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GENERALIZED SEMI-IDEALS IN TERNARY SEMIRINGS

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Abstract. We introduce the notion of a generalized semi-ideal in a ternary semiring. Various examples to establish relationships between ideals, bi-ideals, quasi-ideals and generalized semi-ideals are furnished. A criterion for a commutative ternary semiring without any divisor of zero to a ternary division semiring is given.

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1. Introduction

Ternary rings and their structures were investigated by Lister [4] in 1971. In fact, Lister characterized those additive subgroups of rings which are closed under the triple product. In 2003, T. K. Dutta and S. Kar [3] introduced the notion of a ternary semiring as a generalization of a ternary ring. Ternary semiring arises naturally as follows- consider the subset $\mathbb{Z}^-$of all negative integers of $\mathbb{Z}$. Then, $\mathbb{Z}^-$is an additive semigroup which is closed under the triple product. $\mathbb{Z}^-$ is a ternary semiring. Note that $\mathbb{Z}^-$ does not form a semiring. In [3], T. K. Dutta and S. Kar introduced the notions of left, right lateral ideals of ternary semirings and also characterized regular ternary semirings. In 2005, S. Kar [1] introduced the notions of quasi-ideals and bi-ideals in a ternary semiring. The notion of a generalized semi-ideal in a ring has been introduced and studied by T. K. Dutta in [2]. In this paper we introduce the notion of generalized semi-ideals in a ternary semiring and study them. Also, we establish the relationship between generalized semi-ideals, ideals, bi-ideals, etc. in a ternary semiring and study some properties of generalized semi-ideals in ternary semirings.

2. Preliminaries

For preliminaries we refer to ([1] and [3]).

Definition 2.1. An additive commutative semigroup S, together with a ternary multiplication denoted by [ ] is said to be a ternary semiring if

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i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,
ii) $[(a + b)cd] = [acd] + [bcd]$,
iii) $[a(b + c)d] = [abd] + [acd]$,
iv) $[ab(c + d)] = [abc] + [abd]$ for all $a, b, c, d, e \in S$.

Throughout S will denote a ternary semiring unless otherwise stated.

Definition 2.2. If there exists an element $0 \in S$ such that $0 + x = x$ and $[0xy] = [xy0] = [x0y] = 0$ for all $x, y \in S$, then 0 is called the zero element of S. In this case we say that S is a ternary semiring with zero.

Definition 2.3. S is called a commutative ternary semiring if $[abc] = [bac] = [bca]$, for all $a, b, c \in S$.

Definition 2.4. An additive subsemigroup T of S is called a ternary subsemiring of S if $[t_1t_2t_3] \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.5. An element a in a ternary semiring S is called regular if there exists an element $x \in S$ such that $[axa] = a$. A ternary semiring S is called regular if all of its elements are regular.

Definition 2.6. A ternary semiring S is said to be zero divisor free (ZDF) if for $a, b, c \in S$, $[abc] = 0$ implies that $a = 0$ or $b = 0$ or $c = 0$.

Definition 2.7. A ternary semiring S with $|S| \geq 2$ is called a ternary division semiring if for any non-zero element a of S, there exists a nonzero element $b \in S$ such that $[abx] = [bax] = [xab] = [xba] = x$, for all $x \in S$.

Definition 2.8. A left (right/lateral) ideal I of S is an additive subsemigroup of S such that $[s_1s_2i] \in I$ ($[is_1s_2] \in I/[s_1is_2] \in I$) for all $i \in I$,for all $s_1, s_2 \in S$. If I is a left, a right and a lateral ideal of S, then I is called an ideal of S.

Definition 2.9. An additive subsemigroup Q of a ternary semiring S is called a quasi-ideal of S if $[QSS] \bigcap ([SQS] + [SSQSS]) \bigcap [SSQ] \subseteq Q$.

Definition 2.10. A ternary subsemiring B of a ternary semiring S is called a bi-ideal of S if $[BSBSB] \subseteq B$.

3. Generalized semi-ideals in ternary semirings

Generalized semi-ideals in semirings are introduced and studied by T. K. Dutta in [1].As a generalization, we define generalized semi-ideals in ternary semirings.

Definition 3.1. Let S be a ternary semiring. A non-empty subset A of S satisfying the condition $a + b \in A$, for all $a, b \in A$ is called

i) generalized left semi-ideal of S if $[[xxx]xa] \in A$ for all $a \in A$ for all $x \in S$,

ii) generalized right semi-ideal of S if $[axx]xx] \in A$ for all $a \in A$, for all $x \in S$,

iii) generalized lateral semi-ideal of S if $[xxa]xx] \in A$ for all $a \in A$, for all $x \in S$,

iv) generalized semi-ideal of S if it is a generalized left semi-ideal, a generalized right semi-ideal and a generalized lateral semi-ideal of S.

Example 3.2. Consider a ternary semiring $\mathbb{Z}$ of all integers. The subset A of $\mathbb{Z}$ containing all non-negative integers and the set B of all non-positive integers are generalized semi-ideals of $\mathbb{Z}$.

Remark 3.3. The concepts of generalized semi-ideal and ternary subsemiring are independent in a ternary semiring. That is, every ternary subsemiring of ternary semiring need not be a generalized semi-ideal of ternary semiring and every generalized semi-ideal of ternary semiring need not be a ternary subsemiring of ternary semiring. For this consider the following examples.

Example 3.4. Let $S = M_2\ (\mathbb{Z}_0^-)$ be the ternary semiring of the set of all 2x2 square matrices over $\mathbb{Z}_0^-$, the set of all non-positive integers.
Let $T = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} / a \in \mathbb{Z}_0^-\}$. T is a ternary subsemiring of S, but T is not a generalized semi-ideal of S.

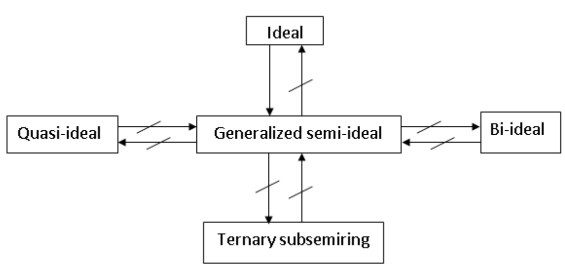
Example 3.5. Let $S = \{\dots, -2i, -i, 0, i, 2i, \dots\}$ be a ternary semiring with respect to addition and complex triple multiplication. Let $A = \{0, i, 2i, \dots\}$. A is a generalized semi-ideal of S, but not a ternary subsemiring of S.

Every ideal is a generalized semi-ideal of S but converse need not be true.

Remark 3.6. Every quasi-ideal need not be a generalized semi-ideal and every generalized semi-ideal i need not be quasi-ideal in S. (in Example 3.4), T is a quasi-ideal of S, but T is not a generalized semi-ideal of S. (in Example 3.5), A is generalized semi-ideal of S, but not a quasi-ideal of S.

Every quasi-ideal is a bi-ideal in S [2]. Hence, bi-ideals and generalized semi-ideals in S are independent concepts.

The flow chart of the relationship between ideals, bi-ideals, quasi-ideals, ternary subsemiring and generalized semi-ideals in a ternary semiring is given below.



4. Properties of generalized semi-ideals

The intersection of an arbitrary collection of generalized semi-ideals of a ternary semiring is a generalized semi-ideal of a ternary semiring. But, the union of two generalized semi-ideals of a ternary semiring may not be a generalized semi-ideal of a ternary semiring. This we establish in the following example.

Let $S = \{\dots, -2i, -i, 0, i, 2i, \dots\}$ be a ternary semiring with respect to addition and complex triple multiplication. Then $I = \{\dots, -4i, -2i, 0, 2i, \dots\}$ and $J = \{\dots, -10i, -5i, 0, 5i, 10i, \dots\}$ are two generalized semi-ideals of S, but $I \bigcup J$ is not a generalized semi-ideal of S.

Theorem 4.1. *Let A be a generalized semi-ideal of a ternary semiring S and let T be a ternary subsemiring of S. If $A \bigcap T \neq \emptyset$, then $A \bigcap T$ is a generalized semi-ideal of T.*

Proof. Let $a, b \in A \bigcap T$. Then $a + b \in A \bigcap T$. For $x \in T$ and $a \in A \bigcap T$ we have $[[xxx]xa] \in A \bigcap T$, $[[axx]xx] \in A \bigcap T$, $[[xxa]xx] \in A \bigcap T$. Hence, $A \bigcap T$ is generalized semi-ideal of S. □

Theorem 4.2. *If A and B are generalized semi-ideals of a ternary semiring S, then $A + B = \{a + b/a \in A, b \in B\}$ is a generalized semi-ideal of S.*

Proof. Let $x, y \in A + B$. Hence $x = a + b, y = c + d$, for $a, c \in A$ and $b, d \in B$. Then $x + y = (a + b) + (c + d) = (a + c) + (b + d) \in A + B$. Let $t \in S$ and $x \in A + B$, hence $x = a + b$ for some $a \in A$ and $b \in B$. Therefore, $[[ttt]tx] = [[ttt]t(a + b)] = [[ttt](ta)] + [[ttt]tb] \in A + B$. Similarly, we have $[[ttx]tt] = [[tt(a + b)]tt] = [([tta] + [ttb])tt] = [[tta]tt] + [[ttb]tt] \in A + B$ and $[[xtt]tt] = [[(a + b)tt]tt] = [([att] + [btt])tt] = [[att]tt] + [[btt]tt] \in A + B$. Thus, $A + B$ is a generalized semi -ideal of S. □

Theorem 4.3. *Let S be a ternary semiring with zero. Let A and B be two generalized semi-ideals with zero. Then $A + B$ is the smallest generalized semi-ideal of S containing both A and B.*

Proof. From Theorem 4.2 $A + B$ is a generalized semi-ideal of S. Since $0 \in A$, $0 \in B$ we get $0 \in A + B$ and for $a \in A, a = a + 0 \in A + B$. Hence, $A \subseteq A + B$. Similarly, $B \subseteq A + B$. Let I be any other generalized semi-ideal containing both A and B. Let $x \in A + B$. Then $x = a + b$, for some $a \in A$ and $b \in B$. Hence $x = a + b \in I$. Therefore $A + B \subseteq I$. Thus, $A + B$ is the smallest generalized semi-ideal containing both A and B. □

If A, B, C are subsets of S, then by $[ABC]$ we mean the set of all finite sums of the form $\sum [a_i b_i c_i]$ where $a_i \in A, b_i \in B, c_i \in C$ ([2]).

Theorem 4.4. *Let A be a generalized left semi-ideal of a ternary semiring S. Then $[ABC]$ is a generalized left semi-ideal, for any non-empty subsets B and C of S.*

Proof. For $x, y \in [ABC]$, let $x = \sum_{i=1}^{n} [a_i b_i c_i]$ and $y = \sum_{j=1}^{m} [a_i b_i c_i]$. Obviously, $x + y$ is a finite sum of the form $\sum [a_i b_i c_i]$. Hence $x + y \in [ABC]$. For $t \in S$, we have $[[ttt]tx] = [[ttt]t \sum_{i=1}^{n} [a_i b_i c_i]] = \sum_{i=1}^{n} [[ttt]t[a_i b_i c_i]] = \sum_{i=1}^{n} [[[ttt]ta_i] b_i c_i] \in [ABC]$. Since A is generalized left semi-ideal. Therefore, $[ABC]$ is a generalized left semi-ideal of S. □

Theorem 4.5. *Let A be a generalized left (right) semi-ideal and B be a bi-ideal of a ternary semiring S. Then $[ABB]$ ($[BBA]$)is a generalized left (right) semi-ideal as well as bi-ideal of S.*

Proof. Let $x, y, z \in [ABB]$. Hence $x = \sum_{i=1}^{n}[a_i b_i c_i]$, $y = \sum_{i=n+1}^{m}[a_i b_i c_i]$, $z = \sum_{i=m+1}^{p}[a_i b_i c_i]$ for all $a_i \in A$ and $b_i, c_i \in B$. Thus $x + y$ is the finite sum of the form $\sum[a_i b_i c_i]$. Hence $x + y \in [ABB]$. Let $t \in S$ and $x = \sum_{i=1}^{n}[a_i b_i c_i] \in [ABB]$. Then $[[ttt]tx] = [[ttt]t\sum_{i=1}^{n}[a_i b_i c_i]] = \sum_{i=1}^{n}[[[ttt]ta_i]b_i c_i] \in [ABB]$. Hence $[ABB]$ is generalized left semi-ideal of S. Now $[[ABB][ABB][ABB]] = [A[[B[BAB]B]AB]B] \subseteq [A[BSBSB]B] \subseteq [ABB]$. (Since $[BAB] \subseteq S$ and B is a bi-ideal).This shows that $[ABB]$ is ternary subsemiring of S.
Again,$[[ABB]S[ABB]S[ABB]] = [A[B[BSA]B[BSA]B]B] \subseteq [A[BSBSB]B] \subseteq [ABB]$(Since B is a bi-ideal). Hence $[ABB]$ is bi ideal of S. □

Theorem 4.6. *Let A and B be ternary subsemirings of a ternary semiring S such that $A^3 = A$ and A be a left ideal of B and B be a generalized left semi-ideal of S. Then A is a generalized left semi-ideal of S.*

Proof. Let $a \in A$, therefore $a = [a_1 a_2 a_3]$, where $a_1, a_2, a_3 \in A$. Now for any $x \in S$,$[xxx]xa] = [[xxx]x[a_1 a_2 a_3]] = [[[xxx]xa_1]a_2 a_3] \in [Ba_2 a_3] \subseteq A$ (Since A is a left ideal of B, $a_1 \in A \subset B$, B is a generalized left semi-ideal of S). Therefore, A is a generalized left semi-ideal of S. □

Theorem 4.7. *If G is a generalized left (right) semi-ideal of S and T_1, T_2 are two ternary subsemirings of S, then $[GT_1T_2]$ ($[T_1T_2G]$) is a generalized left (right) semi-ideal of S.*

Proof. For any $a, b \in [GT_1T_2], a = \sum_{i=1}^{n}[g_i t_i t_i^{'}]$ and $b = \sum_{i=n+1}^{m}[g_i t_i t_i^{'}]$, for $g_i \in G, t_i \in T_1, t_i^{'} \in T_2$. Therefore $a + b$ is the finite sum of the form $\sum[g_i t_i t_i^{'}]$ will imply $a + b \in [GT_1T_2]$. Let $a = \sum_{i=1}^{n}[g_i t_i t_i^{'}] \in [GT_1T_2]$ and let $x \in S$. Then $[[xxx]xa] = [[xxx]x\sum_{i=1}^{n}[g_i t_i t_i^{'}]] = \sum_{i=1}^{n}[[[xxx]xg_i]t_i t_i^{'}] \in [GT_1T_2]$. Hence, $[GT_1T_2]$ is a generalized left semi-ideal of S. □

A necessary and sufficient condition for a commutative ternary semiring S without any divisors of zero to be ternary division semiring is given in the following theorem.

Theorem 4.8. *A commutative ternary semiring S without any divisors of zero will be ternary division semiring iff for any generalized semi-ideal A, $a \in S\backslash A$ (the complement of A in S) and $x(\neq 0) \in S$ implies $[[xxx]xa] \in S\backslash A$.*

Proof. Suppose a commutative ternary semiring S without any divisor of zero will be ternary division semiring. Let A be a generalized semi-ideal of S. Select $a \in S\backslash A$ and $x(\neq 0) \in S$. Hence, $\exists y(\neq 0) \in S$ such that
$[xyz] = [yxz] = [zxy] = [zxy] = z$, for all $z \in S$. Therefore, $[xya] = [yxa] = [axy] = [ayx] = a$. This proves that $[[xxx]xa] \in S\backslash A$. Assume that $[[xxx]xa] = x^4 a \in A$. Therefore, $a = [[yxy]^4 ax^4] \in A$. (Since S is commutative, A is generalized semi-ideal), which is a contradiction. Hence, $[[xxx]xa] \in S\backslash A$.

Conversely, suppose that for any generalized semi-ideal A, $a \in S\backslash A$ and $x \neq 0 \in S$ implies $[xxx]xa] \in S\backslash A$. To prove that S is a ternary division semiring, that is to prove that for $a(\neq 0) \in S \exists b(\neq 0) \in S$ such that $[abS] = S$. If possible, let $[abS] \neq S$ and $y \in S\backslash[abS]$, then $[[aaa]ay] = [a^3ay] = [aa^3y] = [aby] \in [abS]$, where $b = a^3(\neq 0) \in S$ which is contradiction because $[a^3ay] \in S\backslash[abS]$. Hence, $[abS] = S$. Therefore, S is a ternary division semiring. □

Suppose A is a generalized semi-ideal of a commutative ternary semiring S. Let $\beta(A)$ denote the set of all those elements a for which there exists a nonzero element $x \in S$ such that $[[xxx]xa] \in A$. It is then clear that $A \subseteq \beta(A)$. Further we have the following theorem.

Theorem 4.9. *Let S be a commutative ternary semiring without any divisor of zero. If A is a generalized semi-ideal of S, then $\beta(A)$ is also a generalized semi-ideal of S.*

Proof. Let $a, b \in \beta(A)$. So, there exist non-zero elements $x, y \in S$ such that $p = [[xxx]xa] \in A$,$q = [[yyy]yb] \in A$. Now

$$\begin{aligned} \varepsilon &= [[xxx]x[yyy]y(a+b)] \\ &= [[xxx]x[yyy]ya] + [[xxx]x[yyy]yb] \\ &= [[yyy]y[[xxx]xa]] + [[xxx]x[[yyy]yb]] \\ &= [[yyy]yp] + [[xxx]xq] \in A. \end{aligned}$$

For $z(\neq 0) \in S$, $[[z\ z\ z]\ z\varepsilon] \in A$ (Since A is a generalized semi-ideal of S)

Therefore, $[[[x\ y\ z][x\ y\ z][x\ y\ z]][x\ y\ z](a+b)] \in A$. Hence $(a+b) \in \beta(A)$. For $a \in \beta(A)$, $[[xxx]xa] \in A$. Let $z \in S$, hence

$$[[xxx]x[[zzz]za]] = [[zzz]z[[xxx]xa]] \in A.$$

Therefore, $[[zzz]za] \in \beta(A)$ for all $z \in S$. Therefore, $\beta(A)$ is a generalized semi-ideal of S. □

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