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NOTE ON $\star-$CONNECTED IDEAL SPACES

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Abstract. In [1], Ekici and Noiri introduced and studied $\star-$connected and $\star_s-$connected ideal spaces. We further study the properties of $\star-$connected and $\star_s-$connected ideal spaces.

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1. Introduction and preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski [3] and Vaidyanathaswamy [4]. An *ideal* $\mathcal{I}$ on a set X is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^\star$:$\wp(X) \to \wp(X)$, called a *localfunction* [3] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X, A^\star(\mathcal{I}, \tau)=\{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local function [2, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star()$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called the $\star-$*topology*, finer than τ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$ [4]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an *ideal space*. $\star-$connected ideal space and $\star_s-$connected sets are introduced and studied by Ekici and Noiri in [1]. In section 2 of this paper, we further study the properties of these sets and give a characterization of $\star-$connected ideal spaces.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $cl(A)$ and $int(A)$ will, respectively, denote the *closure* and *interior* of A in (X, τ) and $int^\star(A)$ will denote the *interior* of A in $(X, \tau^\star)$. Subsets of X closed in $(X, \tau^\star)$ are called $\star-$*closed* sets. A subset A of X in an ideal space $(X, \tau, \mathcal{I})$ is $\star-$*closed* if and only if $A^\star \subset A$ [2]. An ideal space $(X, \tau, \mathcal{I})$ is called $\star-$*connected* [1] if X cannot be written as the disjoint union of a nonempty open set and a nonempty $\star-$open set. Clearly, every $\star-$connected space is connected but the converse is not true [1, Remark 5]. If $\mathcal{I} = \{\emptyset\}$, then $\tau = \tau^\star$ and so $\star-$connectedness coincides with connectedness. Nonempty

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subsets A and B of an ideal space $(X, \tau, \mathcal{I})$ are said to be $\star-$ *separated* [1] if $cl^\star(A) \cap B = A \cap cl(B) = \emptyset$. Clearly, separated sets are $\star-$separated but one can easily show that the converse is not true. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is called $\star_s-$*connected* [1] if A is not the union of two $\star-$separated sets in $(X, \tau, \mathcal{I})$. The ideal space $(X, \tau, \mathcal{I})$ is called $\star_s-$ *connected* if X is $\star_s-$ *connected* as a subset. The following lemma will be useful in the sequel.

Lemma 1.1. *[1, Theorem 14] Let $(X, \tau, \mathcal{I})$ be an ideal space. If A is a $\star_s-$connected set of X and H, G are $\star-$separated sets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

2. Main Results

In this section, we give the properties of $\star-$separated, $\star-$connected subsets. The following Theorem 2.1 deals with $\star-$separated sets. If $\mathcal{I} = \{\emptyset\}$, in Theorem 2.1, we get Corollary 2.3.

Theorem 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If* A *and* B *are nonempty disjoint sets such that A is open and B is $\star-$open, then* A *and* B *are $\star-$separated.*

Proof. $A \cap B = \emptyset$ implies that $A \subset X - B$ and so $cl^\star(A) \subset cl^\star(X - B) = X - B$ which implies that $cl^\star(A) \cap B = \emptyset$. Again, $B \subset X - A$ implies that $cl(B) \subset cl(X - A) = X - A$ and so $cl(B) \cap A = \emptyset$. Therefore, A and B are $\star-$separated. □

Corollary 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the disjoint nonempty open sets of X are $\star-$separated.*

Corollary 2.3. *In any space (X, τ), the disjoint nonempty open sets are separated.*

Theorem 2.4. *If every pair of distinct points of a subset* E *of an ideal space $(X, \tau, \mathcal{I})$ are elements of some $\star_s-$connected subset of* E*, then* E *is a $\star_s-$connected subset of* X.

Proof. Suppose E is not $\star_s-$connected. Then there exist nonempty subsets A and B of X such that $cl^\star(A) \cap B = \emptyset = A \cap cl(B)$ and $E = A \cup B$. Since A and B are nonempty, there exists a point $a \in A$ and a point $b \in B$. By hypothesis, a and b must be elements of a $\star_s-$connected subset C of E. Since $C \subset A \cup B$, by Lemma 1.1, either $C \subset A$ or $C \subset B$. Consequently, either a and b are both in A or both in B. Let $a, b \in A$. Then, $A \cap B \neq \emptyset$, a contradiction to the fact that A and B are disjoint. Therefore, E must be $\star_s-$connected. □

Theorem 2.5. *[1] Let Y be an open subset of an ideal space $(X, \tau, \mathcal{I})$. Then, the following are equivalent.*

(a) Y is $\star_s-$connected in X.
(b) Y is $\star-$connected in X.

Theorem 2.6. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then X is $\star-$connected if and only if X is $\star_s-$connected.*

Proof. The proof follows from Theorem 2.5. □

The following theorem gives a property of $\star-$separated sets. If $\mathcal{I} = \{\emptyset\}$ in Theorem 2.7, we get Corollary 2.8.

Theorem 2.7. *Let A and B be two $\star-$separated sets in an ideal space $(X, \tau, \mathcal{I})$. If C and D are nonempty subsets such that $C \subset A$ and $D \subset B$, then C and D are also $\star-$separated.*

Proof. Since A and B are $\star-$separated, $cl^\star(A) \cap B = \emptyset = A \cap cl(B)$. Now, $cl^\star(C) \cap D \subset cl^\star(A) \cap B = \emptyset$ and so $cl^\star(C) \cap D = \emptyset$. Similarly, we can prove that $C \cap cl(D) = \emptyset$. Hence C and D are $\star-$separated. □

Corollary 2.8. *Let A and B be two separated sets in (X, τ). If C and D are nonempty subsets such that $C \subset A$ and $D \subset B$, then C and D are also separated.*

The following theorem gives a property of $\star_s-$connected subsets. If $\mathcal{I} = \{\emptyset\}$ in Theorem 2.9, we get Corollary 2.11.

Theorem 2.9. *If A is a $\star_s-$connected subset of a $\star_s-$connected ideal topological space $(X, \tau, \mathcal{I})$ such that $X - A$ is the union of two $\star-$separated sets B and C, then $A \cup B$ and $A \cup C$ are $\star_s-$connected.*

Proof. Suppose $A \cup B$ is not $\star_s-$connected. Then there exist two nonempty $\star-$separated sets G and H such that $A \cup B = G \cup H$. Since A is $\star_s-$connected, $A \subset A \cup B = G \cup H$, by Lemma 1.1, either $A \subset G$ or $A \subset H$. Suppose $A \subset G$. Since $A \cup B = G \cup H$, $A \subset G$ implies that $A \cup B \subset G \cup B$ and so $G \cup H \subset G \cup B$. Hence $H \subset B$. Since B and C are $\star-$separated, H and C are also $\star-$separated. Thus, H is $\star-$separated from G as well as C. Now, $cl^\star(H) \cap (G \cup C) = (cl^\star(H) \cap G) \cup (cl^\star(H) \cap C) = \emptyset$ and $H \cap cl(G \cup C) = H \cap (cl(G) \cup cl(C)) = (H \cap cl(G)) \cup (H \cap cl(C)) = \emptyset$. Therefore, H is $\star-$separated from $G \cup C$. Since $X - A = B \cup C$, $X = A \cup (B \cup C) = (A \cup B) \cup C = (G \cup H) \cup C$, since $A \cup B = G \cup H$ and so $X = (G \cup C) \cup H$. Thus, X is the union of two nonempty $\star-$separated sets $G \cup C$ and H, which is a contradiction. Similar contradiction will arise if $A \subset H$. Hence, $A \cup B$ is $\star_s-$connected. Similarly, we can prove that $A \cup C$ is $\star_s-$connected. □

Corollary 2.10. *If A is a connected subset of a connected space (X, τ) such that $X - A$ is the union of two separated sets B and C, then $A \cup B$ and $A \cup C$ are connected.*

The following example shows that the union of two $\star_s-$connected sets is not a $\star_s-$connected set. Theorem 2.12 shows that the union of two $\star_s-$connected sets is a $\star_s-$connected set, if none of them is $\star-$separated.

Example 2.11. Consider the ideal space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{b, c\}, \{a, b, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. If $A = \{a, b\}$ and $B = \{a, d\}$, then A and B are $\star_s-$connected. But $A \cup B = \{a, b, d\} = \{b\} \cup \{a, d\}$. Here $cl^\star(\{b\}) \cap \{a, d\} = \{b\} \cap \{a, d\} = \emptyset$ and $\{b\} \cap cl(\{a, d\}) = \{b\} \cap \{a, d\} = \emptyset$ and so $\{b\}$ and $\{a, d\}$ are $\star-$separated sets. Hence, $A \cup B$ is not $\star_s-$connected.

Theorem 2.12. *If A and B are $\star_s-$connected sets of an ideal space $(X, \tau, \mathcal{I})$ such that none of them is $\star-$separated, then $A \cup B$ is $\star_s-$connected.*

Proof. Let A and B be $\star_s-$connected in X. Suppose $A \cup B$ is not $\star_s-$connected. Then, there exist two nonempty disjoint $\star-$seperated sets G and H such that $A \cup B = G \cup H$. Since A and B are $\star_s-$connected, by Lemma 1.1, either $A \subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$. Now if $A \subset G$ and $B \subset H$, then $A \cap H = B \cap G = \emptyset$. Therefore, $(A \cup B) \cap G = (A \cap G) \cup (B \cap G) = (A \cap G) \cup \emptyset = A \cap G = A$. Also, $(A \cup B) \cap H = (A \cap H) \cup (B \cap H) = B \cap H = B$. Similarly, if $A \subset H$ and $B \subset G$, then $(A \cup B) \cap G = A$ and $(A \cup B) \cap H = B$. Now, $((A \cup B) \cap H) \cap cl((A \cup B) \cap G) \subset (A \cup B) \cap H \cap cl(A \cup B) \cap cl(G) = (A \cup B) \cap H \cap cl(G) = \emptyset$ and $cl^\star((A \cup B) \cap H) \cap ((A \cup B) \cap G) \subset cl^\star(A \cup B) \cap cl^\star(H) \cap (A \cup B) \cap G = (A \cup B) \cap cl^\star(H) \cap G = \emptyset$. Therefore, $(A \cup B) \cap G$ and $(A \cup B) \cap H$ are $\star-$seperated sets. Thus, A and B are $\star-$seperated, which is a contradiction. Hence, $A \cup B$ is $\star_s-$connected. □

Theorem 2.13. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\{A_\alpha\}$ be a family of $\star_s-$connected subspaces of X, and A be a $\star_s-$connected subspace of X. If $A \cap A_\alpha \neq \emptyset$ for every α, then $A \cup (\bigcup A_\alpha)$ is $\star_s-$connected.*

Proof. Let $Y = A \cup (\bigcup A_\alpha)$. Suppose that Y is not $\star_s-$connected. Then there exist $\star-$separated sets H and G such that $Y = G \cup H$. Since A is $\star_s-$connected, by Lemma 1.1, either $A \subset G$ or $A \subset H$. Suppose $A \subset H$. For each α, since $A \cap A_\alpha \neq \emptyset$, there exists $x_\alpha \in A \cap A_\alpha$ which implies that $x_\alpha \in A$ and $x_\alpha \in A_\alpha$. Since $A \subset H$, $x_\alpha \in H$ and so $x_\alpha \notin cl(G)$, which implies that $x_\alpha \notin G$. Since each A_α is $\star_s-$connected, either $A_\alpha \subset G$ or $A_\alpha \subset H$. Now, $x_\alpha \in A_\alpha$ and $x_\alpha \notin G$ implies that $A_\alpha \subset H$, which in turn implies that $Y = A \cup (\bigcup A_\alpha) \subset H$ and so $G = \emptyset$, a contradiction. Thus, Y is $\star_s-$connected. □

If $\mathcal{I} = \{\emptyset\}$ in the above Theorem 2.13, we have the following corollary.

Corollary 2.14. *Let (X, τ) be a space, $\{A_\alpha\}$ be a family of connected subsets of X, and A be a connected subset of X. If $A \cap A_\alpha \neq \emptyset$ for every α, then $A \cup (\bigcup A_\alpha)$ is connected.*

Theorem 2.15. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. If C is $\star_s-$connected subspace of* X *that intersects both A and $X - A$, then C intersects $Bd(A)$, the boundary of A.*

Proof. Suppose $C \cap Bd(A) = \emptyset$. Then $C \cap cl(A) \cap cl(X - A) = \emptyset$. Now, $C = C \cap X = C \cap (A \cup (X - A)) = (C \cap A) \cup (C \cap (X - A))$. Also, $cl^\star(C \cap A) \cap (C \cap (X - A)) \subset cl^\star(C) \cap cl^\star(A) \cap C \cap (X - A) = C \cap cl^\star(A) \cap (X - A) = \emptyset$ and $(C \cap A) \cap cl(C \cap (X - A)) \subset C \cap A \cap cl(C) \cap cl(X - A) = C \cap cl(X - A) \cap A = \emptyset$. Thus, $C \cap A$ and $C \cap (X - A)$ form a $\star-$separation for C, which is a contradiction. Hence, $C \cap Bd(A) \neq \emptyset$. □

Corollary 2.16. *Let $(X, \tau, \mathcal{I})$ be a $\star-$connected ideal space. Then every nonempty proper subset has a nonempty boundary.*

Theorem 2.17. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If the union of two $\star-$separated sets is a closed set, then one set is closed and the other is $\star-$closed.*

Proof. Let A and B be two $\star-$separated sets such that $A\cup B$ is closed. Then $A\cap cl^\star(B)=\emptyset=cl(A)\cap B$. Since $A\cup B$ is closed, $A\cup B=cl(A)\cup cl(B)$. Now, $cl(A)=cl(A)\cap(cl(A)\cup cl(B))=cl(A)\cap(A\cup B)=(cl(A)\cap A)\cup(cl(A)\cap B)=A\cup\emptyset=A$ and so A is closed. Also, $B\subset A\cup B$ implies that $cl^\star(B)\subset cl^\star(A\cup B)\subset cl(A\cup B)=A\cup B$ and so $cl^\star(B)=cl^\star(B)\cap(A\cup B)=(cl^\star(B)\cap A)\cup(cl^\star(B)\cap B)=\emptyset\cup B=B$. Hence B is $\star-$closed. □

Theorem 2.18. *An ideal space $(X,\tau,\mathcal{I})$ is $\star-$connected if and only if no nonempty proper subset of X is both open and $\star-$closed.*

Proof. The proof follows from the definition of $\star-$connected ideal spaces. □

If $\mathcal{I}=\{\emptyset\}$ in the above Theorem 2.18, we have the following corollary.

Corollary 2.19. *A space (X,τ) is connected if and only if no nonempty proper subset of X is both open and closed.*

The union of all $\star_s-$connected subset of X in an ideal space $(X,\tau,\mathcal{I})$ containing a point $x\in X$ is said to be the $\star-$*component* [1] of X containing x. Moreover, in [1], it is established that every $\star-$component is a maximal $\star_s-$connected $\star-$closed subset of X. The following theorem gives a property of $\star_s-$connected subset of X.

Theorem 2.20. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then, each $\star_s-$connected subset of X which is both open and $\star-$closed is a $\star-$component of X.*

Proof. Let A be a $\star_s-$connected subset of X such that A is both open and $\star-$closed. Let $x\in A$. Since A is a $\star_s-$connected subset of X containing x, if C is the $\star-$component containing x, then $A\subset C$. Let A be a proper subset of C. Then C is nonempty and $C\cap(X-A)\neq\emptyset$. Since A is open and $\star-$closed, $X-A$ is closed and $\star-$open. $(A\cap C)\cap((X-A)\cap C)=\emptyset$. Also, $(A\cap C)\cup((X-A)\cap C)=(A\cup(X-A))\cap C=C$. Again, A and $X-A$ are two nonempty disjoint open and $\star-$open sets respectively such that $A\cap cl(X-A)=\emptyset$ and $cl^\star(A)\cap(X-A)=\emptyset$. Thus, A and $X-A$ form a $\star-$separation for C. Hence, C is not $\star_s-$connected, a contradiction. Hence, A is not a proper subset of C and so $A=C$. This completes the proof. □

If $\mathcal{I}=\{\emptyset\}$ in the above Theorem 2.20, we have the following Corollary 2.21.

Corollary 2.21. *Let (X,τ) be a space. Then, each connected subset of X which is both open and closed is a component of X.*

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