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NUMERICAL SOLVING OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS WITH DISCONTINUITIES[1]

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Abstract

One-dimensional convection-diffusion problem with interior layers caused by the discontinuity of data is considered. Though standard Galerkin finite element method (FEM) generates oscillations in the numerical solutions, we prove its convergence in the ε-weighted norm of the first order on a class of layer-adapted meshes. We use streamline-diffusion finite element method (SDFEM) in order to stabilize Galerkin FEM and prove ε-uniform convergence of the second order.

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1 Introduction

On the interval $\Omega = (0,1)$, a one-dimensional convection-diffusion equation with a discontinuous source term is considered. The source term has a jump discontinuity at an interior point d of the domain. A discontinuity is also allowed in the convection coefficient at the same point. If we introduce the notation $\Omega_1 = (0,d)$ and $\Omega_2 = (d,1)$, then the considered problem is: find a function $u \in C^2(\Omega_1 \cup \Omega_2) \cap C^1(\overline{\Omega})$ such that

$$(1) \qquad \begin{cases} \mathcal{L}u := -\varepsilon u'' + bu' = f \quad \text{on} \quad \Omega_1 \cup \Omega_2, \\ u(0) = u(1) = 0, \\ |[b](d)| \le C, \quad |[f](d)| \le C, \end{cases}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter, d is an ε-independent point and C is a generic constant independent of ε and a discretization mesh. The jump of a function g at d is denoted with $[g](d) = g(d^+) - g(d^-)$.

Convection-diffusion problems with sufficiently smooth data have been examined by many authors; see for instance [2], [4], [5] and references therein. For

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such problems, a boundary layer appears in the solution near the point $x = 1$. Here, due to the discontinuity of the functions b and f, the solution of the problem (1) has an additional interior layer near the point $x = d$.

The problem of this type, but with a layer in the vicinity of $x = 0$, is examined in [3]. In that paper, an ε-uniformly convergent method of the first order is constructed. This method involves a piecewise-uniform Shishkin mesh, which is fitted to the interior and boundary layers, and the standard upwind finite difference scheme.

Here we extend the analysis from [6] and [7], where in (1), the function b is continuous and the right-hand side has a point source. Similarly to [6] and [7], we use a class of layer-adapted meshes and prove the robust first-order of accuracy for the standard Galerkin finite element method and the second order for the streamline diffusion FEM. We remark that these estimates directly hold for the continuous b as well.

Throughout the paper we assume that b and f are sufficiently smooth on $\overline{\Omega} \setminus \{d\}$ and

$$b(x) \geq \beta > 0, \quad -b'(x)/2 \geq \gamma > 0, \quad x \in \overline{\Omega} \setminus \{d\}. \tag{2}$$

2 Solution Decomposition

In order to construct layer-adapted meshes and perform error analysis, we need a priori information on the behavior of the solution u of the problem (1) and its derivatives.

When b and f in (1) are sufficiently smooth functions, the properties of the corresponding solution are introduced in [5]. Here we have the following theorem.

Theorem 2.1. *Solution u of (1) can be decomposed as $u = S + E$, where for $k = 0, 1, \ldots, q$ it holds*

$$|S^{(k)}(x)| \leq C, \quad x \in \Omega_1 \cup \Omega_2, \tag{3}$$

$$|E^{(k)}(x)| \leq \begin{cases} C\varepsilon^{-k} e^{-\beta(d-x)/\varepsilon}, & x \in \Omega_1, \\ C\varepsilon^{-k} e^{-\beta(1-x)/\varepsilon}, & x \in \Omega_2, \end{cases} \tag{4}$$

and the maximal order q depends on the smoothness of the functions b and f on $\overline{\Omega} \setminus \{d\}$.

Proof. The proof can be easily derived following the arguments from [4, Section 3.4] and [7, Theorem 2.3.1]. □

3 Layer-adapted meshes

For discretization of the domain we use a class of piecewise-uniform meshes of Shishkin type which are fitted to the layers at $x = d$ and $x = 1$. More on layer-adapted meshes for singularly perturbed problems can be found in [4].

Let $N > 4$ be an even integer and set

$$\lambda_1 = \min\left\{\frac{d}{2}, \frac{\tau}{\beta}\,\varepsilon\ln N\right\}, \quad \lambda_2 = \min\left\{\frac{1-d}{2}, \frac{\tau}{\beta}\,\varepsilon\ln N\right\}, \quad \tau \geq 1.$$

Assumption 1. We assume that $\lambda_1 = \lambda_2 = \lambda = \tau/\beta\,\varepsilon\ln N$, since otherwise N is exponentially large relative to ε, which is rare in practice.

We construct the discretization mesh such that it is equidistant on $\overline{\Omega}_c$ and gradually subdivided it on $\overline{\Omega}_f$, where $\Omega_c = (0, d-\lambda) \cup (d, 1-\lambda)$ and $\Omega_f = (d-\lambda, d) \cup (1-\lambda, 1)$. The mesh transition points are defined to be

$$x_{\frac{N}{4}} = d - \lambda, \quad x_{\frac{N}{2}} = d, \quad x_{\frac{3N}{4}} = 1 - \lambda.$$

As a result of the existence of additional interior layer at the point $x = d$, we need two mesh generating functions ϕ_1 and ϕ_2. The functions ϕ_1 and ϕ_2 are both continuous, piecewise continuously differentiable, strictly decreasing with the properties

$$\phi_1(1/4) = \ln N, \quad \phi_1(1/2) = 0, \quad \phi_2(3/4) = \ln N, \quad \phi_2(1) = 0.$$

The mesh characterizing functions are defined by $\psi_k = e^{-\phi_k}$, $k = 1, 2$.

Remark 3.1. In numerical experiments (Section 8) we shall use standard Shishkin (S-) mesh, Bakhvalov-Shishkin (BS-) and modified Bakhvalov-Shishkin (mBS-) mesh. One can find the corresponding mesh characterizing functions in [7]. Here we just mention that the values for $\max|\psi'|$ ($\psi = \psi_k$, $k = 1, 2$) for those specially chosen meshes are

- S-mesh: $\max|\psi'| = C\ln N$,
- BS-mesh, mBS-mesh: $\max|\psi'| = C$.

The term $\max|\psi'|$ appears further in the results on the interpolation error (Theorem 4.1), as well as in the error bounds for the Galerkin FEM (Theorem 4.2) and SDFEM (Theorems 7.1 and 7.2).

The points x_i of the layer-adapted discretization mesh Ω_ε^N are given by

$$(5) \qquad x_i = \begin{cases} 4(d-\lambda)t_i, & i \in I_1 \cup \{0\} = \left\{1, \ldots, \frac{N}{4}\right\} \cup \{0\}, \\ d - \frac{\tau\varepsilon}{\beta}\,\phi_1(t_i), & i \in I_2 = \left\{\frac{N}{4} + 1, \ldots, \frac{N}{2}\right\}, \\ d + 4(1-d-\lambda)\left(t_i - \frac{1}{2}\right), & i \in I_3 = \left\{\frac{N}{2} + 1, \ldots, \frac{3N}{4}\right\}, \\ 1 - \frac{\tau\varepsilon}{\beta}\,\phi_2(t_i), & i \in I_4 = \left\{\frac{3N}{4} + 1, \ldots, N\right\}, \end{cases}$$

with $t_i = i/N$, $i = 0, 1, \ldots, N$.

Assumption 2. Assume that the mesh generating functions satisfy

$$N^{-1}\max|\phi'| \leq C,$$

with the appropriate function $\phi = \phi_k$, $k = 1, 2$.

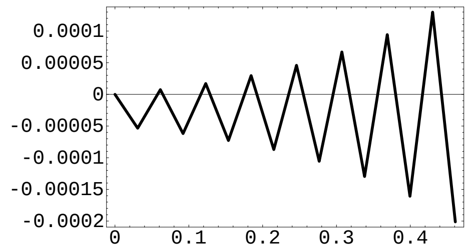

Figure 1: Numerical solution of the problem $-10^{-3}u''+u' = \delta_{0.5}$, $u(0) = u(1) = 0$ depicted outside the layers, obtained with the Galerkin FEM on S-mesh with $N = 2^6$

4 The Galerkin Finite Element Method

We start our investigation with the variational formulation of problem (1)

$$\text{(6)} \qquad \text{find} \quad u \in V = H_0^1(\Omega) \quad \text{such that} \quad a(u,v) = l(v) \quad \text{for all} \quad v \in V,$$

where

$$a(v,w) = (\varepsilon v', w') + (bv', w), \quad l(w) = (f,w), \quad v,w \in V,$$

and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Since a is a continuous, coercive bilinear form and l is a continuous linear functional, there exists a unique solution u of (6) by the Lax-Milgram theorem.

Let $\Omega^N = \{x_0, x_1, \ldots, x_N\}$, $N \in \mathbb{N}$, be an arbitrary mesh on the domain Ω with $h_i = x_i - x_{i-1}$ the local mesh step size. Let $\{\varphi_i : 0 \le i \le N\}$ be the set of standard piecewise linear hat functions that satisfy $\varphi_i(x_j) = \delta_{ij}$. Let the finite element space be $V_h = span\{\varphi_0, \varphi_1, \ldots, \varphi_N\}$.

The discrete problem which defines conforming FEM for problem (1) is

$$\text{(7)} \qquad \text{find} \quad u_h \in V_h \subset V \quad \text{such that} \quad a(u_h, v_h) = l(v_h) \quad \text{for all} \quad v_h \in V_h.$$

This method is called the Galerkin finite element method (FEM). Again the Lax-Milgram theorem guarantees the exsistence of a unique solution of problem (7).

According to the numerical results (see Figure 1), the Galerkin FEM applied to (1) produces oscillatory numerical solution, even if we use a layer-adapted mesh. The oscillations are a direct outcome of the instability of the Galerkin FEM. However, the results in Table 1 show the second order of convergence of the Galerkin FEM on S-mesh. Therefore, we choose streamline-diffusion finite element method (see Section 5) in order to stabilize the Galerkin FEM. In the sequel, we first study the convergence of the Galerkin FEM on the layer-adapted mesh (5). For that purpose we introduce the estimates for the interpolation error in L^∞-norm.

Table 1: L^∞-norm of the discrete error E^N, order of convergence p^N and Shishkin order of convergence p_S^N for the problem $-10^{-3}u'' + u' = \delta_{0.5}$, $u(0) = u(1) = 0$ (Galerkin FEM, S-mesh)

N	E^N	p^N	p_S^N	N	E^N	p^N	p_S^N
16	0.070040	1.43969	2.1232	512	0.000296	1.70172	2.00682
32	0.025820	1.55343	2.10787	1024	0.000091	1.72896	2.00451
64	0.008796	1.59322	2.04887	2048	0.000027	1.75013	2.00381
128	0.002915	1.62864	2.01725	4096	0.000008	1.76754	2.00504
256	0.000942	1.66817	2.00969	8192	0.000002	-	-

Theorem 4.1. *Suppose that Assumptions 1 and 2 are satisfied. Then the linear interpolant* $u^I \in V_h$ *of the function* u *on mesh (5) with* $\tau \geq 2$ *satisfies*

$$|u(x) - u^I(x)| \leq \begin{cases} CN^{-2} \max |\psi'|^2, & x \in \overline{\Omega}_f, \\ CN^{-2}, & x \in \overline{\Omega}_c, \end{cases}$$

$$\varepsilon |(u - u^I)'(x)| \leq \begin{cases} CN^{-1} \max |\psi'|, & x \in \overline{\Omega}_f, \\ CN^{-2}, & x \in \overline{\Omega}_c. \end{cases}$$

Proof. Follows from the arguments of Theorem 2.3.2 from [7]. □

Let us introduce the ε-weighted H^1-norm

$$\|v\|_\varepsilon^2 := \varepsilon \|v'\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}, \quad \|v\|^2_{L^2(\Omega)} = \int_\Omega v(x)^2 \, dx.$$

The second assumption in (2) ensures that the bilinear form a is coercive with respect to the energy norm.

Theorem 4.2. *Suppose that the mesh generating and mesh characterizing functions in (5) with* $\tau \geq 2$ *satisfy Assumption 2 and*

$$\max |\psi'| \ln^{1/2} N \leq CN. \tag{8}$$

Then

$$\|u - u_h\|_\varepsilon \leq CN^{-1} \max |\psi'|$$

for the error of the Galerkin FEM.

Proof. Let $\eta = u - u^I$ i $\chi = u^I - u_h$. Using the technique from [5] and Theorem 4.1 we get

$$\|\eta\|_\varepsilon \leq CN^{-1} \max |\psi'|. \tag{9}$$

To bound χ we use coercivity of a with respect to $\|\cdot\|_\varepsilon$, the Galerkin orthogonality property and integration by parts:

$$\min\{1,\gamma\}\|\chi\|_\varepsilon^2 \le a(\chi,\chi) = -a(\eta,\chi) = -(\varepsilon\eta',\chi') + (b'\eta,\chi) + (b\eta,\chi').$$

For the first term, integration by parts gives

$$(\varepsilon\eta',\chi') = \varepsilon\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\eta'\chi'\,dx = \varepsilon\sum_{k=1}^{N}\left(\eta\chi'\Big|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k}\eta\chi''\,dx\right) = 0,$$

since $\eta(x_{k-1}) = \eta(x_k) = 0$ and χ is a linear function. For the second term, it holds

$$(b'\eta,\chi) \le C\int_\Omega |\eta\chi|\,\,dx \le C\|\eta\|_{L^2(\Omega)}\|\chi\|_{L^2(\Omega)} \le C\|\eta\|_\varepsilon\|\chi\|_\varepsilon.$$

The third term is bounded using Hölder's inequality

$$\begin{aligned}(b\eta,\chi') \quad \le \quad & C\left(\|\eta\|_{L^\infty(0,d-\lambda)}\|\chi'\|_{L^1(0,d-\lambda)} + \|\eta\|_{L^\infty(d-\lambda,d)}\|\chi'\|_{L^1(d-\lambda,d)}\right.\\ & \left. + \|\eta\|_{L^\infty(d,1-\lambda)}\|\chi'\|_{L^1(d,1-\lambda)} + \|\eta\|_{L^\infty(1-\lambda,1)}\|\chi'\|_{L^1(1-\lambda,1)}\right).\end{aligned}$$

On $(0,d-\lambda)$, analogously on $(d,1-\lambda)$, we get

$$\begin{aligned}\|\chi'\|_{L^1(0,d-\lambda)} &= \sum_{k=1}^{N/4}\int_{x_{k-1}}^{x_k}|\chi'|\,dx \le C\sum_{k=1}^{N/4}\frac{1}{h_k}\int_{x_{k-1}}^{x_k}|\chi|\,dx\\ &= CN\int_0^{d-\lambda}|\chi|\,dx \le CN\sqrt{\int_0^{d-\lambda}1^2\,dx}\sqrt{\int_0^{d-\lambda}\chi^2\,dx}\\ &= CN\sqrt{d-\lambda}\,\|\chi\|_{L^2(0,d-\lambda)} \le CN\|\chi\|_\varepsilon.\end{aligned}$$

On $(d-\lambda,d)$, analogously on $(1-\lambda,1)$, we get

$$\begin{aligned}\|\chi'\|_{L^1(d-\lambda,d)} &\le \sqrt{\int_{d-\lambda}^{d}1^2\,dx}\sqrt{\int_{d-\lambda}^{d}(\chi')^2\,dx}\\ &= C\sqrt{\varepsilon}\sqrt{\ln N}\|\chi'\|_{L^2(d-\lambda,d)}\\ &\le C\sqrt{\ln N}\|\chi\|_\varepsilon,\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and $\lambda = C\varepsilon\ln N$. These bounds yield

$$\min\{1,\gamma\}\|\chi\|_\varepsilon \le C\left(\|\eta\|_\varepsilon + N\|\eta\|_{L^\infty(\Omega_c)} + \sqrt{\ln N}\|\eta\|_{L^\infty(\Omega_f)}\right).$$

Then, from Theorem 4.1 and (9) we get

$$\|\chi\|_\varepsilon \le C\left(N^{-1}\max|\psi'| + N^{-1} + N^{-2}\max|\psi'|^2\sqrt{\ln N}\right).$$

From the inequality (8) we get

$$\|\chi\|_\varepsilon \le CN^{-1}\max|\psi'|. \tag{10}$$

Using a triangle inequality, (9) and (10), we complete the proof. □

Theorem 4.2 proves that the Galerkin FEM for the problem (1) on layer-adapted meshes of Shishkin type (5) has the first order of convergence with respect to the ε-weighted energy norm.

5 Streamline-diffusion Finite Element Method

Our aim is to create a method that is more stable than the Galerkin FEM when applied to (1). The idea is to add weighted residuals to the Galerkin FEM. Such method is called the streamline-diffusion finite element method (SDFEM).

The discrete problem of SDFEM for problem (1) is

$$\text{(11)} \quad \text{find} \quad u_h \in V_h \subset V \quad \text{such that} \quad a_h(u_h, v_h) = l_h(v_h) \quad \text{for all} \quad v_h \in V_h,$$

where

$$a_h(v_h, w_h) = (\varepsilon v_h', w_h') + (b v_h', w_h) + \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_k(-\varepsilon v_h'' + b v_h') b w_h' \, dx,$$

$$l_h(w_h) = (f, w_h) + \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_k f b w_h' \, dx.$$

Here, $\delta_i \in \mathbb{R}^+$, $i = 1, 2, \ldots, N$, is called the streamline-diffusion parameter (SD-parameter).

We define a streamline-diffusion norm (SD-norm) in the following way

$$\|v\|_{SD}^2 = \varepsilon \|v'\|_{L^2(\Omega)}^2 + \gamma \|v\|_{L^2(\Omega)}^2 + \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_k b^2 (v')^2 \, dx, \quad v \in V.$$

The bilinear form a_h is coercive and continuous with respect to the SD-norm. Therefore, there exists a unique solution of problem (11) by the Lax-Milgram theorem.

Take V_h to be the space of piecewise linear functions on an arbitrary mesh Ω^N, $N \in \mathbb{N}$. Then, the SDFEM reduces to the scheme

$$\text{(12)} \qquad \begin{cases} \mathcal{L}^N u_i := -\varepsilon(D^+ u_i - D^- u_i) + \alpha_i D^+ u_i + \beta_i D^- u_i = l_h(\varphi_i), \\ \\ u_0 = u_N = 0, \end{cases}$$

where $u_i = u_h(x_i)$, $i = 1, 2, \ldots, N-1$ and

$$D^+ u_i = \frac{u_{i+1} - u_i}{h_{i+1}}, \quad D^- u_i = \frac{u_i - u_{i-1}}{h_i},$$

$$\alpha_i = h_{i+1} \int_{x_i}^{x_{i+1}} (b \varphi_{i+1}' \varphi_i + \delta_{i+1} b^2 \varphi_{i+1}' \varphi_i') \, dx, \;\; \beta_i = h_i \int_{x_{i-1}}^{x_i} (b \varphi_i' \varphi_i + \delta_i b^2 \varphi_i' \varphi_i') \, dx.$$

Denote $\bar{b} = \|b\|_{L^\infty(\overline{\Omega}\setminus\{d\})}$. Now we are studying properties of the matrix $A = [a_{ij}] = [a_h(\varphi_j, \varphi_i)]$ which corresponds to the scheme (12). By choosing the SD-parameter to be

$$\delta_i = 0, \quad \text{if} \quad h_i \le \frac{2\varepsilon}{\bar{b}}, \tag{13}$$

$$\delta_i = -\left(\int_{x_{i-1}}^{x_i} b\varphi_i'\varphi_{i-1}\,dx\right)\left(\int_{x_{i-1}}^{x_i} b^2\varphi_i'\varphi_{i-1}'\,dx\right)^{-1}, \quad \text{if} \quad h_i > \frac{2\varepsilon}{\bar{b}}. \tag{14}$$

we get that A is an L-matrix. It can be easily shown that $x = [1, \dots, 1]^T$ is a majorizing element for the matrix A, i.e. A is an inverse-monotone matrix. Hence, A is an M-matrix.

Considering the behavior of h_i on the mesh of Shishkin type (5), we obtain

$$\delta_i \le CN^{-1}, \quad i = 1, 2, \dots, N. \tag{15}$$

6 Discrete Green's Functions

In this section we study the discrete Green's functions for (12). One can define a discrete Green's function λ^j by $a_h(\varphi_i, \lambda^j) = \delta_{ij}$, $i = 0, 1, \dots, N$, where δ_{ij} is the Kronecker symbol. Their existence follows from the coercivity of the bilinear form a_h.

Write (12) in the form

$$\begin{cases} -\dfrac{\varepsilon}{\hbar_i}(p_{i+1}D^+u_i - p_iD^-u_i) + r_i\dfrac{u_i - u_{i-1}}{\hbar_i} + q_iu_i = l_h^i, \\ u_0 = u_N = 0, \end{cases}$$

with the appropriate right-hand side l_h^i and coefficients p_i, q_i and r_i which have the following properties

$$p_i \ge p > 0, \quad q_i \ge 0, \quad r_i \ge r > 0. \tag{16}$$

It can be easily shown that for difference scheme (12) we have

$$p_i = 1 - \frac{\alpha_{i-1}}{\varepsilon}, \quad q_i = 0, \quad r_i = \frac{\alpha_{i-1} + \beta_i}{h_i}.$$

Lemma 6.1. *If the conditions*

$$\varepsilon < \min\{d, 1-d\}\bar{b}N^{-1}, \tag{17}$$

$$N^{-1}\max|\phi'| \le \frac{2\beta}{\tau\bar{b}}(1-p), \tag{18}$$

are satisfied for some $0 < p < 1$, *then* $p_i \ge p > 0$, $r_i \ge \beta > 0$.

Proof. Let $i \in I_1 \cup I_3$. From (17) we get

$$h_i = 4(d-\lambda)(t_i - t_{i-1}) = \frac{4}{N}(d-\lambda) \geq 2dN^{-1} > \frac{2\varepsilon}{\overline{b}}, \quad i \in I_1,$$

$$h_i = 4(1-d-\lambda)(t_i - t_{i-1}) = \frac{4}{N}(1-d-\lambda) \geq 2(1-d)N^{-1} > \frac{2\varepsilon}{\overline{b}}, \quad i \in I_3.$$

According to the choice (14) of parameters δ_i, we have $\alpha_{i-1} = 0$. Hence $p_i = 1 > p > 0$.

Let $i \in I_2 \cup I_4$. From (18) we get

$$h_i = \frac{\tau\varepsilon}{\beta}\left(\phi(t_{i-1}) - \phi(t_i)\right) \leq \frac{\tau\varepsilon}{\beta} N^{-1} \max|\phi'| \leq \frac{2\varepsilon}{\overline{b}}(1-p) \leq \frac{2\varepsilon}{\overline{b}}.$$

Now $\delta_i = 0$ and

$$p_i = 1 - \frac{\alpha_{i-1}}{\varepsilon} \geq 1 - \frac{h_i}{2\varepsilon}\overline{b} \geq 1 - \frac{\tau\overline{b}}{2\beta} N^{-1} \max|\phi'| \geq p > 0.$$

Finally we get

$$r_i \geq \frac{\beta}{h_i^2} \int_{x_{i-1}}^{x_i} (x_i - x)\, dx + \frac{\beta}{h_i^2} \int_{x_{i-1}}^{x_i} (x - x_{i-1})\, dx = \beta > 0,$$

that completes the proof. □

When the conditions (16) are satisfied, we can use Lemma 5.2 and 5.3 from [1] to conclude

$$\|\lambda^i\|_{L^\infty(\overline{\Omega})} \leq C, \quad \|(\lambda^i)'\|_{L^1(\Omega)} \leq C. \tag{19}$$

Remark 6.1. The bound (17) does not constitute a major restriction, since we already have $\varepsilon \leq CN^{-1}$ from Assumption 1.

Remark 6.2. For S- and mBS-mesh (18) holds true, but it is not valid for BS-mesh. However, the numerical results in Section 8 show ε-uniform convergence of second order for SDFEM on the BS-mesh.

Remark 6.3. Following the steps in the proof of the previous lemma, we conclude that if $i \in I_1 \cup I_3$ then $\delta_i \leq CN^{-1}$. Also, we have $\delta_i = 0$ for $i \in I_2 \cup I_4$.

7 Error analysis for SDFEM

The SDFEM is a consistent method. Hence, the error at the arbitrary mesh point is

$$u(x_i) - u_h(x_i) = a_h(u^I - u, \lambda^i), \quad x_i \in \Omega_\varepsilon^N.$$

Theorem 7.1. *Let u and u_h be solutions of problems (1) and (11), respectively, and let assumptions from Lemma 6.1 hold true. Then on the layer-adapted (5) with $\tau \geq 2$, the error at the mesh points has the property*

$$|u(x_i) - u_h(x_i)| \leq CN^{-2} \max|\psi'|^2, \quad x_i \in \Omega_\varepsilon^N. \tag{20}$$

Proof. Let x_i be an arbitrary mesh point. Using definition of the bilinear form a_h and properties of the linear interpolant u^I we get

$$\begin{aligned} a_h(u^I-u,\lambda^i) &= \varepsilon\int_0^1 (u^I-u)'(\lambda^i)'\,dx + \int_0^1 b(u^I-u)'\lambda^i\,dx \\ &+\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\varepsilon\delta_k u'' b(\lambda^i)'\,dx + \sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\delta_k b^2(u^I-u)'(\lambda^i)'\,dx. \end{aligned} \tag{21}$$

For the first term on the right-hand side in (21), we obtain

$$\varepsilon\int_0^1 (u^I-u)'(\lambda^i)'\,dx = 0, \tag{22}$$

where we have used integration by parts, $u^I(x_i)=u(x_i)$, $i=0,1,\dots,N$, and $(\lambda^i)''=0$. For the second term, the integration by parts gives

$$\begin{aligned} \left|\int_0^1 b(u^I-u)'\lambda^i\,dx\right| &= \left|-\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}(u^I-u)(b'\lambda^i+b(\lambda^i)')\,dx\right| \\ &\le C\|u-u^I\|_{L^\infty(\overline{\Omega})}\left(\|\lambda^i\|_{L^\infty(\overline{\Omega})}+\|(\lambda^i)'\|_{L^1(\Omega)}\right) \\ &\le C\|u-u^I\|_{L^\infty(\overline{\Omega})}. \end{aligned} \tag{23}$$

For the third term in (21), we use the decomposition $u=S+E$ from Theorem 2.1. For the regular component, we obtain

$$\left|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\varepsilon\delta_k S'' b(\lambda^i)'\,dx\right| \le C\varepsilon N^{-1}\|S''\|_{L^\infty(\overline{\Omega})}\|(\lambda^i)'\|_{L^1(\Omega)} \le C\varepsilon N^{-1},$$

where we have applied (15) and (19). Using the properties of SD-parameters δ_i from Remark 6.3 and smoothness of the function b, we get

$$\begin{aligned} &\left|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\varepsilon\delta_k E'' b(\lambda^i)'\,dx\right| \le \\ &\le C\varepsilon N^{-1}\left(\sum_{k=1}^{N/4}\int_{x_{k-1}}^{x_k}|E''||(\lambda^i)'|\,dx + \sum_{k=N/2+1}^{3N/4}\int_{x_{k-1}}^{x_k}|E''||(\lambda^i)'|\,dx\right). \end{aligned}$$

The local mesh step size h_i can be bounded with $h_i \ge 2\max\{d,1-d\}N^{-1}$, $i\in I_1\cup I_3$. Hence, the functions $\lambda^i\in V_h$ have the property

$$|(\lambda^i)'(x)| = \frac{1}{h_k}|\lambda^i(x_k)-\lambda^i(x_{k-1})| \le CN\|\lambda^i\|_{L^\infty(\overline{\Omega})} \le CN,$$

for all $x \in [x_{k-1}, x_k] \subset \overline{\Omega}_c$. Using the previous estimates and (4), we get

$$\begin{aligned}
&\left|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} \varepsilon\delta_k E'' b(\lambda^i)'\,dx\right| \\
&\le C\varepsilon\left(\sum_{k=1}^{N/4}\int_{x_{k-1}}^{x_k}|E''|\,dx + \sum_{k=N/2+1}^{3N/4}\int_{x_{k-1}}^{x_k}|E''|\,dx\right) \\
&\le C\varepsilon^{-1}\left(\sum_{k=1}^{N/4}\int_{x_{k-1}}^{x_k} e^{-\beta(d-x)/\varepsilon}\,dx + \sum_{k=N/2+1}^{3N/4}\int_{x_{k-1}}^{x_k} e^{-\beta(1-x)/\varepsilon}\,dx\right) \\
&= C\left(\sum_{k=1}^{N/4} e^{-\beta(d-x)/\varepsilon}\Big|_{x_{k-1}}^{x_k} + \sum_{k=N/2+1}^{3N/4} e^{-\beta(1-x)/\varepsilon}\Big|_{x_{k-1}}^{x_k}\right) \\
&\le CN^{-\tau}.
\end{aligned}$$

Finally, for the third term in (21) we obtain

$$\left|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\varepsilon\delta_k u'' b(\lambda^i)'\,dx\right| \le C(\varepsilon N^{-1} + N^{-\tau}). \tag{24}$$

It remains to estimate the last sum in (21). We get

$$\begin{aligned}
\left|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\delta_k b^2(u^I-u)'(\lambda^i)'\,dx\right| &= \left|-\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\delta_k(u^I-u)(b^2(\lambda^i)')'\,dx\right| \\
&\le CN^{-1}\|u-u^I\|_{L^\infty(\overline{\Omega})},
\end{aligned} \tag{25}$$

where we have used integration by parts and properties of δ_i, b and λ^i. Collecting (22)-(25) and using assumptions $\varepsilon \le CN^{-1}$ and $\tau \ge 2$ we get

$$\begin{aligned}
|u(x_i) - u_h(x_i)| &= |a_h(u^I - u, \lambda^i)| \\
&\le C\left(\varepsilon N^{-1} + N^{-\tau} + \|u-u^I\|_{L^\infty(\overline{\Omega})}\right) \\
&\le CN^{-2}\max|\psi'|^2.
\end{aligned}$$

□

A direct consequence of Remark 3.1 when applied to (20) is the following corollary.

Colorallary 7.1. *The SDFEM when applied to* (1) *is of second order of accuracy on the BS- and mBS-mesh, and of almost second order of convergence on the S-mesh.*

We can now state the main result on the ε-uniform error estimate for SDFEM applied to (1) on the discretization mesh (5).

Theorem 7.2. *Let u and u_h be solutions of the problems (1) and (11), respectively, and let assumptions from Lemma 6.1 hold true. Then, on the mesh (5) with $\tau \geq 2$, the error $u - u_h$ has the property*

$$|u(x) - u_h(x)| \leq CN^{-2} \max |\psi'|^2, \quad x \in \overline{\Omega}.$$

Proof. Using the technique from [7], one can prove the following estimate

$$|u(x) - u_h(x)| \leq \|u - u^I\|_{L^\infty[x_{i-1},x_i]} + \max_{0\leq i\leq N} |u_h(x_i) - u(x_i)|, \quad x \in [x_{i-1}, x_i].$$

Using Theorems 4.1 and 7.1 we complete the proof. □

Colorallary 7.2. *Let u and u_h be solutions of the problems (1) and (11), respectively, and let assumptions from Lemma 6.1 hold true. Then for $x \in \overline{\Omega}$ we have*

$$|u(x) - u_h(x)| \leq \begin{cases} CN^{-2} \ln^2 N, & \text{on } S\text{-mesh}, \\ CN^{-2}, & \text{on } BS\text{- and } mBS\text{-mesh}. \end{cases}$$

8 Numerical results

In this section we experimentally verify the assertion of Theorem 7.1.

For two test problems, in Tables 2-3 we present maximum pointwise errors E^N given with

$$E^N = \max_{\varepsilon} \max_{x_i \in \Omega_\varepsilon^N} |u(x_i) - u_h(x_i)|\,,$$

where $\varepsilon = 10^{-2}, \ldots, 10^{-9}$ and $\tau = 2$. We also compute the rates of convergence using the standard formula

$$p^N = \log_2(E^N/E^{2N}),$$

and the Shishkin-rate of convergence for the S-mesh

$$p_S^N = \frac{\ln(E^N/E^{2N})}{\ln(2\ln N/\ln(2N))}.$$

Test problem 1. Let

$$\text{(26)} \qquad \begin{cases} -\varepsilon u''(x) + b(x)u'(x) = f(x), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where

$$b(x) = \begin{cases} 1, & x \leq 0.5, \\ 4, & x > 0.5, \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 0.7, & x \leq 0.5, \\ -0.6, & x > 0.5. \end{cases}$$

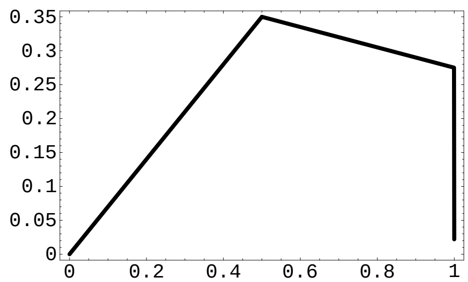

Figure 2: The exact solution of problem (26)

The exact solution is

$$u(x) = \begin{cases} \dfrac{7x}{10} + \dfrac{e^{-\frac{d}{\varepsilon}}\left(e^{\frac{x}{\varepsilon}} - 1\right)\left((12 - 68d + 17\varepsilon)\, e^{\frac{4d}{\varepsilon}} - 17\varepsilon e^{\frac{4}{\varepsilon}}\right)}{20\left(e^{\frac{4}{\varepsilon}} - 4e^{\frac{3d}{\varepsilon}} + 3e^{\frac{4d}{\varepsilon}}\right)}, & x \le 0.5, \\ \dfrac{1}{20}\left(3 - 3x + \dfrac{\left(e^{\frac{4}{\varepsilon}} - e^{\frac{4x}{\varepsilon}}\right)\left(e^{\frac{d}{\varepsilon}}(-3 + 17d - 17\varepsilon) + 17\varepsilon\right)}{-4e^{\frac{4d}{\varepsilon}} + 3e^{\frac{5d}{\varepsilon}} + e^{\frac{4+d}{\varepsilon}}}\right), & x > 0.5, \end{cases}$$

presented in Figure 2. This problem is solved numerically using the SDFEM and the layer-adapted mesh (5). Table 2 verifies the convergence results.

Table 2: Discrete norm E^N and the rates of convergence p^N, p_S^N for problem (26)

	S-mesh			BS-mesh		mBS-mesh	
N	E^N	p^N	p_S^N	E^N	p^N	E^N	p^N
64	0.008796	1.59322	2.04887	0.002401	2.08593	0.001854	1.94409
128	0.002915	1.62864	2.01725	0.000059	2.00786	0.000482	1.93552
256	0.000942	1.66817	2.00969	0.000149	1.99353	0.000126	1.95156
512	0.000296	1.70172	2.00682	0.000037	1.99462	0.000032	1.96006
1024	0.000091	1.72896	2.00451	0.000009	1.99756	0.000008	1.96587
2048	0.000027	1.75013	2.00381	0.000002	1.99785	0.000002	1.96786
4096	0.000008	1.76754	2.00504	5×10^{-7}	1.99542	5×10^{-7}	1.95857
8192	0.000002	-	-	1×10^{-7}	-	1×10^{-7}	-

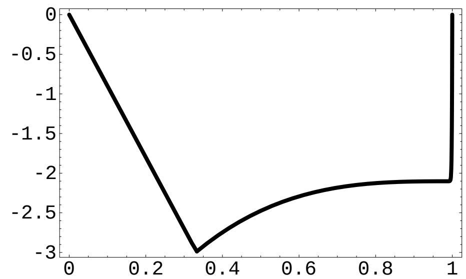

Figure 3: The exact solution of problem (27)

Test problem 2. Consider the second example

$$\left\{ \begin{array}{l} -\varepsilon u''(x) + u'(x) = f(x), \quad x \in (0,1), \\ \\ u(0) = u(1) = 0, \end{array} \right. \tag{27}$$

where

$$f(x) = \left\{ \begin{array}{ll} -9, & x \leq \dfrac{1}{3}, \\ \\ 9(x-1)^2, & x > \dfrac{1}{3}. \end{array} \right.$$

The exact solution is

$$u(x) = \left\{ \begin{array}{ll} -9x + \varepsilon\, C_1\, e^{x/\varepsilon} + C_2, & x \leq \dfrac{1}{3}, \\ \\ 3x^3 + 9x^2(\varepsilon - 1) + 9x(1 - 2\varepsilon + 2\varepsilon^2) + \varepsilon\, C_3\, e^{x/\varepsilon} + C_4, & x > \dfrac{1}{3}, \end{array} \right.$$

depicted in Figure 3, where

$$C_1 = \frac{19 + 9\left(13e^{\frac{2}{3\varepsilon}} - 9\right)\varepsilon - 108e^{\frac{2}{3\varepsilon}}\varepsilon^2 + 162\left(e^{\frac{2}{3\varepsilon}} - 1\right)\varepsilon^3}{9\left(e^{\frac{1}{\varepsilon}} - 1\right)\varepsilon},$$

$$C_2 = -\varepsilon C_1,$$

$$C_3 = \frac{e^{-\frac{1}{3\varepsilon}}\left(9\varepsilon(13 - 12\varepsilon + 18\varepsilon^2) + e^{\frac{1}{3\varepsilon}}(19 - 81\varepsilon - 162\varepsilon^3)\right)}{9\left(e^{\frac{1}{\varepsilon}} - 1\right)\varepsilon},$$

$$C_4 = \frac{1}{9}(e^{\frac{1}{\varepsilon}} - 1)^{-1}\Big(27(1 - 3\varepsilon + 6\varepsilon^2) - 9e^{\frac{2}{3\varepsilon}}\varepsilon(13 - 12\varepsilon + 18\varepsilon^2) + 2e^{\frac{1}{\varepsilon}}(-23 + 81\varepsilon - 81\varepsilon^2 + 81\varepsilon^3)\Big).$$

Table 3: Discrete norm E^N and rates of convergence p^N, p_S^N for problem (27)

	S-mesh			BS-mesh		mBS-mesh	
N	E^N	p^N	p_S^N	E^N	p^N	E^N	p^N
64	0.018001	1.58798	2.04182	0.002663	1.97212	0.003459	1.92682
128	0.005987	1.61993	2.00646	0.000678	1.98738	0.000909	1.93882
256	0.001948	1.66264	2.0021	0.000171	1.99402	0.000237	1.95054
512	0.000065	1.69703	1.99988	0.000042	1.9971	0.000061	1.95987
1024	0.000189	1.72488	1.99993	0.000010	1.99857	0.000015	1.96674
2048	0.000057	1.74887	1.99989	0.000002	1.9993	0.000004	1.97128
4096	0.000017	1.75893	1.9997	6×10^{-7}	1.99976	1×10^{-7}	1.97382
8192	0.000005	-	-	1×10^{-7}	-	2×10^{-7}	-

Numerical results for this problem are presented in Table 3, which are in agreement with the theoretical results from Section 7.

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