertions are equivalent.*

*(1) $H$ is an almost valuation monoid.*

*(2) For all $x \in G(H)\backslash\{0\}$, we have $x^n \in H$ or $x^{-n} \in H$.*

*Proof.* (1)$\Longrightarrow$(2) Suppose that $x = a^{-1}b$, where $a, b \in H\backslash\{0\}$ and $\{a^n, b^n\}_s = a^nH \cup b^nH = d^nH$, $n \in \mathbb{Z}^+$ and $d \in H$. This implies that $d^n \mid a^n$, $d^n \mid b^n$ and either $a^n \mid d^n$ or $b^n \mid d^n$. Thus we have either $a^n \mid b^n$ that is $x^n \in H$ or $b^n \mid a^n$ that is $x^{-n} \in H$.

(2)$\Longrightarrow$(1) It follows from the definition of an almost valuation monoid. □

**Proposition 2.** *A monoid $H$ with quotient groupoid $G(H)$ is an almost valuation monoid if and only if for each $x \in G(H)$, there exist $n \in \mathbb{Z}^+$ such that $x^nH \subset H$ or $H \subset x^nH$.*

*Proof.* If $H$ is an almost valuation monoid, then clearly for each $x \in G(H)$ either $x^n \in H$ or $x^{-n} \in H$ for $n \in \mathbb{Z}^+$. If $x^n \in H$, then $x^n H \subset H$ and if $x^{-n} \in H$, then $x^{-n}H \subset H$. Conversely, for any $x \in G(H)$, let $x^n H \subset H$, that is $x^n \in H$. If $H \subset x^n H$, then this implies $x^{-n}H \subset H$ and hence $x^{-n} \in H$. □

## 4. Almost pseudo-valuation monoid

We begin by defining an almost pseudo valuation monoid but pseudo-valuation monoid has already been established in [11, Definition 16.7]. First we recall [11, Definition 16.8] that "an $r-$ideal $P \in I_r(H)$ is primary or a primary $r-$ideal if $P \neq H$, and $a, b \in H$, $ab \in P$ implies $a \in P$ or $b \in rad(P)$".

**Definition 2.** (a) Let $G(H)$ be a quotient monoid of $H$ then $r-$ideal $P \in I_r(H)$ is strongly primary $r-$ideal if $a, b \in G(H)$ such that $ab \in P$ implies $a \in P$ or $b \in rad(P)$.

(b) If $H$ is a monoid and $G(H)$ its quotient groupoid, then $H$ is an almost pseudo-valuation monoid if every $r-$prime ideal $P$ of $H$ is strongly $r-$primary, that is, $P$ satisfies the property; $x$, $y \in G(H)$ such that $xy \in P$ and if $x \notin P$ implies some power of $y$ belongs to $P$.

Recall that an $r$-ideal $M \in I_r(H)$ is called $r$-maximal if $M \neq H$ and there is no $r$-ideal $J$ such that $M \subseteq J \subseteq H$ [11, Definition 6.4] and a monoid $H$ is called $r-$local, if $H$ possesses exactly one $r-$maximal $r-$ideal [11, Definition 6.5].

As an ad-hoc notation we say that a monoid $H$ is $r-$quasi $r-$local if it is not $r-$Noetherian but possesses exactly one $r-$maximal $r-$ideal.

The following theorem extends [11, Theorem 16.7] for almost pseudo-valuation monoid.

**Theorem 1.** *Let $r$ be a finitary ideal system on $H$ and $M = H \backslash H^{\times}$, then the following statements are equivalent:*

*(1) $H$ is an almost pseudo-valuation monoid.*

*(2) If $P \in r - spec(H)$ and $x$, $y \in G(H)$, then $xy \in P$ implies $x \in P$ or $y^n \in P$.*

*(3) For all $P \in r - spec(H)$ and $x \in G(H) \backslash H$, we have $x^{-n} \in (P : P)$.*

*(4) $H$ is $r-$local and or all $x \in G(H) \backslash H$, we have $x^{-n} \in (M : M)$.*

*(5) $H$ is $r-$local and $(M : M)$ is a valuation monoid with maximal primary $s-$ideal $M$.*

*(6) $H$ is $r$-local and there exists a valuation monoid $V$ for $H$ such that $\sqrt{M}$ is maximal $s$-ideal of $V$.*

*Proof.* (1) ⇒ (2) Suppose $P \in r - spec(H)$, and $x, y \in G(H)$ and $xy \in P$. If both $x$ and $y$ lie in $H$, we are done since $P$ is prime. We may assume that $y = x^{-n}(x^n y) \notin H$ and hence $x^n \notin H^{\times}$. Since $xy \notin H$ our assumption implies $x^{n-1} = y^{-1}(x^{n-1}y) \in H$, and since $x^{-(n-1)} \notin H$, it also implies that $y^{-1}x^{n-1} \in H$. Consequently, $x^n = (xy)(y^{-1}x^{n-1}) \in P$, and hence $x^n \in P$.

(2) ⇒ (3) Let $P \in r - spec(H)$ and $x \in G(H) \backslash H$ be given. If $p \in P$, then $p = (px^{-n})x^n \in P$ implies $px^{-n} \in P$. Consequently, $x^{-n}P \in P$, and therefore $x^{-n} \in (P : P)$.

(3) $\Rightarrow$ (4) We must prove that $P \subset \sqrt{Q}$ for all $P \in r - spec(H)$ and $Q \in r-\max(H)$. Let $P \neq Q$ and fix some element $q \in Q \backslash P$, if $p \in P$ then $p^{-n}q \notin H$ implies $p^n q^{-1} Q \subset \sqrt{Q}$ and hence $p^n = (p^n q^{-1})q \in \sqrt{Q}$.

(4) $\Rightarrow$ (5) If $x \in G(H) \backslash (M : M) \subset G \backslash H$, then $x^{-n} \in (M : M)$, and therefore $(M : M)$ is a valuation monoid. Since $M(M : M) \subset M$, $M$ is an $s-$ideal of $(M : M)$. If $x \in (M : M) \backslash (M : M)^{\times}$ then $x^{-n} \notin (M : M)$ implies $x^n \in H$, and since $x \notin H^{\times}$, we obtain $x^n \in M$. Therefore $M$ is the maximal primary $s-$ideal of $(M : M)$.

(5) $\Rightarrow$ (6) It is very clear.

(6) $\Rightarrow$ (1) If $x \in G(H) \backslash H$ and $a^n \in H \backslash H^{\times} = M$, then $x^{-1} \in V$, and consequently $x^{-1}a^n \in M \subset H$. □

## 5. Pseudo-almost valuation monoid

In this section we introduced some terminology, mainly as a part the motivations from a pseudo-almost valuation domain. Further, we also characterizes pseudo-almost valuation monoids.

**Definition 3.** (a) An $r-$prime ideal $P$ of $H$ is said to be a pseudo-strongly $r-$prime ideal if, whenever $x$, $y \in G$(quotient groupoid of $H$) and $xyP \subseteq P$, then there is a positive integer $m \geq 1$ such that either $x^m \in H$ or $y^m P \subseteq P$.

(b) If each prime ideal of a monoid $H$ is a pseudo-strongly $r-$prime ideal, then $H$ is called a pseudo-almost valuation domain.

Like a pseudo-almost valuation domain as in [2, Theorem 2.8] we can also define a pseudo-almost valuation monoid as follows.

**Definition 4.** A monoid $H$ is said to be pseudo-almost valuation monoid if and only if for every nonzero $x \in G(H)$, there is a positive integer $n \geq 1$ such that either $x^n \in H$ or $ax^{-n} \in H$ for every nonunit $a \in H$.

**Proposition 3.** *If $H$ is a pseudo-almost valuation monoid, then for every pseudo-strongly $r-$prime ideal $P$ of $H$, $H' = (P : P)$ is an almost valuation monoid for every $r-$prime ideal $P$ of $H$.*

*Proof.* Let $H' = (P : P)$ and $G(H')$ be the quotient monoid of $H'$ and $x \in G(H') \backslash H'$ such that $x^n \notin H'$ Hence $x^n \notin H$. Since $P$ is a pseudo-strongly $r-$prime ideal there is an $n \geq 1$ such that $x^{-n}P \subseteq P$. Hence $x^{-n} \in H'$. Thus $H'$ is an almost valuation domain. □

**Proposition 4.** *Every almost valuation monoid is a pseudo-almost valuation monoid.*

*Proof.* Let $H$ be an almost valuation monoid and $G(H)$ be a quotient groupoid of a monoid $H$ then for all $x \in G$ either $x^n \in H$ or $x^{-n} \in H$. If $x^n \in H$ then we are done otherwise $x^{-n} \in H$, let $a \in H$ be a nonunit of $H$, then $ax^{-n} \in H$ as $H$ is a monoid. □

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