NOVI SAD J. MATH.
VOL. 42, NO. 1, 2012, 21-25

LINEAR SPAN OF THE CONJUGACY CLASS OF A MATRIX

Marcin Skrzyński[1]

Abstract. Let $\mathcal{M}_n$ be the vector space of all $n \times n$-matrices over an algebraically closed field $\mathbb{F}$ of characteristic zero. We describe the linear span of the conjugacy class (with respect to the full linear group $\mathcal{GL}_n$) of an arbitrary matrix $A \in \mathcal{M}_n$, and derive the existence of some particular bases of $\mathcal{M}_n$. Moreover, we propose certain observations on finite sequences of nilpotent matrices and their linear spans.

AMS Mathematics Subject Classification (2010): primary 15A03, secondary 14L35

Key words and phrases: linear span of a set, conjugacy class of a matrix, trace, nilpotent matrix, basis

0. Introduction

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero and let $n \in \mathbb{N} \setminus \{0, 1\}$. We consider the vector space $\mathcal{M}_n$ of all $n \times n$-matrices over $\mathbb{F}$. Let $I, O \in \mathcal{M}_n$ be the unit matrix and the zero matrix, respectively. We define $\mathfrak{sl}_n = \{A \in \mathcal{M}_n : \operatorname{tr}(A) = 0\}$ and $\mathcal{GL}_n = \{U \in \mathcal{M}_n : \det(U) \neq 0\}$. We denote by $\mathcal{O}(A)$ the conjugacy class of a matrix $A \in \mathcal{M}_n$, i. e., $\mathcal{O}(A) = \{U^{-1}AU : U \in \mathcal{GL}_n\}$. Finally, we define $\mathrm{S}_2(A)$ to be the sum of all principal minors of size 2 of the matrix A, and denote by $\operatorname{Span}\mathcal{E}$ the linear span (over $\mathbb{F}$) of a set $\mathcal{E} \subseteq \mathcal{M}_n$. (If $\mathcal{E} = \{A_1, \ldots, A_d\}$, then we write $\operatorname{Span}(A_1, \ldots, A_d)$ instead of $\operatorname{Span}\mathcal{E}$).

It is well known that a matrix $A \in \mathcal{M}_n$ is nilpotent iff the conjugacy class $\mathcal{O}(A)$ is a cone, in the sense that $\lambda B \in \mathcal{O}(A)$ for each $B \in \mathcal{O}(A)$ and each $\lambda \in \mathbb{F} \setminus \{0\}$. Notice also that $\mathrm{S}_2(A) = 0$ whenever A is nilpotent.

The only topology we consider on the spaces $\mathcal{M}_n \cong \mathbb{F}^{n^2}$ and

$$\underbrace{\mathcal{M}_n \times \ldots \times \mathcal{M}_n}_{d \text{ times}} \cong \mathbb{F}^{dn^2},$$

where $d \geq 2$, and on all their subsets is the Zariski topology.

The key fact we will use is

Theorem 0.1 (Gerstenhaber – Hesselink)**.** *Let $A, B \in \mathcal{M}_n$. Suppose that A is a nilpotent matrix. Then the following conditions are equivalent:*

(1) $B \in \overline{\mathcal{O}(A)}$*, the Zariski closure of* $\mathcal{O}(A)$ *in the space* $\mathcal{M}_n$*,*

(2) $\operatorname{rk}(B^k) \leq \operatorname{rk}(A^k)$ *for all* $k \in \mathbb{N} \setminus \{0\}$.

[1]Institute of Mathematics, Cracow University of Technology, ul. Warszawska 24, 31–155 Kraków, Poland, e-mail:pfskrzyn@cyf-kr.edu.pl

Proofs of the theorem can be found, for instance, in [2, 4, 5].

We will also use the following version of Gerstenhaber's theorem on linear spaces of nilpotent matrices.

Theorem 0.2. *If $A \in \mathcal{M}_n$ is a nilpotent matrix, then*

$$\max\{\dim \mathcal{L} : \mathcal{L} \text{ is a linear subspace of } \mathcal{M}_n,\ \mathcal{L} \subseteq \overline{\mathcal{O}(A)}\} = \frac{1}{2}\left(n^2 - \sum_{k=0}^{\infty}\left(\mathrm{rk}(A^k) - \mathrm{rk}(A^{k+1})\right)^2\right).$$

In particular,

$$\max\{\dim \mathcal{L} : \mathcal{L} \text{ is a linear subspace of } \mathcal{M}_n,\ \mathcal{L} \text{ consists of nilpotent matrices only}\} = n(n-1)/2.$$

Proofs and other versions of Theorem 0.2 can be found in [1, 3, 6].

We refer for all information needed on algebraic geometry to [7].

In the present note, we describe the linear span of the conjugacy class of an arbitrary (non-scalar) matrix $A \in \mathcal{M}_n$, and derive the existence of some particular bases of the space $\mathcal{M}_n$. Moreover, given a nilpotent matrix $A \in \mathcal{M}_n \setminus \{O\}$, we propose a necessary and sufficient condition for the existence of a finite sequence $(A_1, \ldots, A_d)$ of elements of $\mathcal{O}(A)$ with the property that $\mathrm{Span}(A_1, \ldots, A_d)$ consists of nilpotent matrices only. We also propose a necessary and sufficient condition for the existence of such a sequence with the property that its linear span is contained in $\overline{\mathcal{O}(A)}$.

1. Results

Our main observation is that the linear subspace of $\mathcal{M}_n$ spanned by the conjugacy class of a non-scalar matrix is always as large as possible. More precisely, the following theorem holds true.

Theorem 1.1. *If $A \in \mathcal{M}_n \setminus \mathbb{F}I$, then*

$$\mathrm{Span}\,\mathcal{O}(A) = \begin{cases} \mathfrak{sl}_n, & \text{if } \mathrm{tr}(A) = 0, \\ \mathcal{M}_n, & \text{otherwise.} \end{cases}$$

Proof. Define $\mathcal{L} = \mathrm{Span}\,\mathcal{O}(A)$, and consider first the case where A is nilpotent. Notice that in this case $\mathrm{tr}(A) = 0$ which yields $\mathcal{L} \subseteq \mathfrak{sl}_n$. For each pair (p, q) of different positive integers not greater than n define $B_{pq} = [\beta_{ij}] \in \mathcal{M}_n$ by

$$\beta_{ij} = \begin{cases} 1, & \text{if } (i, j) = (p, q), \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for each integer r such that $1 < r \leq n$ define $C_r = [\gamma_{ij}] \in \mathcal{M}_n$ by

$$\gamma_{ij} = \begin{cases} 1, & \text{if } (i, j) \in \{(1, 1), (1, r)\}, \\ -1, & \text{if } (i, j) \in \{(r, 1), (r, r)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $\mathrm{rk}(B_{pq}) = \mathrm{rk}(C_r) = 1$ and $B_{pq}^2 = C_r^2 = O$ for all suitable $(p,\, q)$ and r. Consequently, by the Gerstenhaber – Hesselink theorem, $B_{pq} \in \overline{\mathcal{O}(A)}$ and $C_r \in \overline{\mathcal{O}(A)}$, again for all suitable $(p,\, q)$ and r. It is quite easy to verify that all the matrices B_{pq} and C_r form a linearly independent set with $n^2 - 1$ elements. Since $\overline{\mathcal{O}(A)} \subseteq \mathcal{L} \subseteq \mathfrak{sl}_n$, we finally obtain $\dim \mathcal{L} \geq n^2 - 1 = \dim \mathfrak{sl}_n$ and $\mathcal{L} = \mathfrak{sl}_n$.

Consider now the case where A is not nilpotent. Notice that the assumption $A \notin \mathbb{F}I$ yields $\dim \mathcal{O}(A) \geq 1$, and the non-nilpotency implies that $\mathcal{O}(A) \neq \mathbb{F}\mathcal{O}(A)$. Consequently, $\dim \overline{\mathbb{F}\mathcal{O}(A)} \geq 2$. By the "in particular" part of [8, Lemma 3.1], we have $\dim\{C \in \overline{\mathbb{F}\mathcal{O}(A)} : C \text{ is nilpotent}\} = -1 + \dim \overline{\mathbb{F}\mathcal{O}(A)} \geq 1$. Therefore, there is a non-zero nilpotent matrix $B \in \overline{\mathbb{F}\mathcal{O}(A)}$. Observe that $\mathcal{O}(B) \subseteq \overline{\mathbb{F}\mathcal{O}(A)} \subseteq \mathcal{L}$. Applying to B the just-proved nilpotent case of our theorem, we obtain $\mathrm{Span}\, \mathcal{O}(B) = \mathfrak{sl}_n \subseteq \mathcal{L}$. If $\mathrm{tr}(A) = 0$, then $\mathcal{L} \subseteq \mathfrak{sl}_n$ and the equality $\mathcal{L} = \mathfrak{sl}_n$ follows. If $\mathrm{tr}(A) \neq 0$, then $\mathcal{L} \supseteq \mathfrak{sl}_n \oplus \mathbb{F}A = \mathcal{M}_n$ which yields $\mathcal{L} = \mathcal{M}_n$. □

The above theorem immediately implies the existence of some particular bases of the space $\mathcal{M}_n$.

Corollary 1.2. *Let $A \in \mathcal{M}_n \setminus \{O\}$ be a nilpotent matrix and let $B \in \mathcal{M}_n$ satisfy the condition $\mathrm{tr}(B) \neq 0$. Then, there are matrices $A_1, \ldots, A_{n^2-2} \in \mathcal{O}(A)$ such that $(A,\, B,\, A_1, \ldots, A_{n^2-2})$ is a basis of $\mathcal{M}_n$.*

Corollary 1.3. *Let $A \in \mathcal{M}_n \setminus \mathbb{F}I$ be such that $\mathrm{tr}(A) \neq 0$. Then there are matrices $A_1, \ldots, A_{n^2} \in \mathcal{O}(A)$ which form a basis of the space $\mathcal{M}_n$.*

Let us note down an alternative version of Corollary 1.2.

Corollary 1.4. *Let $A,\, B \in \mathcal{M}_n \setminus \{O\}$. Assume that A is a nilpotent matrix and that $\mathrm{tr}(B) = 0$. Then, there is an integer d satisfying inequalities $1 \leq d \leq n^2 - 1$ and there are linearly independent matrices $A_1, \ldots, A_d \in \mathcal{O}(A)$ such that $B = \sum_{i=1}^{d} A_i$.*

We conclude the present note with the following

Theorem 1.5. *Let $A \in \mathcal{M}_n \setminus \{O\}$ be a nilpotent matrix and let d be an integer such that $2 \leq d \leq n^2 - 1$. Define*

$$\mathcal{U} = \{(A_1, \ldots, A_d) \in \underbrace{\mathcal{O}(A) \times \ldots \times \mathcal{O}(A)}_{d \text{ times}} : A_1, \ldots, A_d \text{ are linearly independent}\},$$

$$\mathcal{U}_0 = \{(A_1, \ldots, A_d) \in \mathcal{U} : \mathrm{Span}(A_1, \ldots, A_d) \text{ consists of nilpotent matrices}\},$$

$$\mathcal{U}_A = \{(A_1, \ldots, A_d) \in \mathcal{U} : \mathrm{Span}(A_1, \ldots, A_d) \subseteq \overline{\mathcal{O}(A)}\}.$$

Then, the following are true:

(i) *$\mathcal{U}$ is a nonempty open subset of* $\underbrace{\mathcal{O}(A) \times \ldots \times \mathcal{O}(A)}_{d \text{ times}}$,

(ii) $\mathcal{U}_A \subseteq \mathcal{U}_0 \neq \mathcal{U}$,

(iii) *$\mathcal{U}_0$ and $\mathcal{U}_A$ are closed subsets of $\mathcal{U}$,*

(iv) $\mathcal{U}_0 \neq \emptyset \Leftrightarrow d \leq n(n-1)/2$,

(v) $\mathcal{U}_A \neq \emptyset \Leftrightarrow d \leq \frac{1}{2}\left(n^2 - \sum_{k=0}^{\infty}\big(\mathrm{rk}(A^k) - \mathrm{rk}(A^{k+1})\big)^2\right)$.

Proof. The nonemptiness of $\mathcal{U}$ follows immediately from Theorem 1.1. The openness is obvious, as well as the inclusion in *(ii)*.

To see the non-equality in *(ii)* consider the Jordan canonical form J of the matrix A, and notice that the matrices J and J^{T} are linearly independent, $\{J, J^{\mathrm{T}}\} \subseteq \mathcal{O}(A)$ and $\mathrm{S}_2(J + J^{\mathrm{T}}) \neq 0$. (If $d \geq 3$, then in virtue of Theorem 1.1 the pair (J, J^{T}) can be completed to an element of $\mathcal{U}$).

Assertion *(iii)* follows from the closedness of the sets

$$\{B \in \mathcal{M}_n : B \text{ is nilpotent}\} \text{ and } \overline{\mathcal{O}(A)}$$

and from the fact that the map

$$\underbrace{\mathcal{M}_n \times \ldots \times \mathcal{M}_n}_{d \text{ times}} \ni (B_1, \ldots, B_d) \mapsto \sum_{i=1}^{d} \lambda_i B_i \in \mathcal{M}_n$$

is regular (hence continuous) for each $(\lambda_1, \ldots, \lambda_d) \in \mathbb{F}^d$.

Implications "$\Rightarrow$" in *(iv)* and *(v)* follow immediately from the Gerstenhaber theorem on linear spaces of nilpotent matrices. To see implication "$\Leftarrow$" in *(iv)* consider the set

$$\mathcal{T} \overset{\text{def.}}{=} \{B \in \mathcal{M}_n : B \text{ is nilpotent and upper triangular}\},$$

and observe that $\mathcal{T}$ is a linear subspace of $\mathcal{M}_n$, $\dim \mathcal{T} = n(n-1)/2$ and $\mathcal{T} \subset \mathfrak{sl}_n$. Consequently, if $d \leq \frac{1}{2}n(n-1)$, then there is a d-dimensional linear subspace $\mathcal{L}$ of $\mathcal{M}_n$ such that $\mathcal{L} \subseteq \mathcal{T}$. Thus, by Theorem 1.1, we have $\mathcal{L} \subset \mathfrak{sl}_n = \mathrm{Span}\,\mathcal{O}(A)$ which implies that $\mathcal{L} = \mathrm{Span}(A_1, \ldots, A_d)$ for some $(A_1, \ldots, A_d) \in \mathcal{U}$. Therefore, $(A_1, \ldots, A_d) \in \mathcal{U}_0$. The implication "$\Leftarrow$" in *(v)* can be proved in an analogous way, with $\mathcal{T}$ replaced by a linear subspace $\mathcal{S}$ of $\mathcal{M}_n$ such that $\dim \mathcal{S} = \frac{1}{2}\left(n^2 - \sum_{k=0}^{\infty}\big(\mathrm{rk}(A^k) - \mathrm{rk}(A^{k+1})\big)^2\right)$ and $\mathcal{S} \subset \overline{\mathcal{O}(A)}$. (The existence of such a subspace follows from the Gerstenhaber theorem on linear spaces of nilpotent matrices). □

Notice that Gerstenhaber's Theorems 0.1 and 0.2 remain true over an arbitrary field of characteristic zero. Consequently, Theorem 1.1, with an additional assumption that A is a nilpotent matrix, is valid over an arbitrary field of characteristic zero. This implies that Corollary 1.2, Corollary 1.4 and Theorem 1.5 remain true over an arbitrary field of characteristic zero.

References

[1] Brualdi, R. A., Chavey, K. L., Linear spaces of Toeplitz and nilpotent matrices. J. Comb. Theory, Ser. A 63 (1993), 65–78.

[2] Gerstenhaber, M., On dominance and varieties of commuting matrices. Ann. Math. 73 (1961), 324–348.

[3] Gerstenhaber, M., On nilalgebras and linear varieties of nilpotent matrices IV. Ann. Math. 75 (1962), 382–418.

[4] Hazewinkel, M., Martin, C. F., Representations of the symmetric group, the specialization order, systems and Grassmann manifolds. L'Enseignement Mathématique 29 (1983), 53–87.

[5] Hesselink, W., Singularities in the nilpotent scheme of a classical group. Trans. Amer. Math. Soc. 222 (1976), 1–32.

[6] Mathes, B., Omladič, M., Radjavi, H., Linear spaces of nilpotent matrices. Linear Algebra Appl. 149 (1991), 215–225.

[7] Shafarevich, I. R., Basic Algebraic Geometry. Berlin – Heidelberg – New York: Springer-Verlag 1977.

[8] Skrzyński, M., On the linear capacity of algebraic cones. Math. Bohem. 127 (No. 3) (2002), 453–462.

Received by the editors November 10, 2009