Novi Sad J. Math.
Vol. 42, No. 1, 2012, 49-57

COMMON FIXED POINT THEOREM IN QUASI-UNIFORMIZABLE SPACES

Tatjana Došenović[1]

Abstract. In this paper a common fixed point theorem in sequentially complete Hausdorff quasi-uniformizable spaces is proved. Using the additional condition for the quasi-metric d_λ the condition (2) of Theorem 3.92 in the paper [4] is improved. Since Menger space $(S, \mathcal{F}, T)$, where $\sup_{a<1} T(a,a) = 1$ is a quasi-uniformizable space a corollaries on common fixed points in Menger spaces is obtained.

AMS Mathematics Subject Classification (2010): 54H25, 55M20, 47H10

Key words and phrases: t-norm, quasi-uniformizable space, common fixed point, Menger space

1. Introduction

D.H. Tan in [8] introduced the definition of quasi-uniformizable spaces. Using a similar method as in [4] (see also [2], [3] and [9]) we prove in this paper a common fixed point theorem for three mappings in sequentially complete Hausdorff quasi-uniformizable spaces. As a corollary we prove a common fixed point theorem in Menger spaces. Menger spaces (see [6]) are a generalization of the notion of a metric space (M, d) in which the distance $d(p, q)$ $(p, q \in M)$ between p and q is replaced by a distribution function $F_{p,q} \in \Delta^+$. $F_{p,q}(x)$ can be interpreted as the probability that the distance between p and q is less than x. Since then, the theory of probabilistic metric spaces has been developed in many directions ([4], [7]).

2. Preliminaries

D.H. Tan in [8] introduced the notion of quasi-uniformizable spaces.

Definition 1. Let S and I be arbitrary sets, $g : I \to I$ and for every $i \in I$, $d_i : S \times S \to \mathbb{R}^+$. The triplet $(S, (d_i)_{i\in I}, g)$ is said to be a quasi-uniformizable space if for every $x, y, z \in S$ and $i \in I$ the following hold:

a) $d_i(x, y) \geq 0$, $d_i(x, x) = 0$
b) $d_i(x, y) = d_i(y, x)$
c) $d_i(x, y) \leq d_{g(i)}(x, z) + d_{g(i)}(z, y)$.

[1]Faculty of Technology, University of Novi Sad, Bulevar Cara Lazara 1, Novi Sad, Serbia, e-mail: tatjanad@tf.uns.ac.rs

A quasi-uniformizable space $(S,\,(d_i)_{i\in I},\,g)$ is Hausdorff if the relation $d_i(x,\,y)=0$, for all $i\in I$ implies that $x=y$. A Hausdorff quasi-uniformizable space $(S,\,(d_i)_{i\in I},\,g)$ becomes a Hausdorff topological space if the fundamental system of neighbourhoods of $x\in S$ is given by the family $\mathcal{U}_x=(B(x,\,\varepsilon,\,i))_{i\in I}^{\varepsilon>0}$

$$B(x,\,\varepsilon,\,i)=\{y\,|\,y\in S,\,d_i(x,\,y)<\varepsilon\},\quad i\in I,\,\varepsilon>0.$$

Let Δ^+ be the set of all distribution functions F such that $F(0)=0$ (F is a nondecreasing, left continuous mapping from $\mathbb{R}$ into $[0,1]$ such that $\sup\limits_{x\in\mathbb{R}}F(x)=1$).

The ordered pair $(S,\mathcal{F})$ is said to be a probabilistic metric space if S is a nonempty set and $\mathcal{F}:S\times S\to\Delta^+$ ($\mathcal{F}(p,q)$ written by $F_{p,q}$ for every $(p,q)\in S\times S$) satisfies the following conditions:

1. $F_{u,v}(x)=1$ for every $x>0\Rightarrow u=v$ $(u,v\in S)$.
2. $F_{u,v}=F_{v,u}$ for every $u,v\in S$.
3. $F_{u,v}(x)=1$ and $F_{v,w}(y)=1\Rightarrow F_{u,w}(x+y)=1$ for $u,v,w\in S$ and $x,y\in\mathbb{R}^+$.

A Menger space (see [7]) is an ordered triple $(S,\mathcal{F},T)$, where $(S,\mathcal{F})$ is a probabilistic metric space, T is a triangular norm (abbreviated t-norm), and the following inequality holds

$$F_{u,v}(x+y)\geq T(F_{u,w}(x),F_{w,v}(y))\text{ for every }u,v,w\in S\text{ and every }x>0,y>0.$$

Recall that a mapping $T:[0,1]\times[0,1]\to[0,1]$ is called a triangular norm (a t-norm) if the following conditions are satisfied:

(i) $T(a,1)=a$ for every $a\in[0,1]$,

(ii) $T(a,b)=T(b,a)$ for every $a,b\in[0,1]$,

(iii) $a\geq b,\ c\geq d\Rightarrow\ T(a,c)\geq T(b,d)$ $(a,b,c,d\in[0,1])$,

(iv) $T(a,T(b,c))=T(T(a,b),c)$ $(a,b,c\in[0,1])$.

The following are the four basic t-norms:

$$\begin{aligned}T_M(x,y)&=\min(x,y)\\T_P(x,y)&=x\cdot y\\T_L(x,y)&=\max(x+y-1,0)\\T_D(x,y)&=\begin{cases}\min(x,y)&\text{if }\max(x,y)=1,\\0&\text{otherwise.}\end{cases}\end{aligned}$$

As regards the pointwise ordering, we have the inequalities

$$T_D<T_L<T_P<T_M.$$

The $(\epsilon,\lambda)-$topology in S is introduced by the family of neighbourhoods $\mathcal{U}=\{U_v(\epsilon,\lambda)\}_{(v,\epsilon,\lambda)\in S\times\mathbb{R}_+\times(0,1)}$, where

$$U_v(\epsilon,\lambda)=\{u;\ F_{u,v}(\epsilon)>1-\lambda\}.$$

If a t-norm T is such that $\sup_{x<1} T(x,x) = 1$, then S is in the (ϵ,λ)-topology a metrizable topological space.

In [1], the following class of t-norms is introduced, which is useful in the fixed point theory in probabilistic metric spaces.

Let T be a t-norm and $T_n : [0,\,1] \to [0,\,1]$ $(n \in \mathbb{N})$ is defined in the following way:

$$T_1(x) = T(x,\,x),\; T_{n+1}(x) = T(T_n(x),\,x)\; (n \in \mathbb{N},\, x \in [0,\,1]).$$

We say that t-norm T is of the H-type if T is continuous and the family $\{T_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$.

The family $\{T_n(x)\}_{n\in\mathbf{N}}$ is equicontinuous at $x = 1$, if for every $\lambda \in (0,\,1)$ there exists $\delta(\lambda) \in (0,\,1)$ such that the following implication holds:

$$x > 1 - \delta(\lambda) \Rightarrow T_n(x) > 1 - \lambda,\ \text{for all } n \in \mathbb{N}.$$

A trivial example of t-norms of H-type is $T = T_M$. A nontrivial example is given in the paper [1].

Each t-norm T can be extended (by associativity) (see [5]) in a unique way to an n-ary operation taking for $(x_1, \ldots, x_n) \in [0,1]^n$ the values

$$\mathbf{T}^0_{i=1} x_i = 1,\; \mathbf{T}^n_{i=1} x_i = T(\mathbf{T}^{n-1}_{i=1} x_i, x_n).$$

We can extend T to a countable infinitary operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $[0,1]$ the values

$$\mathbf{T}^\infty_{i=1} x_i = \lim_{n\to\infty} \mathbf{T}^n_{i=1} x_i.$$

Limit of the right side exists since the sequence $(\mathbf{T}^n_{i=1} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below.

In [4], the following lemma is given.

Lemma 1. *A Menger space* $(S,\,\mathcal{F},\,T)$ *such that* $\sup_{x<1} T(x,\,x) = 1$ *is a quasi-uniformizable space* $(S,\,(d_\lambda)_{\lambda\in J},\,g)$*, where* $J = (0,\,1)$*,* $g : J \to J$*,*

$$d_\lambda(x,\,y) = \sup\{s\,|F_{x,\,y}(s) \le 1 - \lambda\},\ \lambda \in J,\ x,\ y \in S. \tag{1}$$

Every locally convex space $(X,\,(p_i)_{i\in I})$, where $(p_i)_{i\in I}$ is the familly of seminorms on X, is a quasi-uniformizable space, where $g : I \to I$ is defined by $g(i) = i$, for every $i \in I$.

3. Common fixed point theorem

In this paper we prove a common fixed point theorem in some special class of quasi-uniformizable spaces. This theorem is a generalization of a theorem proved in [4].

Lemma 2. *Let* $(S, \mathcal{F}, T)$ *be a Menger space and* T *be a t-norm of H-type. Then* $(S, (d_\lambda)_{\lambda\in J}, g)$, $J = (0, 1)$ *is a quasi-uniformizable space such that for every* $\alpha \in (0, 1)$ *there exists* $\beta \in (0, 1)$ *such that for all* $x_1, \dots x_n \in S$

$$d_\alpha(x_1, x_n) \le d_\beta(x_1, x_2) + d_\beta(x_2, x_3) + \cdots + d_\beta(x_{n-1}, x_n). \tag{2}$$

Proof. Let $\alpha \in (0, 1)$. Since T is a t-norm of H-type for a given $\alpha \in (0, 1)$, there exists $\beta \in (0, 1)$ such that $\mathbf{T}_{i=1}^{n-1}(1-\beta) > 1-\alpha$, for all $n \in \mathbb{N}$.

In order to prove (2) we shall suppose that $\alpha_1, \alpha_2, \dots \alpha_{n-1}$, $n \in \mathbb{N}$ is such that

$$\begin{array}{rcl} d_\beta(x_1, x_2) & < & \alpha_1 \\ d_\beta(x_2, x_3) & < & \alpha_2 \\ & \dots & \\ d_\beta(x_{n-1}, x_n) & < & \alpha_{n-1}. \end{array} \tag{3}$$

We need to prove $d_\alpha(x_1, x_n) < \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$.

From (3) it follows

$$\begin{array}{rcl} F_{x_1, x_2}(\alpha_1) & > & 1-\beta \\ F_{x_2, x_3}(\alpha_2) & > & 1-\beta \\ & \dots & \\ F_{x_{n-1}, x_n}(\alpha_{n-1}) & > & 1-\beta \end{array} \tag{4}$$

and

$$\begin{array}{rcl} F_{x_1, x_n}(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}) & \ge & T(F_{x_1, x_2}(\alpha_1), F_{x_2, x_n}(\alpha_2 + \cdots + \alpha_{n-1})) \\ & \ge & \underbrace{T(T(\dots T}_{(n-2)-time}(F_{x_1,x_2}(\alpha_1), \dots (F_{x_{n-1},x_n}(\alpha_{n-1}))) \\ & \ge & \underbrace{T(T(\dots T}_{(n-2)-time}(1-\beta, 1-\beta)\dots, 1-\beta) \\ & = & \mathbf{T}_{i=1}^{n-1} 1-\beta \\ & > & 1-\alpha, \end{array}$$

so $d_\alpha(x_1, x_n) < \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$ i.e. (2) holds. □

Definition 2. Let $(S, (d_\lambda)_{\lambda\in I}, f)$ be a sequentially complete Hausdorff quasiuniformizable space. We say that the space satisfies condition (*), if for a given $\alpha \in I$ there exists $\beta \in I$ such that for all $x_1, \dots, x_n \in S$

$$d_\alpha(x_1, x_n) \le d_\beta(x_1, x_2) + d_\beta(x_2, x_3) + \cdots + d_\beta(x_{n-1}, x_n). \tag{*}$$

We observe that, a Menger space $(S, \mathcal{F}, T)$, where T is a t-norm of H-type, satisfies the condition (*).

Theorem 1. *Let $(S, (d_\lambda)_{\lambda \in I}, f)$ be a sequentially complete Hausdorff quasi-uniformizable space, family $(d_\lambda)_{\lambda \in I}$ satisfies the condition (2), $f : I \to I$, $L_1, L_2 : S \to S$ continuous mapping, $L : S \to L_1 S \cap L_2 S$ a continuous mapping which commutes with L_1 and L_2, and the following conditions are satisfied:*

(i) for every $i \in I$ there exists $q_i : \mathbb{R}^+ \to [0, 1)$ which is a non-decreasing function, for which $\overline{\lim_{n\to\infty}} q_{f^n(i)}(t) < 1$, for every $t \in \mathbb{R}^+$ and every $i \in I$ and

$$d_i(Lx, Ly) \le q_i(d_{f(i)}(L_1 x, L_2 y)) \cdot d_{f(i)}(L_1 x, L_2 y) \text{ for every } x, y \in S.$$

(ii) there exists $x_0 \in S$ such that for every $i \in I$

$$\sup_{j \in O(i, f)} d_j(Lx_0, Lx_1) \le K_i \quad (K_i \in \mathbb{R}^+)$$

where $O(i, f) = \{f^n(i) : n \in \mathbb{N}\}$, and the sequence $(x_p)_{p \in \mathbb{N}}$ is defined by

$$L_1 x_{2n-1} = Lx_{2n-2} \text{ and } L_2 x_{2n} = Lx_{2n-1}, (n \in \mathbb{N}).$$

Then there exists $z \in S$ such that $Lz = L_1 z = L_2 z$.

If for every $i \in I$

$$\sup_{j \in O(i, f)} d_j(L^3 x_1, L^2 x_0) \le M_i, \ (M_i \in \mathbb{R}^+),$$

then Lz is a common fixed point for L, L_1 and L_2, and Lz is the unique common fixed point for L, L_1, L_2 in the set

$$\{u \,|\, u \in S, (\forall i \in I), (\exists V_i \in \mathbb{R}^+) (\sup_{j \in O(i, f)} d_j(Lz, u) \le V_i)\}.$$

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be such sequence from S that

$$L_2 x_{2k} = Lx_{2k-1} \text{ and } L_1 x_{2k-1} = Lx_{2k-2}, \text{ for every } k \in \mathbb{N}.$$

We will prove that $(Lx_n)_{n \in \mathbf{N}}$ is a Cauchy sequence which means that for every $i \in I$ and every $\varepsilon > 0$ there exists $n(\varepsilon, i) \in \mathbb{N}$ such that

$$d_i(Lx_n, Lx_{n+p}) < \varepsilon, \text{ for every } n \ge n(\varepsilon, i), \text{ and every } p \in \mathbb{N} \tag{5}$$

In order to prove (5), we will estimate

$$d_i(Lx_{2k}, Lx_{2k-1}) \text{ and } d_i(Lx_{2k+1}, Lx_{2k}), \text{ for every } i \in I \text{ and every } k \in \mathbb{N}.$$

From condition (i) it follows that

$$\begin{aligned}
&d_i(Lx_{2k},\, Lx_{2k-1})\\
&\quad\leq\quad q_i(d_{f(i)}(L_2x_{2k},\, L_1x_{2k-1}))\cdot d_{f(i)}(L_2x_{2k},\, L_1x_{2k-1})\\
&\quad=\quad q_i(d_{f(i)}(Lx_{2k-1},\, Lx_{2k-2}))\cdot d_{f(i)}(Lx_{2k-1},\, Lx_{2k-2})\\
&\quad\leq\quad q_i(d_{f(i)}(Lx_{2k-1},\, Lx_{2k-2}))\cdot q_{f(i)}(d_{f^2(i)}(L_1x_{2k-1},\, L_2x_{2k-2}))\\
&\quad\cdot\quad (d_{f^2(i)})(L_1x_{2k-1},\, L_2x_{2k-2})\\
&\quad\dots\\
&\quad\leq\quad q_i(d_{f(i)}(Lx_{2k-1},\, Lx_{2k-2}))\\
&\qquad\quad\cdot\prod_{s=0}^{2k-3} q_{f^{s+1}(i)}(d_{f^{s+2}(i)}(Lx_{2k-s-1},\, Lx_{2k-s-2}))\cdot d_{f^{2k-1}(i)}(Lx_1,\, Lx_0)
\end{aligned}$$

and similarly

$$\begin{aligned}
d_i(Lx_{2k+1},\, Lx_{2k}) \quad\leq\quad & q_i(d_{f(i)}(Lx_{2k},\, Lx_{2k-1}))\\
&\cdot\prod_{s=0}^{2k-2} q_{f^{s+1}(i)}(d_{f^{s+2}(i)}(Lx_{2k-s-1},\, Lx_{2k-s-2}))\\
&\cdot d_{f^{2k}(i)}(Lx_1,\, Lx_0)
\end{aligned}$$

Since $q_i(t) < 1$ for every $i \in I$ and every $t \in \mathbb{R}^+$, it follows that for every $i \in I$ and every $n \in \mathbb{N}$

$$d_j(Lx_n,\, Lx_{n-1}) \leq K_i,$$

for every $j \in O(i,\, f)$ and therefore

$$d_i(Lx_{2k},\, Lx_{2k-1}) \leq \prod_{s=0}^{2k-2} q_{f^s(i)}(K_i)K_i$$

$$d_i(Lx_{2k+1},\, Lx_{2k}) \leq \prod_{s=0}^{2k-1} q_{f^s(i)}(K_i)K_i$$

where $f^0(i) = i$ for every $i \in I$.

Since $\overline{\lim_{n\to\infty}} q_{f^n(i)}(K_i) \leq Q_i < 1$, for all $i \in I$ there exists $n_i \in \mathbb{N}$ such that for some $S_i \in \mathbb{R}^+$ $(i \in I)$ it follows

$$d_i(Lx_n,\, Lx_{n-1}) \leq S_iQ_i^n,$$

for every $i \in I$.

From Lemma 2, for every $i \in I$ there exists $j \in I$ such that

$$\begin{aligned}
d_i(Lx_n,\, Lx_{n+p}) \quad\leq\quad & d_j(Lx_n,\, Lx_{n+1}) + d_j(Lx_{n+1},\, Lx_{n+2}) + \dots\\
& \dots + d_j(Lx_{n+p-1},\, Lx_{n+p})\\
\leq\quad & S_j\sum_{i=n+1}^{\infty} Q_j^i.
\end{aligned}$$

From the condition $Q_j < 1$, $(j \in I)$, it follows that the sequence $(Lx_n)_{n\in\mathbf{N}}$ is a Cauchy sequence. Let

$$z = \lim_{n\to\infty} Lx_n = \lim_{n\to\infty} Lx_{2n-1} = \lim_{n\to\infty} Lx_{2n}.$$

Since L and L_1 are continuous functions, we have

$$L_1 z = \lim_{n\to\infty} L_1 Lx_{2n+1} = \lim_{n\to\infty} LL_1 x_{2n+1} = Lz$$

and similarly $L_2 z = Lz$.

We are going to prove that Lz is a common fixed point for the mappings L, L_1 and L_2 if

$$\sup_{j\in O(i,f)} d_j(L^3 x_1,\, L^2 x_0) \le M_i$$

for all $i \in I$.

First we shall prove that for every $i \in I$

$$\sup_{j\in O(i,f)} d_j(L^2 z,\, Lz) \le M_i.$$

Since

$$d_j(L^2 z,\, Lz) = \lim_{n\to\infty} d_j(L^2 Lx_{2n+1},\, LLx_{2n}),$$

it is enough to prove that for every $i \in I$

$$d_j(L^3 x_{2n+1},\, L^2 x_{2n}) \le M_i \text{ for every } n \in \mathbb{N} \text{ and } j \in O(i,\, f).$$

The pairs L, L_1 and L, L_2 commute, and we obtain that

$$\begin{aligned}
&d_j(L^3 x_{2n+1},\, L^2 x_{2n})\\
&\quad\le q_j(d_{f(j)}(L_2(Lx_{2n}),\, L_1(L^2 x_{2n+1})))d_{f(j)}(L_2(Lx_{2n}),\, L_1(L^2 x_{2n+1}))\\
&\quad= q_j(d_{f(j)}(L(L_2 x_{2n}),\, L^2(L_1 x_{2n+1})))d_{f(j)}(L(L_2 x_{2n}),\, L^2(L_1 x_{2n+1}))\\
&\quad= q_j(d_{f(j)}(LLx_{2n-1},\, L^2 Lx_{2n}))d_{f(j)}(LLx_{2n-1},\, L^2 Lx_{2n})\\
&\quad= q_j(d_{f(j)}(L^2 x_{2n-1},\, L^3 x_{2n}))d_{f(j)}(L^2 x_{2n-1},\, L^3 x_{2n})\\
&\quad\le d_{f(j)}(L^2 x_{2n-1},\, L^3 x_{2n})\\
&\quad\quad\vdots\\
&\quad\le d_{f(j)}(L^3 x_1,\, L^2 x_0)\\
&\quad\le M_i
\end{aligned}$$

for every $n \in \mathbb{N}$ since $f^{2n}(j) \in O(i,\, f)$.

Now we have

$$\begin{aligned}
d_i(L^2 z,\, Lz) &\le q_i(d_{f(i)}(L_1 Lz,\, L_2 z)) \cdot d_{f(i)}(L_1 Lz,\, L_2 z)\\
&= q_i(d_{f(i)}(L^2 z,\, Lz)) \cdot d_{f(i)}(L^2 z,\, Lz)\\
&\quad\vdots\\
&\le q_i(M_i) q_{f(i)}(M_i) \ldots q_{f^n(i)}(M_i) M_i
\end{aligned}$$

for every $i \in I$.

Since $\overline{\lim_{n\to\infty}} q_{f^n(i)}(M_i) < 1$ it follows that $d_i(L^2z, Lz) = 0$ for every $i \in I$, which implies that $Lz = L^2z$.

Hence, from $Lz = L_1z = L_2z$ it follows that

$$L^2z = LL_1z = L_1Lz = L_2Lz = Lz$$

which means that Lz is a common fixed point for the mappings L, L_1 and L_2.

Suppose that $y = Ly = L_1y = L_2y$ and for every $i \in I$

$$\sup_{j\in O(i,f)} d_j(Lz, y) \le V_i \qquad (V_i \in \mathbb{R}^+).$$

We prove that $y = Lz$. For every $i \in I$ we have

$$\begin{aligned} d_i(Lz, y) &= d_i(L(Lz), Ly) \\ &\le q_i(d_{f(i)}(L_1(Lz), L_2y)) \cdot d_{f(i)}(L_1(Lz), L_2y) \\ &= q_i(d_{f(i)}(Lz, y)) \cdot d_{f(i)}(Lz, y) \\ &\vdots \\ &\le q_i(d_{f(i)}(Lz, y)) \dots q_{f^n(i)}(d_{f^{n+1}(i)}(Lz, y)) \cdot d_{f^{n+1}(i)}(Lz, y) \\ &\le q_i(V_i) q_{f(i)}(V_i) \dots q_{f^n(i)}(V_i) V_i \end{aligned}$$

Since $\overline{\lim_{n\to\infty}} q_{f^n(i)}(V_i) < 1$ it follows that $d_i(Lz, y) = 0$ for every $i \in I$, which means that $y = Lz$. □

Corollary 1. *Let $(S, \mathcal{F}, T)$ be a complete Menger space and t-norm T is of H-type, $f : (0, 1) \to (0, 1)$, $L_1, L_2 : S \to S$ continuous mappings, $L : S \to L_1S \cap L_2S$ a continuous mapping which commutes with L_1 and L_2, and the following conditions are satisfied:*
(a) for every $\lambda \in (0, 1)$, there exists a right continuous, non-decreasing mapping

$$q_\lambda : \mathbb{R}^+ \to [0, 1) \;\; \text{for which} \; \overline{\lim_{n\to\infty}} q_{f^n(\lambda)}(t) < 1 \; \text{for every} \; t \in \mathbb{R}^+$$

and for every $\lambda \in (0, 1)$, every $r > 0$ and every $x, y \in S$

$$F_{L_1x, L_2y}(r) > 1 - f(\lambda) \Rightarrow F_{Lx, Ly}(q_\lambda(r)r) > 1 - \lambda. \tag{6}$$

(b) there exists $x_0 \in S$ such that for every $\lambda \in (0, 1)$, there exists $K_\lambda \in \mathbb{R}^+$ such that for every $n \in \mathbb{N}$

$$F_{Lx_0,\, Lx_1}(K_\lambda) > 1 - f^n(\lambda)$$

where the sequence $(x_n)_{n\in\mathbb{N}}$ is defined by

$$L_1x_{2n-1} = Lx_{2n-2} \; \text{and} \; L_2x_{2n} = Lx_{2n-1} \quad (n \in \mathbb{N}).$$

Then, there exists $z \in S$ such that

$$Lz = L_1z = L_2z.$$

If, in addition, for every $\lambda \in (0, 1)$ there exists $M_\lambda \in \mathbb{R}^+$ such that for every $n \in \mathbb{N}$

$$F_{L^3 x_1, L^2 x_0}(M_\lambda) > 1 - f^n(\lambda),$$

then, Lz is a common fixed point for the mapping L, L_1 and L_2, which is the unique common fixed point in the set

$$\{u \,|\, u \in S, (\forall \lambda \in (0, 1)) (\exists P_\lambda \in \mathbb{R}^+) (\forall n \in \mathbb{N}) (F_{Lz,u}(P_\lambda) > 1 - f^n(\lambda))\}.$$

Proof. We have only to prove that condition (6) implies that for every $\lambda \in (0, 1)$ and every $x, y \in S$

$$d_\lambda(Lx, Ly) \leq q_\lambda(d_{f(\lambda)}(L_1 x, L_2 y)) \cdot d_{f(\lambda)}(L_1 x, L_2 y) \tag{7}$$

Suppose that $d_{f(\lambda)}(L_1 x, L_2 y) < r$. Then, $F_{L_1 x, L_2 y}(r) > 1 - f(\lambda)$ and by (6) we obtain that $F_{Lx,Ly}(q_\lambda(r)r) > 1 - \lambda$. Hence, $d_\lambda(Lx, Ly) < q_\lambda(r)r$ and since q_λ is right-continuous, the inequality (7) holds.
From Lemma 2 it follows that for every $\alpha \in (0, 1)$ there exists $\beta \in (0, 1)$ such that for every $x_1, \ldots, x_n \in S$

$$d_\alpha(x_1, x_n) \leq d_\beta(x_1, x_2) + d_\beta(x_2, x_3) + \cdots + d_\beta(x_{n-1}, x_n)$$

and all conditions of the previous theorem are satisfied. □

Acknowledgment

The work was supported by the Ministry of Education and Science of the Republic of Serbia under grants 174009.

References

[1] Hadžić O., A fixed point theorem in Menger spaces. Publ. Inst. Math. Beograd T 20 (1979), 107–112.

[2] Hadžić O., On a common fixed point in quasi-uniformizable spaces. Review of Research Faculty of Science-University of Novi Sad, Vol. 13 (1983), 31–39.

[3] Hadžić O., On Common Fixed Points in Uniformizable Spaces. Anal. Num. Appr., 12, 1(1983), 45–54.

[4] Hadžić O., Pap E., Fixed Point Theory in Probabilistic Metric Spaces. Theory in Probabilistic Metric Spaces. Dordrecht: Kluwer Academic Publishers, 2001.

[5] Klement E.P., Mesiar R., Pap E., Triangular Norms, Trends in Logic 8. Dordrecht: Kluwer Academic Publishers, 2000.

[6] Menger K., Statistical metric. Proc. Nat. Acad. U.S.A., 28 (1942), 535–537.

[7] Schweizer B., Sklar A., Probabilistic metric spaces. New York: Elsevier North-Holland, 1983.

[8] Tan D.H., On the contraction principle in uniformizable spaces, Acta Mathematica Vietnamica, Tom 5 (1980), 88–99.

[9] Žikić T., Existence of fixed point in fuzzy structures, PhD Thesis 2002., University of N. Sad, Faculty of Sciences and Mathematics.

Received by the editors December 18, 2010