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ON COMPLETELY REGULAR TERNARY SEMIRING

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Abstract. We introduce and study the notion of a completely regular ternary semiring. Necessary and sufficient conditions for a ternary semiring to be completely regular are furnished.

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1. Introduction

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, and the like. This provides sufficient motivation to researchers to review various concepts and results. The theory of ternary algebraic system was introduced by D. H. Lehmer [2]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. The notion of ternary semigroups was introduced by S. Banach. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. In W. G. Lister [3] characterized additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. T. K. Dutta and S. Kar [1] introduced and studied some properties of ternary semirings which is a generealization of ternary ring.

Ternary semiring arises naturally as follows, consider the ring of integers Z which plays a vital role in the theory of ring. The subset Z^+ of all positive integers of Z is an additive semigroup which is closed under the ring product, i.e. $Z^+$is a semiring. Now, if we consider the subset Z^- of all negative integers of Z, then we see that Z^- is an additive semigroup which is closed under the triple ring product (however, Z^- is not closed under the binary ring product), i.e. Z^- forms a ternary semiring. Thus, we see that in the ring of integers Z, Z^+ forms a semiring whereas Z^- forms a ternary semiring . Z^+ forms a semiring where as Z^- forms a ternary semiring in the fuzzy settings too.

Sen, Maity and Shum [4] have discussed in detail completely regular semiring. Our main purpose in this paper is to introduce the notion of completely regular ternary semiring and to obtain various characterizations of it.

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2. Preliminaries

In this section we review some definitions and results which will be used in later sections.

Definition 2.1. A set R, together with associative binary operations called addition (+) and multiplication (.), will be called a semiring provided:

i) Addition is a commutative operation,

ii) There exists $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = x$ for each $x \in R$,

iii) Multiplication distributes over addition both form the left and the right, i.e. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for $a, b, c \in R$.

Definition 2.2. An element a in R is called completely regular if there exists x in R satisfying the following conditions:

i) $a = a + x + a$

ii) $a + x = x + a$

iii) $a(a + x) = a + x$.

Definition 2.3. We call R to be a completely regular semiring if every element of R is completely regular.

Definition 2.4. An additive commutative semigroup S, together with a ternary multiplication denoted by [ ], is said to be a ternary semiring if

i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,

ii) $[(a + b)cd] = [acd] + [bcd]$,

iii) $[a(b + c)d] = [abd] + [acd]$,

iv) $[ab(c + d)] = [abc] + [abd]$ for all $a, b, c, d, e \in S$.

From now onward, unless stated otherwise, S will denote a ternary semiring.

Definition 2.5. If there exists an element $0 \in S$ such that $0 + x = x$ and $[0xy] = [xy0] = [x0y] = 0$ for all $x, y \in S$, then 0 is called the zero element of S. In this case we say that S is a ternary semiring with zero.

Definition 2.6. An additive subsemigroup T of S is called a ternary subsemiring of S if $[t_1t_2t_3] \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.7. S is called a commutative ternary semiring if $[abc] = [bac] = [bca]$, for all $a, b, c \in S$.

Definition 2.8. An element a in S is called regular if there exists an element $x \in S$ such that $[axa] = a$. S is called regular if all of its elements are regular.

3. Completely regular ternary semiring

We define a completely regular element in S as

Definition 3.1. An element a of S is completely regular if there exists x in S satisfying the following conditions:

i) $a = a + x + a$,
ii) $[aa(a + x)] = a + x$.

Definition 3.2. S is a completely regular ternary semiring if every element of S is completely regular.

We first show that the condition ii) in Definition 3.1 can be replaced by the condition ii') of the following theorem.

Theorem 3.3. *S is a completely regular ternary semiring iff for any $a \in S$ $\exists$ $x \in S$ such that the following conditions are satisfied:*

i) $a = a + x + a$
ii') $[(a + x)aa] = a + x$.

Proof. Suppose that S is a completely regular ternary semiring. Then for any $a \in S$ $\exists$ $x \in S$ satisfying the conditions i) and ii). We only need to verify that the condition ii') holds.

We have $[aaa] + [(a+x)aa] = [(a+x)aa] + [aaa] = [(a+x+a)aa] = [aaa]$. Hence $[aax] + ([aaa] + [(a + x)aa]) = [aax] + [aaa]$. This shows that $([aax] + [aaa]) + [(a+x)aa] = [aa(x+a)]$. Thus $[aa(x+a)] + [(a+x)aa] = [aa(a+x)]$. By ii),

$$a + x + [(a + x)aa] = a + x. \tag{1}$$

Since $[aa(a+x)] + [aaa] = [aa(a+x+a)] = [aaa]$, as $a = a+x+a$. By ii), we get $(a+x) + [aaa] = [aaa]$. Hence we get $a+x+[aaa]+[xaa] = [aaa]+[xaa]$. This shows that

$$a + x + [(a + x)aa] = [(a + x)aa]. \tag{2}$$

From (1) and (2) we get $[(a+x)aa] = a+x$. This proves that the condition ii') holds.

Proof of the converse part being similar, so we omit it. □

It is also interesting to see that the condition ii) in Definition 3.1 can be also replaced by the condition ii") of the following theorem.

Theorem 3.4. *S is a completely regular ternary semiring iff for all $a \in S$ $\exists$ $x \in S$ such that the following conditions are satisfied :*

i) $a = a + x + a$,

ii") $[a(a + x)a] = a + x$.

Proof. Suppose that S is a completely regular ternary semiring. Then for any $a \in S$ $\exists$ $x \in S$ such that the conditions i) and ii) hold. We only need to verify that the condition ii") holds. We have

$$[aaa] + [a(a + x)a] = [a(a + a + x)a] = [a(a + x + a)a] = [aaa].$$

Hence $[aax] + ([aaa] + [a(a + x)a]) = [aax] + [aaa]$. This shows that $([aax] + [aaa]) + [a(a + x)a] = [aa(x + a)]$. $[aa(x + a)] + [a(a + x)a] = [aa(a + x)]$. By ii) we get

$$a + x + [a(a + x)a] = a + x. \tag{3}$$

Since $[aa(a+x)]+[aaa] = [aa(a+x+a)] = [aaa]$. By ii) we get $a+x+[aaa] = [aaa]$. Therefore $a + x + [aaa] + [axa] = [aaa] + [axa]$. Hence

$$a + x + [a(a + x)a] = [a(a + x)a]. \tag{4}$$

From (3) and (4) we have $[a(a + x)a] = a + x$. This proves that the condition ii") holds.

Proof of the converse part being similar, so we omit it. □

In the following theorem we characterize a completely regular ternary semiring.

Theorem 3.5. *S is a completely regular ternary semiring iff for each $a \in S$ $\exists$ $x \in S$ satisfying the following conditions:*

i) $a = a + x + a$;
ii) $[aa(a + x)] = a + x$;
iii) $[a(a + x)a] = a + x$;
iv) $[(a + x)aa] = a + x$;
v) $a + [(a + x)aa] = a$;
vi) $a + [aa(a + x)] = a$;
vii) $a + [a(a + x)a] = a$;
viii) $[aa(a + x)] = [a(a + x)a] = [(a + x)aa]$.

Proof. First assume that $a \in S$ is a completely regular element. Then, there exists an element $x \in S$ such that $a = a+x+a$ and $[aa(a+x)] = [a(a+x)a] = [(a + x)aa] = a + x$, by Theorems 3.3 and 3.4.

Also $a+[(a+x)aa] = a+a+x = a+x+a = a$. Similarly, $a+[aa(a+x)] = a = a+[a(a+x)a]$. Thus, there exists an element $x \in S$ such that $a = a+x+a$ and satisfying the conditions ii) to viii). The converse part is obviously true. □

Let $V^+(a)$ be the set of all inverse elements of $a \in S$ in the regular semigroup $(S, +)$ As usual, we denote the Green's relations on S by L, R, D, J and H and correspondingly, the L-relation, R-relation, D-relation, J-relation and H-relation on $(S, +)$ are L^+, R^+, D^+, J^+ and H^+respectively.

We characterize completely regular elements in S in the following theorem

.

Theorem 3.6. *The following statements are equivalent for any element $a \in S$.*

i) *a is completely regular.*
ii) *There exists a unique element $y \in V^+(a)$ such that $[aa(a+y)] = a + y, a + [(a+y)aa] = a, [aa(a+y)] + a = a, [a(a+y)a] + a = a$ and $[aa(a+y)] = [a(a+y)a] = [(a+y)aa]$.*
iii) *There exists an unique element $y \in V^+(a)$ such that $[aa(a+y)] = a + y$.*
iv) *H_a^+ is a ternary subring of S, where H_a^+ is the H-class on $(S,+)$ containing $a \in S$.*

Proof. Let $a \in S$ be completely regular.

Hence by Theorem 3.4, there exists a element $x \in S$ satisfying the following conditions :

$$\begin{aligned} a+x+a &= a, & [aa(a+x)] &= a+x, & [aa(a+x)]+a &= a,\\ [(a+x)aa] &= a, & a+[(a+x)aa] &= a, & [a(a+x)a] &= a+x,\\ [a(a+x)a] &= a, & [aa(a+x)] &= [a(a+x)a] = [(a+x)aa]. \end{aligned}$$

Let $y = x + a + x$. Hence

$$a+y+a = a+(x+a+x)+a = (a+x+a)+x+a = a+x+a = a$$

and

$$\begin{aligned} y+a+y &= (x+a+x)+a+(x+a+x) = x+(a+x+a)+x+a+x\\ &= x+a+x+a+x = x+(a+x+a)+x = x+a+x = y. \end{aligned}$$

Therefore, y is the inverse of a. Hence $y \in V^+(a)$.

Therefore $a+y+a = a$ and

$$\begin{aligned} [aa(a+y)] &= [aa(a+x+a+x)] = [aa(a+x)]\\ &= a+x = (a+x+a)+x = a+(x+a+x) = a+y \end{aligned}$$

and

$$[aa(a+y)]+a = a+y+a = a.$$

Further

$$\begin{aligned} [(a+y)aa] &= [(a+x+a+x)aa] = [(a+x)aa]\\ &= (a+x) = (a+x+a)+x = a+(x+a+x) = a+y,\\ [a(a+y)a] &= [a(a+x+a+x)a] = [a(a+x)a] = a+x = a+y\\ [(a+y)aa]+a &= a+y+a = a\\ [a(a+y)a]+a &= a+y+a = a. \end{aligned}$$

Hence, $[aa(a+y)] = [(a+y)aa] = [a(a+y)a]$ and $a+[(a+y)aa] = a+a+y = 2a+y = y+2a$.

Uniqueness: Let $z \in V^+(a)$ be another element satisfying the conditions. Hence

$$\begin{aligned} y &= y + a + y = 2y + a = 2y + a + z + a = 2y + a + z + a + z + a \\ &= 2y + 3a + 2z = 2a + y + y + a + 2z = a + y + a + 2z \\ &= a + 2z = z + a + z = z. \end{aligned}$$

Thus i) $\Rightarrow$ ii).

ii) $\Rightarrow$ iii) is obviously true.

Let us prove that iii) $\Rightarrow$ iv). We have $a = a + (y + a) = (a + y) + a$, therefore $aH^+(a + y)$. Hence, H_a^+ contains an additive idempotent element $a + y(= y + a)$. Therefore, H_a^+ is a group.

Now, $a = (a + y) + a = [aa(a + y)] + a = [aaa] + ([aay] + a)$. Also, $[aaa] = [aa(a + y + a)] = [aa(a + y)] + [aaa] = a + y + [aaa] = a + (y + [aaa])$. This implies that aR^+a^3. Similarly we have aL^+a^3. Therefore

$$aH^+a^3 \tag{5}$$

Let $b, c, d \in H_a^+$. Therefore, $b, c, d \in L_a^+$ and $b, c, d \in R_a^+$. Hence, there exists $x, y, z, u, v, w \in S$ such that $a = x + b, b = u + a, a = y + c, c = v + a, a = z + d, d = w + a$. Now

$$\begin{aligned} [bcd] &= [(u + a)(v + a)(w + a)] \\ &= [(u + a)(v + a)w] + [(u + a)(v + a)a] \\ &= [u(v + a)w] + [a(v + a)w] + [u(v + a)a] + [a(v + a)a] \\ &= [uvw] + [uaw] + [avw] + [aaw] + [uva] + [uaa] + [ava] + [aaa]. \end{aligned}$$

Also

$$\begin{aligned} [aaa] =& [(x + b)(y + c)(z + d)] \\ &= [xyz] + [xyd] + [xcz] + [xcd] + [byz] + [byd] + [bcz] + [bcd]. \end{aligned}$$

Therefore, $[bcd]L^+[aaa] \Rightarrow [bcd] \in L^+_{[aaa]} = L^+_{a^3} = L^+_a$. Similarly, $[bcd] \in R^+_{[aaa]} = R^+_{a^3} = R^+_a$. Hence, $[bcd] \in H^+_{[aaa]} = H^+_{a^3} = H^+_a$.

From i) this shows that $(H_a^+, \cdot)$ is a ternary semigroup. Hence $(H_a^+, +, \cdot)$ is a ternary ring.

Let us prove that iv) $\Rightarrow$ i). Let $(H_a^+, +, \cdot)$ is a ternary subring of S. Every element of a ternary ring is being completely regular. Hence, $a \in S$ is completely regular. $\square$

Let a be a completely regular element in S. Denote the unique element in $V^+(a)$ satisfying condition iii) of Theorem 3.5 by a'.

We obtain the following theorem of completely regular ternary semirings.

Theorem 3.7. *If S is a completely regular ternary semiring, then $E^+(S) = \{a + a' : a \in S\}$ and $e^3 = [eee] = e$ for all $e \in E^+(S)$, where $E^+(S)$ is the set of all additive idempotents of S.*

Proof. Let $e \in E^+(S)$. Then, there exists $x \in V^+(e)$ such that $e+x+e = e$,and $[ee(e+x)] = e+x$. Now $e = e+e+x = e+x$. Also $x = x+e+x = x+e = e+x = e$. Hence, $E^+(S) = \{a + a' : a \in S\}$ and $e^3 = [eee] = [ee(e + x)] = e + x = e$ for all $e \in E^+(S)$. $\square$

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