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# ON THE $[p,q]$-ORDER OF ANALYTIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC

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**Abstract.** In this paper, we study the growth of analytic solutions of homogeneous linear differential equations in which the coefficients are analytic functions of $[p,q]-$order in the unit disc.

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## 1. Introduction and main results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's theory in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ (see [10, 11, 18, 21]). Recently, there has been an increasing interest in studying the growth of analytic solutions of linear differential equations in the unit disc by making use of the Nevanlinna theory (see [2, 3, 5, 6, 7, 9, 11, 12, 19]).

Before we state our results we need to give some definitions and discussions. Firstly, let us a give a definition of the degree of small growth order of functions in $\Delta$ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in $\Delta$ (i.e., see [7, 8]).

**Definition 1.1.** For a meromorphic function $f$ in $\Delta$ let

$$D(f) := \limsup_{r\to 1^-} \frac{T(r,\ f)}{\log \frac{1}{1-r}},$$

where $T(r,f)$ is the characteristic function of Nevanlinna of $f$. If $D(f) < \infty$, we say that $f$ is of finite degree $D(f)$ (or is non-admissible); if $D(f) = \infty$, we say that $f$ is of infinite degree (or is admissible). If $f$ is an analytic function in $\Delta$, and

$$D_M(f) := \limsup_{r\to 1^-} \frac{\log^+ M(r,\ f)}{\log \frac{1}{1-r}},$$

in which $M(r,\ f) = \max_{|z|=r} |f(z)|$ is the maximum modulus function, then we say that $f$ is a function of finite degree $D_M(f)$ if $D_M(f) < \infty$; otherwise, $f$ is of an infinite degree.

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Now, we give the definitions of iterated order and growth index to classify generally the functions of fast growth in $\Delta$ as those in $\mathbb{C}$ (see, [4, 16, 17]). Let us define inductively, for $r \in [0,1)$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all $r$ sufficiently large in $(0,1)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

**Definition 1.2.** (see [5, 6, 18]) Let $f$ be a meromorphic function in $\Delta$. Then, the iterated $p-$order of $f$ is defined by

$$\rho_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r,\ f)}{\log \frac{1}{1-r}} \quad (p \geqslant 1 \ \text{ is an integer}),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+ \log_p^+ x$. For $p = 1$, this notation is called order and for $p = 2$ hyper-order [11, 19]. If $f$ is analytic in $\Delta$, then the iterated $p-$order of $f$ is defined by

$$\rho_{M,p}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,\ f)}{\log \frac{1}{1-r}} \quad (p \geqslant 1 \text{ is an integer}).$$

*Remark* 1.1. It follows by M. Tsuji [21, p. 205] that if $f$ is an analytic function in $\Delta$, then we have the inequalities

$$\rho_1(f) \leqslant \rho_{M,1}(f) \leqslant \rho_1(f) + 1,$$

which are the best possible in the sense that there are analytic functions $g$ and $h$ such that $\rho_{M,1}(g) = \rho_1(g)$ and $\rho_{M,1}(h) = \rho_1(h) + 1$, see [8]. However, it follows by Proposition 2.2.2 in [17] that $\rho_{M,p}(f) = \rho_p(f)$ for $p \geqslant 2$.

**Definition 1.3.** ([5]) The growth index of the iterated order of a meromorphic function $f(z)$ in $\Delta$ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{j \in \mathbb{N} : \rho_j(f) < \infty\} & \text{if } f \text{ is admissible,} \\ \infty, & \text{if } \rho_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

For an analytic function $f$ in $\Delta$, we also define

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{j \in \mathbb{N} : \rho_{M,j}(f) < \infty\} & \text{if } f \text{ is admissible,} \\ \infty, & \text{if } \rho_{M,j}(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

*Remark* 1.2. If $\rho_p(f) < \infty$ or $i(f) \leqslant p$, then we say that $f$ is of the finite iterated $p-$order; if $\rho_p(f) = \infty$ or $i(f) > p$, then we say that $f$ is of infinite iterated $p-$order. In particular, we say that $f$ is of finite order if $\rho_1(f) < \infty$ or $i(f) \leqslant 1$; $f$ is of infinite order if $\rho_1(f) = \infty$ or $i(f) > 1$.

Now, we introduce the concept of $[p,q]$-order of meromorphic and analytic functions in the unit disc.

**Definition 1.4.** Let $p \geqslant q \geqslant 1$ be integers. Let $f$ be a meromorphic function in $\Delta$, the $[p,q]$-order of $f(z)$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}}.$$

For an analytic function $f$ in $\Delta$, we also define

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log_q \frac{1}{1-r}}.$$

*Remark* 1.3. It is easy to see that $0 \leqslant \rho_{[p,q]}(f) \leqslant \infty$. If $f(z)$ is non-admissible, then $\rho_{[p,q]}(f) = 0$ for any $p \geqslant q \geqslant 1$. By Definition 1.4, we have that $\rho_{[1,1]}(f) = \rho_1(f) = \rho(f)$, $\rho_{[2,1]}(f) = \rho_2(f)$ and $\rho_{[p+1,1]}(f) = \rho_{p+1}(f)$.

**Proposition 1.1.** *Let $p \geqslant q \geqslant 1$ be integers, and let $f$ be an analytic function in $\Delta$ of $[p,q]$-order. The following two statements hold:*
*(i) If $p = q$, then*

$$\rho_{[p,q]}(f) \leqslant \rho_{M,[p,q]}(f) \leqslant \rho_{[p,q]}(f) + 1.$$

*(ii) If $p > q$, then*

$$\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f).$$

*Proof.* By the standard inequalities (see [17], p. 26)

$$T(r,f) \leqslant \log^+ M(r,f) \leqslant \frac{1+3r}{1-r} T\left(\frac{1+r}{2}, f\right),$$

we deduce that (i) and (ii) hold. □

Consider for $k \geqslant 2$ the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

where $A_0, A_1, \cdots, A_{k-1}$ are analytic functions in $\Delta$. It is well-known that all solutions of equation (1.1) are analytic functions in $\Delta$ and that there are exactly $k$ linearly independent solutions of (1.1) (see [11] ). In [14, 15], Juneja, Kapoor and Bajpai have investigated some properties of the entire functions of $[p,q]$-order, and obtained some results. Recently, in [20], by using the concept of $[p,q]$-order Liu, Tu and Shi have considered equation (1.1) with entire coefficients and obtained different results concerning the growth of its solutions.

In this paper, we continue to consider this subject and investigate the complex linear differential equation (1.1) when the coefficients $A_0, A_1, \cdots, A_{k-1}$ are analytic functions in $\Delta$. For $F \subset [0,1)$, the upper and lower densities of $F$ are defined by

$$\overline{dens}_\Delta F = \limsup_{r\to 1^-} \frac{m(F\cap[0,r))}{m([0,r))} \text{ and } \underline{dens}_\Delta F = \liminf_{r\to 1^-} \frac{m(F\cap[0,r))}{m([0,r))},$$

respectively, where $m(G) = \int_G \frac{dt}{1-t}$ for $G \subset [0,1)$. We obtain the following results.

**Theorem 1.1.** *Let $p \geqslant q \geqslant 1$ be integers. Let $H$ be a set of complex numbers satisfying $\overline{dens}_{\Delta}\{|z| : z \in H \subseteq \Delta\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$ be analytic functions in the unit disc $\Delta$ such that for the real constants $\alpha$, $\beta$ where $0 \leqslant \beta < \alpha$, we have*

$$T(r, A_0) \geqslant \exp_p \left\{ \alpha \log_q \left( \frac{1}{1-|z|} \right) \right\} \tag{1.2}$$

*and*

$$T(r, A_j) \leqslant \exp_p \left\{ \beta \log_q \left( \frac{1}{1-|z|} \right) \right\} \quad (j = 1, \cdots, k-1) \tag{1.3}$$

*as $|z| = r \to 1^-$ for $z \in H$. Then, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \geqslant \alpha$.*

**Theorem 1.2.** *Let $p \geqslant q \geqslant 1$ be integers. Let $H$ be a set of complex numbers satisfying $\overline{dens}_{\Delta}\{|z| : z \in H \subseteq \Delta\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$ be analytic functions in the unit disc $\Delta$ satisfying $\max\{\rho_{[p,q]}(A_j) : j = 1, \cdots, k-1\} \leqslant \rho_{[p,q]}(A_0) = \rho$. Suppose that there exists a real number $\mu$ satisfying $0 \leqslant \mu < \rho$ such that for any given $\varepsilon$ $(0 < \varepsilon < \rho - \mu)$ sufficiently small, we have*

$$T(r, A_0) \geqslant \exp_p \left\{ (\rho - \varepsilon) \log_q \left( \frac{1}{1-|z|} \right) \right\} \tag{1.4}$$

*and*

$$T(r, A_j) \leqslant \exp_p \left\{ \mu \log_q \left( \frac{1}{1-|z|} \right) \right\} \quad (j = 1, \cdots, k-1) \tag{1.5}$$

*as $|z| = r \to 1^-$ for $z \in H$. Then, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and*

$$\rho_{[p,q]}(A_0) \leqslant \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leqslant \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \cdots, k-1\}.$$

*Furthermore, if $p > q$, then*

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A_0).$$

**Theorem 1.3.** *Let $p \geqslant q \geqslant 1$ be integers, and let $A_0(z), \cdots, A_{k-1}(z)$ be analytic functions in the unit disc $\Delta$ such that for some integer $s$, $1 \leqslant s \leqslant k-1$ satisfying $\max\{\rho_{[p,q]}(A_j)\ (j \neq s)\} < \rho_{[p,q]}(A_s)$. Then, every admissible solution $f$ of (1.1) with $\rho_{[p,q]}(f) < \infty$ satisfies $\rho_{[p,q]}(f) \geqslant \rho_{[p,q]}(A_s)$.*

**Theorem 1.4.** *Let $p \geqslant q \geqslant 1$ be integers, and let $A_0(z), \cdots, A_{k-1}(z)$ be analytic functions in the unit disc $\Delta$ such that for some integer $s$, $0 \leqslant s \leqslant k-1$, we have $\rho_{[p,q]}(A_s) = \infty$ and $\max\{\rho_{[p,q]}(A_j)\ (j \neq s)\} < \infty$. Then, every solution $f \not\equiv 0$ of (1.1) satisfies $\rho_{[p,q]}(f) = \infty$.*

## 2. Preliminary lemmas

In this section we give some lemmas which are used in the proofs of our theorems.

**Lemma 2.1.** *([11]) Let $f$ be a meromorphic function in the unit disc $\Delta$, and let $k \geqslant 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S\left(r, f\right), \tag{2.1}$$

*where $S(r,f) = O\left(\log^{+} T(r,f) + \log(\frac{1}{1-r})\right)$, possibly outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$.*

In the following, we give the generalized logarithmic derivative lemma.

**Lemma 2.2.** *Let $p \geqslant q \geqslant 1$ be integers. Let $f$ be a meromorphic function in the unit disc $\Delta$ such that $\rho_{[p,q]}\left(f\right) = \rho < \infty$, and let $k \geqslant 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right) \tag{2.2}$$

*holds for all $r$ outside a set $E_2 \subset [0,1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$.*

*Proof.* First for $k=1$. Since $\rho_{[p,q]}\left(f\right) = \rho < \infty$, we have for all $r \to 1^{-}$

$$T\left(r, f\right) \leqslant \exp_p\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}. \tag{2.3}$$

By Lemma 2.1, we have

$$m\left(r, \frac{f'}{f}\right) = O\left(\ln^{+} T\left(r, f\right) + \log(\frac{1}{1-r})\right) \tag{2.4}$$

holds for all $r$ outside a set $E_2 \subset [0,1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$. Hence, we obtain

$$m\left(r, \frac{f'}{f}\right) = O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right),\ r \notin E_2. \tag{2.5}$$

Next, we assume that we have

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right),\ r \notin E_2 \tag{2.6}$$

for some integer $k \geqslant 1$. Since $N\left(r, f^{(k)}\right) \leqslant (k+1)\,N\left(r, f\right)$, it holds that

$$T\left(r, f^{(k)}\right) = m\left(r, f^{(k)}\right) + N\left(r, f^{(k)}\right)$$

$$\leqslant m\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,f\right)+\left(k+1\right)N\left(r,f\right)$$

$$\leqslant m\left(r,\frac{f^{(k)}}{f}\right)+\left(k+1\right)T\left(r,f\right)=O\left(\exp_{p-1}\left\{\left(\rho+\varepsilon\right)\log_q\left(\frac{1}{1-r}\right)\right\}\right)$$

$$+\left(k+1\right)T\left(r,f\right)=O\left(\exp_{p}\left\{\left(\rho+\varepsilon\right)\log_q\left(\frac{1}{1-r}\right)\right\}\right). \tag{2.7}$$

By $(2.4)$ and $(2.7)$, we again obtain

$$m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)=O\left(\exp_{p-1}\left\{\left(\rho+\varepsilon\right)\log_q\left(\frac{1}{1-r}\right)\right\}\right),\ r\notin E_2 \tag{2.8}$$

and hence,

$$m\left(r,\frac{f^{(k+1)}}{f}\right)\leqslant m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+m\left(r,\frac{f^{(k)}}{f}\right)$$

$$=O\left(\exp_{p-1}\left\{\left(\rho+\varepsilon\right)\log_q\left(\frac{1}{1-r}\right)\right\}\right),\ r\notin E_2. \tag{2.9}$$

□

**Lemma 2.3.** ([1]) *Let $g:(0,1)\to\mathbb{R}$ and $h:(0,1)\to\mathbb{R}$ be monotone increasing functions such that $g(r)\leqslant h(r)$ holds outside of an exceptional set $E_3\subset[0,1)$ for which $\int_{E_3}\frac{dr}{1-r}<\infty$. Then, there exists a constant $d\in(0,1)$ such that if $s(r)=1-d(1-r)$, then $g(r)\leqslant h(s(r))$ for all $r\in[0,1)$.*

**Lemma 2.4.** ([13]) *Let $f$ be a solution of equation (1.1), where the coefficients $A_j(z)$ $(j=0,\cdots,k-1)$ are analytic functions in the disc $\Delta_R=\{z\in\mathbb{C}:|z|<R\}$, $0<R\leqslant\infty$. Let $n_c\in\{1,\cdots,k\}$ be the number of nonzero coefficients $A_j(z)$ $(j=0,\cdots,k-1)$, and let $\theta\in[0,2\pi]$ and $\varepsilon>0$. If $z_\theta=\nu e^{i\theta}\in\Delta_R$ is such that $A_j(z_\theta)\neq0$ for some $j=0,\cdots,k-1$, then for all $\nu<r<R$,*

$$\left|f\left(re^{i\theta}\right)\right|\leqslant C\exp\left(n_c\int_\nu^r\max_{j=0,\cdots,k-1}\left|A_j\left(te^{i\theta}\right)\right|^{\frac{1}{k-j}}dt\right), \tag{2.10}$$

*where $C>0$ is a constant satisfying*

$$C\leqslant(1+\varepsilon)\max_{j=0,\cdots,k-1}\left(\frac{\left|f^{(j)}\left(z_\theta\right)\right|}{\left(n_c\right)^j\max\limits_{n=0,\cdots,k-1}\left|A_n\left(z_\theta\right)\right|^{\frac{j}{k-n}}}\right). \tag{2.11}$$

**Lemma 2.5.** *Let $p \geqslant q \geqslant 1$ be integers. If $A_0(z), \cdots, A_{k-1}(z)$ are analytic functions of $[p,q]-$order in the unit disc $\Delta$, then every solution $f \not\equiv 0$ of (1.1) satisfies*

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leqslant \max\left\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \cdots, k-1\right\}. \tag{2.12}$$

*Proof.* Set $\max\left\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \cdots, k-1\right\} = \sigma$. Let $f \not\equiv 0$ be a solution of $(1.1)$. Let $\theta_0 \in [0, 2\pi)$ be such that $\left|f\left(re^{i\theta_0}\right)\right| = M(r,f)$. By Lemma 2.4, we have

$$\begin{aligned} M(r,f) &\leqslant C \exp\left(n_c \int_{\nu}^{r} \max_{j=0,\cdots,k-1} \left|A_j\left(te^{i\theta}\right)\right|^{\frac{1}{k-j}} dt\right) \\ &\leqslant C \exp\left(n_c \int_{\nu}^{r} \max_{j=0,\cdots,k-1} \left(M(r,A_j)\right)^{\frac{1}{k-j}} dt\right) \\ &\leqslant C \exp\left(n_c (r-\nu) \max_{j=0,\cdots,k-1} \left\{M(r,A_j)\right\}\right). \end{aligned} \tag{2.13}$$

By Definition 1.4,

$$M(r,A_j) \leqslant \exp_{p+1}\left\{(\sigma+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\} \quad (j = 0, 1, \cdots, k-1) \tag{2.14}$$

holds for any $\varepsilon > 0$. Hence, from (2.13) and (2.14) we obtain

$$\rho_{M,[p+1,q]}(f) \leqslant \sigma + \varepsilon. \tag{2.15}$$

Since $\varepsilon > 0$ is arbitrary, we have by Proposition 1.1(ii)

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leqslant \sigma = \max\left\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \cdots, k-1\right\}.$$

□

## 3. Proof of Theorem 1.1

Suppose that $f \not\equiv 0$ is a solution of (1.1). By $(1.1)$, we can write

$$A_0(z) = -\left(\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f}\right). \tag{3.1}$$

From the conditions of Theorem 1.1, there is a set $H$ of complex numbers satisfying $\overline{dens}_\Delta\{|z| : z \in H \subseteq \Delta\} > 0$ such that for $z \in H$, we have (1.2) and (1.3) as $|z| \to 1^-$. Set $H_1 = \{r = |z| : z \in H \subseteq \Delta\}$, since $\overline{dens}_\Delta\{|z| : z \in H \subseteq \Delta\} > 0$, then $H_1$ is a set of $r$ with $\int_{H_1} \frac{dr}{1-r} = \infty$. It follows by (1.2), (1.3), (3.1) and Lemma 2.1 that

$$\exp\left\{\alpha \log_q\left(\frac{1}{1-|z|}\right)\right\} \leqslant T(r,A_0) = m(r,A_0)$$

$$\leqslant \sum_{j=1}^{k-1} m\left(r, A_j\right) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O\left(1\right)$$

$$\leqslant (k-1)\exp_p\left\{\beta \log_q\left(\frac{1}{1-|z|}\right)\right\} + O\left(\log^+ T(r,f) + \log(\frac{1}{1-r})\right) \tag{3.2}$$

holds for all $z$ satisfying $|z| = r \in H_1 \backslash E_1$ as $|z| \to 1^-$, where $E_1 \subset [0,1)$ is a set with $\int_{E_1} \frac{dr}{1-r} < \infty$. Noting that $\alpha > \beta \geqslant 0$, by (3.2) we obtain

$$(1 - o\left(1\right))\exp_p\left\{\alpha \log_q\left(\frac{1}{1-|z|}\right)\right\} \leqslant O\left(\log^+ T(r,f) + \log(\frac{1}{1-r})\right) \tag{3.3}$$

for all $z$ satisfying $|z| = r \in H_1 \backslash E_1$ as $|z| \to 1^-$. By Lemma 2.3, we have for any $d \in [0,1)$

$$(1 - o\left(1\right))\exp_p\left\{\alpha \log_q\left(\frac{1}{1-|z|}\right)\right\}$$

$$\leqslant O\left(\log^+ T(1 - d\left(1-r\right), f) + \log(\frac{1}{d\left(1-r\right)})\right) \tag{3.4}$$

as $|z| \to 1^-$. Hence, by $(3.4)$, we obtain $\rho_{[p,q]}\left(f\right) = \rho_{M,[p,q]}\left(f\right) = \infty$ and

$$\rho_{[p+1,q]}\left(f\right) = \rho_{M,[p+1,q]}\left(f\right) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ T\left(r,\ f\right)}{\log_q \frac{1}{1-r}} \geqslant \alpha.$$

Thus Theorem 1.1 is proved.

## 4. Proof of Theorem 1.2

Suppose that $f \not\equiv 0$ is a solution of (1.1). Then for any given $\varepsilon > 0$, by the results of Theorem 1.1, we have $\rho_{[p,q]}\left(f\right) = \rho_{M,[p,q]}\left(f\right) = \infty$ and

$$\rho_{[p+1,q]}\left(f\right) = \rho_{M,[p+1,q]}\left(f\right) \geqslant \rho - \varepsilon. \tag{4.1}$$

Since $\varepsilon > 0$ is arbitrary we get from (4.1) that $\rho_{[p+1,q]}\left(f\right) = \rho_{M,[p+1,q]}\left(f\right) \geqslant \rho = \rho_{[p,q]}\left(A_0\right)$. On the other hand, by Lemma 2.5, we have

$$\rho_{[p+1,q]}\left(f\right) = \rho_{M,[p+1,q]}\left(f\right) \leqslant \max\left\{\rho_{M,[p,q]}\left(A_j\right) : j = 0, 1, \cdots, k-1\right\}. \tag{4.2}$$

It yields

$$\begin{aligned} \rho_{[p,q]}\left(A_0\right) &\leqslant \rho_{[p+1,q]}\left(f\right) = \rho_{M,[p+1,q]}\left(f\right) \\ &\leqslant \max\left\{\rho_{M,[p,q]}\left(A_j\right) : j = 0, 1, \cdots, k-1\right\}. \end{aligned}$$

If $p > q$, then we have

$$\max\left\{\rho_{M,[p,q]}\left(A_j\right) : j = 0, 1, \cdots, k-1\right\} = \rho_{[p,q]}\left(A_0\right).$$

Therefore, we deduce that

$$\rho_{[p+1,q]}\left(f\right) = \rho_{M,[p+1,q]}\left(f\right) = \rho_{[p,q]}\left(A_0\right).$$

## 5. Proof of Theorem 1.3

Set $\max\left\{\rho_{[p,q]}\left(A_j\right)(j\neq s)\right\}=\beta<\rho_{[p,q]}\left(A_s\right)=\alpha$. Suppose that $f$ is an admissible solution of (1.1) with $\rho=\rho_{[p,q]}(f)<\infty$. It follows from (1.1) that

$$A_s(z)=-\frac{f^{(k)}}{f^{(s)}}-A_{k-1}(z)\frac{f^{(k-1)}}{f^{(s)}}-\cdots-A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}}$$

$$-A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}}-\cdots-A_1(z)\frac{f'}{f^{(s)}}-A_0(z)\frac{f}{f^{(s)}}. \tag{5.1}$$

Applying Lemma 2.2, we have

$$m\left(r,\frac{f^{(j+1)}}{f}\right)=O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right)\ (j=0,\cdots,k-1) \tag{5.2}$$

holds for all $r$ outside a set $E_2\subset[0,1)$ with $\int_{E_2}\frac{dr}{1-r}<\infty$. Since $N\left(r,f^{(j+1)}\right)=0$, it holds for $j=0,\cdots,k-1$ that

$$T\left(r,f^{(j+1)}\right)=m\left(r,f^{(j+1)}\right)\leqslant m\left(r,\frac{f^{(j+1)}}{f}\right)+m(r,f)$$

$$\leqslant T(r,f)+m\left(r,\frac{f^{(j+1)}}{f}\right). \tag{5.3}$$

By (5.3), we can obtain from (5.1) and (5.2) that

$$T(r,A_s)\leqslant cT(r,f)+\sum_{j\neq s}T(r,A_j)$$

$$+O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right)\ (r\notin E_2), \tag{5.4}$$

where $c>0$ is a constant. Since $\rho_{[p,q]}(A_s)=\alpha$, there exists a sequence $\{r_n'\}$ $(r_n'\longrightarrow 1^-)$ such that

$$\lim_{r_n'\mapsto 1^-}\frac{\log_p^+T(r_n',A_s)}{\log_q\frac{1}{1-r_n'}}=\alpha. \tag{5.5}$$

Set $\int_{E_2}\frac{dr}{1-r}:=\log\gamma<\infty$. Since $\int_{r_n'}^{1-\frac{1-r_n'}{\gamma+1}}\frac{dr}{1-r}=\log(\gamma+1)$, then there exists a point $r_n\in\left[r_n',1-\frac{1-r_n'}{\gamma+1}\right]-E_2\subset[0,1)$. From

$$\frac{\log_p^+T(r_n,A_s)}{\log_q\frac{1}{1-r_n}}\geqslant\frac{\log_p^+T\left(r_n',A_s\right)}{\log_q\left(\frac{\gamma+1}{1-r_n'}\right)}=\frac{\log_p^+T\left(r_n',A_s\right)}{\log_q\frac{1}{1-r_n'}+\log\left(\frac{\log_{q-1}\left(\frac{\gamma+1}{1-r_n'}\right)}{\log_{q-1}\frac{1}{1-r_n'}}\right)}, \tag{5.6}$$

it follows that

$$\lim_{r_n \to 1^-} \frac{\log_p^+ T(r_n, A_s)}{\log_q \frac{1}{1-r_n}} = \alpha. \tag{5.7}$$

So, for any given $\varepsilon$ $(0 < 2\varepsilon < \alpha - \beta)$, we have

$$T(r_n, A_s) > \exp_p \left\{ (\alpha - \varepsilon) \log_q \left( \frac{1}{1-r_n} \right) \right\} \tag{5.8}$$

and for $j \neq s$

$$T(r_n, A_j) \leqslant \exp_p \left\{ (\beta + \varepsilon) \log_q \left( \frac{1}{1-r_n} \right) \right\} \tag{5.9}$$

hold for $r_n \to 1^-$. By (5.4), (5.8) and (5.9), we obtain for $r_n \to 1^-$

$$\exp_p \left\{ (\alpha - \varepsilon) \log_q \left( \frac{1}{1-r_n} \right) \right\} \leqslant (k-1) \exp_p \left\{ (\beta + \varepsilon) \log_q \left( \frac{1}{1-r_n} \right) \right\}$$

$$+ cT(r_n, f) + O\left( \exp_{p-1} \left\{ (\rho + \varepsilon) \log_q \left( \frac{1}{1-r_n} \right) \right\} \right). \tag{5.10}$$

Noting that $\alpha - \varepsilon > \beta + \varepsilon$, it follows from (5.10) that for $r_n \to 1^-$

$$(1 - o(1)) \exp_p \left\{ (\alpha - \varepsilon) \log_q \left( \frac{1}{1-r_n} \right) \right\} \leqslant cT(r_n, f)$$

$$+ O\left( \exp_{p-1} \left\{ (\rho + \varepsilon) \log_q \left( \frac{1}{1-r_n} \right) \right\} \right). \tag{5.11}$$

Therefore, by (5.11) we obtain

$$\limsup_{r_n \mapsto 1^-} \frac{\log_p^+ T(r_n, f)}{\log_q \frac{1}{1-r_n}} \geqslant \alpha - \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, we get $\rho_{[p,q]}(f) \geqslant \rho_{[p,q]}(A_s) = \alpha$. This proves Theorem 1.3.

## 6. Proof of Theorem 1.4

Suppose that $f \not\equiv 0$ is a solution of (1.1). Setting $\max \left\{ \rho_{[p,q]}(A_j) \, (j \neq s) \right\} = \beta$, then for a given $\varepsilon > 0$, we have

$$T(r, A_j) \leqslant \exp_p \left\{ (\beta + \varepsilon) \log_q \left( \frac{1}{1-r} \right) \right\} \quad (j \neq s) \tag{6.1}$$

for $r \to 1^-$. Now we can write from (1.1)

$$A_s(z) = -\frac{f^{(k)}}{f^{(s)}} - A_{k-1}(z)\frac{f^{(k-1)}}{f^{(s)}} - \cdots - A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}}$$

$$- A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} - \cdots - A_1(z)\frac{f'}{f^{(s)}} - A_0(z)\frac{f}{f^{(s)}}. \tag{6.2}$$

Hence by (5.3) and (6.2) we obtain that

$$T(r, A_s) \leqslant cT(r,f) + \sum_{j=0}^{k-1} m\left(r, \frac{f^{(j+1)}}{f}\right) + \sum_{j\neq s} T(r, A_j), \tag{6.3}$$

where $c > 0$ is a constant. If $\rho = \rho_{[p,q]}(f) < \infty$, then by Lemma 2.2

$$m\left(r, \frac{f^{(j+1)}}{f}\right) = O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right) \quad (j = 0, \cdots, k-1) \tag{6.4}$$

holds for all $r$ outside a set $E_2 \subset [0,1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$. For $r \to 1^-$, we have

$$T(r,f) \leqslant \exp_p\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}. \tag{6.5}$$

Thus, by $(6.1), (6.3), (6.4)$ and $(6.5)$, we get

$$T(r, A_s) \leqslant (k-1)\exp_p\left\{(\beta+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}$$

$$+c\exp_p\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}$$

$$+ O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right) \tag{6.6}$$

for $r \notin E_2$ and $r \to 1^-$. By Lemma 2.3, we have for any $d \in [0,1)$

$$T(r, A_s) \leqslant (k-1)\exp_p\left\{(\beta+\varepsilon)\log_q\left(\frac{1}{d(1-r)}\right)\right\}$$

$$+c\exp_p\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{d(1-r)}\right)\right\}$$

$$+ O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{d(1-r)}\right)\right\}\right) \tag{6.7}$$

for $r \to 1^-$. Therefore,

$$\rho_{[p,q]}(A_s) \leqslant \max\{\beta+\varepsilon, \rho+\varepsilon\} < \infty.$$

This contradicts the fact that $\rho_{[p,q]}(A_s) = \infty$. This proves Theorem 1.4.

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