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SOME INEQUALITIES ON INVARIANT SUBMANIFOLDS IN QUATERNION SPACE FORMS

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Abstract. We establish some inequalities for invariant submanifolds involving totally real sectional curvature and the scalar curvature in a quaternion space form. The equality cases are also discussed.

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1. Introduction

Perhaps one of the most significant aspects of submanifold theory is that which deals with the relations between the main extrinsic invariants and the main intrinsic invariants of a submanifold. B.Y. Chen [3] introduced a series of invariants on a Kaehler manifold and established several general inequalities involving these invariants for Kaehler submanifolds in complex space forms. In [5] authors established similar inequalities for invariant submanifolds in locally conformal almost cosymplectic manifolds. In the present paper, we study invariant submanifolds in a quaternion space form.

2. Preliminaries

Let $\tilde{M}$ be a $4m$-dimensional Riemannian manifold with metric tensor g. Then $\tilde{M}$ is said to be a quaternion Kaehlerian manifold, if there exists a 3-dimensional vector bundle E consisting of tensors of type $(1,1)$ with local basis of almost Hermitian structures J_1, J_2 and J_3 such that [1]

$$\text{(a)} \qquad \begin{aligned} &J_1^2 = -I,\ J_2^2 = -I,\ J_3^2 = -I \\ &J_1J_2 = -J_2J_1 = J_3,\ J_2J_3 = -J_3J_2 = J_1,\ J_3J_1 = -J_1J_3 = J_2 \end{aligned}$$

where I denotes the identity tensor field of type $(1,1)$ on $\tilde{M}$.

(b) for any local cross-section J of E and any vector X tangent to $\tilde{M}$, $\tilde{\nabla}_X J$ is also a local cross-section of E, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$.

The condition (b) is equivalent to the following condition:

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(c) there exist the local 1-forms p, q and r such that

$$\begin{aligned}\tilde{\nabla}_X J_1 &= r(X)J_2 - q(X)J_3,\\ \tilde{\nabla}_X J_2 &= -r(X)J_1 + p(X)J_3,\\ \tilde{\nabla}_X J_3 &= q(X)J_1 - p(X)J_2.\end{aligned}$$

Now, let X be a unit vector tangent to the quaternion manifold $\tilde{M}$, then X, J_1X, J_2X and J_3X form an orthonormal frame. We denote by $Q(X)$ the 4-plane spanned by them and call $Q(X)$ the quaternion section determined by X. For any orthonormal vectors X, Y tangent to $\tilde{M}$, the plane $X \wedge Y$ spanned by X, Y is said to be totally real if $Q(X)$ and $Q(Y)$ are orthogonal. Any plane in a quaternion section is called a quaternion plane. The sectional curvature of a quaternion plane is called a quaternion sectional curvature. A quaternion manifold is called a quaternion space form if its quaternion sectional curvatures are equal to a constant.

Let $\tilde{M}(c)$ be a $4m$-dimensional quaternion space form of constant quaternion sectional curvature c. The curvature tensor of $\tilde{M}(c)$ has the following expression [4]:

$$(2.1)\qquad \begin{aligned}\tilde{R}(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y\\ &+ g(J_1Y,Z)J_1X - g(J_1X,Z)J_1Y + 2g(X,J_1Y)J_1Z\\ &+ g(J_2Y,Z)J_2X - g(J_2X,Z)J_2Y + 2g(X,J_2Y)J_2Z\\ &+ g(J_3Y,Z)J_3X - g(J_3X,Z)J_3Y + 2g(X,J_3Y)J_3Z\},\end{aligned}$$

for any vector fields X, Y, Z tangent to $\tilde{M}$. The equation (2.1) can be written as:

$$(2.2)\qquad \begin{aligned}\tilde{R}(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y\\ &+ \sum_{i=1}^{3}[g(J_iY,Z)J_iX - g(J_iX,Z)J_iY + 2g(X,J_iY)J_iZ]\},\end{aligned}$$

for any vector fields X, Y, Z tangent to $\tilde{M}$.

Let M be an n-dimensional submanifold of a 4m-dimensional quaternion space form $\tilde{M}(c)$. For each $\pi \subset T_pM,\ p \in M$, we denote $K(\pi)$ the sectional curvature of the plane section π. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of the tangent space T_pM. Then, the scalar curvature τ of M is defined by [3]

$$(2.3)\qquad \tau = \sum_{i<j} K(e_i, e_j), \quad i, j = 1, ..., n,$$

where $K(e_i, e_j)$ is the sectional curvature of the section spanned by e_i and e_j.

A plane section $\pi \subset T_pM$ is called totally real if $J_i\pi,\ i = 1, 2, 3$ is perpendicular to π. For each real number k we define an invariant δ_k^r by

$$(2.4)\qquad \delta_k^r(p) = \tau(p) - k \inf K^r(p), \quad p \in M,$$

where $\inf K^r(p) = \inf_{\pi^r}\{K(\pi^r)\}$ and π^r runs over all totally real plane sections in T_pM [3].

The first Chen invariant can be introduced as

$$\delta_M(p) = \tau(p) - (\inf K)(p). \tag{2.5}$$

Let L be a subspace of T_pM of $\dim l \geq 2$ and $\{e_1, ..., e_l\}$ an orthonormal basis of L. Then, the scalar curvature $\tau(L)$ of the $l-$plane section L, by [7], is:

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha, e_\beta), \qquad \alpha,\ \beta = 1, ..., l. \tag{2.6}$$

Given an orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_pM, denote $\tau_{1,.....,l}$ the scalar curvature of the $l-$plane section spanned by $e_1, ..., e_l$. The scalar curvature $\tau(p)$ of M at p is nothing but the scalar curvature of the tangent space of M at p and if L is a $2-$plane section, $\tau(L)$ is nothing but the sectional curvature $K(L)$ of L.

Now, for an integer $k \geq 0$, denote by $S(n,k)$ the finite set which consists of $k-$tuples $(n_1, ..., n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + ... + n_k \leq n$. Let denote by $S(n)$ the set of $k-$tuples with $k \geq 0$ for fixed n. For each $k-$tuples $(n_1, ..., n_k) \in S(n)$, a Riemannian invariant is defined by

$$\delta(n_1, ..., n_k) = \tau(p) - S(n_1, ..., n_k)(p), \tag{2.7}$$

where

$$S(n_1, ..., n_k) = \inf\{\tau(L_1) + ... + \tau(L_k)\}. \tag{2.8}$$

$L_1, ..., L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, ..., k$.

This invariant is different from ours $\delta_k^r(p)$ and it was studied in [8] for certain submanifolds of quaternionic space forms.

For a submanifold M in a quaternion space form $\tilde{M}(c)$, we denote by g the metric tensor of $\tilde{M}(c)$ as well as that induced on M. Let ∇ be the induced covariant differentiation on M. The Gauss and Weingarten formulae for M are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.9}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \tag{2.10}$$

for any vector fields X, Y tangent to M and any vector field V normal to M, where h, A_V and $\nabla^{\perp}$ are the second fundamental form, the shape operator in the direction of V and the normal connection, respectively. The second fundamental form and the shape operator are related by

$$g(h(X,Y),V) = g(A_V X, Y). \tag{2.11}$$

For the second fundamental form h, we define the covariant differentiation $\tilde{\nabla}$ with respect to the connection in $TM \oplus T^{\perp}M$ by

$$(\tilde{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.12}$$

for any vector fields X, Y, Z tangent to M.

The Gauss, Codazzi and Ricci equations of M are given by [2]

$$\begin{aligned} R(X,Y,Z,W) =& \tilde{R}(X,Y,Z,W) + g(h(X,W), h(Y,Z)) \\ &- g(h(X,Z), h(Y,W)), \end{aligned} \tag{2.13}$$

$$(R(X,Y),Z)^{\perp} = (\tilde{\nabla}_X h)(Y,Z) - (\tilde{\nabla}_Y h)(X,Z), \tag{2.14}$$

$$\tilde{R}(X,Y,V,\eta) = R^{\perp}(X,Y,V,\eta) - g([A_V, A_\eta]X, Y), \tag{2.15}$$

for any vector fields X, Y, Z, W tangent to M and V, η normal to M, where R and are $R^{\perp}$ are the curvature tensors with respect to ∇ and $\nabla^{\perp}$ respectively.

The mean curvature vector $H(p)$ at p of M is defined by [6]

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.16}$$

where n denotes the dimension of M. If, we have

$$h(X,Y) = \lambda g(X,Y)H, \tag{2.17}$$

for any vector fields X, Y tangent to M, then M is called totally umbilical submanifold. In particular, if $h = 0$identically, M is called a totally geodesic submanifold.

Also, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r), \; i, j \in \{1, ..., n\}, \; r \in \{n+1, ..., 4m\}, \tag{2.18}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.19}$$

A submanifold M is said to be an invariant submanifold of a quaternion space form $\tilde{M}(c)$ if $J_i(T_pM) \subseteq T_pM, \quad i = 1, 2, 3, \; p \in M$.

We assume that $\dim M = n$, where $n = 4d$.

3. Totally real sectional curvature for invariant submanifolds

Let M $(n \geq 2)$ be an invariant submanifold of a quaternion space form $\tilde{M}(c)$. We choose an orthonormal basis

$$\{e_1, ..., e_d, \bar{e}_1 = J_i e_1, ..., \bar{e}_d = J_i e_d\}, \; i = 1, 2, 3$$

for T_pM and an orthonormal basis

$$\{\alpha_1, \ldots.., \alpha_{(m-d)}, \bar{\alpha}_1 = J_i\alpha_1, \ldots.., \bar{\alpha}_{(m-d)} = J_i\alpha_{(m-d)}\},\ i = 1, 2, 3$$

for $T_p^{\perp}M$. Then with respect to such an orthonormal frame, the complex structure $J_i,\ i = 1, 2, 3$ on M is given by

$$J_1 = \begin{pmatrix} 0 & -I_d & 0 & 0 \\ I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_d \\ 0 & 0 & I_d & 0 \end{pmatrix},$$

$$(3.1)\qquad J_2 = \begin{pmatrix} 0 & 0 & -I_d & 0 \\ 0 & 0 & 0 & -I_d \\ I_d & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & -I_d \\ 0 & 0 & -I_d & 0 \\ 0 & I_d & 0 & 0 \\ I_d & 0 & 0 & 0 \end{pmatrix}$$

where I_d denotes an identity matrix of degree d.

Lemma 3.1. *Let $M(n \geq 2)$ be an invariant submanifold in a quaternion space form $\tilde{M}(c)$. Then, we have*

$$(3.2)\quad \begin{aligned} K(X,Y) + K(X, J_iY) =& \frac{1}{4}\{H(X + J_iY) + H(X - J_iY) \\ &+ H(X+Y) + H(X-Y) - H(X) - H(Y)\}, \\ &\ i = 1, 2, 3, \end{aligned}$$

for all orthonormal vectors X and Y with $g(X, J_iY) = 0$, $i = 1, 2, 3$.

Proof. For each point $p \in M$ and any orthonormal unit tangent vectors X and Y, with $g(X, J_iY) = 0$, $i = 1, 2, 3$, from equation (2.2) and the Gauss equation (2.13), we have

$$(3.3)\qquad K(X,Y) + K(X, J_iY) = \frac{c}{2} - 2\,\|h(X,Y)\|^2,$$

$$(3.4)\qquad H(X) + H(Y) = 2c - 2\,\|h(X,X)\|^2 - 2\,\|h(Y,Y)\|^2.$$

Lemma follows from (3.3) and (3.4). □

Theorem 3.2. *Let $M(n \geq 2)$ be an invariant submanifold in a quaternion space form $\tilde{M}(c)$. Then, we have*

$$(3.5)\qquad \inf\ K^r(p) \leq \frac{c}{4},\quad p \in M.$$

The equality in (3.5) *holds at $p \in M$ if and only if p is a totally geodesic point.*

Proof. For each non-zero tangent vector X to M we denote by $H(X)$ the holomorphic sectional curvature of X i.e., $H(X)$ is the sectional curvature of the plane section spanned by X and J_iX, $i = 1, 2, 3$.

Let $D'M$ denote the unit sphere bundle of M consisting of all unit tangent vectors on M. For each $p \in M$, define

$$D'(p) = \{X \in T_pM | g(X, X) = 1\} \tag{3.6}$$

and

$$U_p = \{(X, Y)| X, Y \in D'(p),\ g(X, Y) = g(X, J_iY) = 0,\ i = 1, 2, 3\}. \tag{3.7}$$

Then U_p is a closed subset of $D'(p) \times D'(p)$. It can be easily seen that if $\{X, Y\}$ spans a totally real plane section, then $\{X + J_iY, X - J_iY\}$ for $i = 1, 2, 3$ and $\{X + Y, X - Y\}$ also span totally real plane sections.

Now, define a function $H' : U_p \to \mathbb{R}$ by

$$H'(X, Y) = H(X) + H(Y), \quad (X, Y) \in U_p, \tag{3.8}$$

then there exists $(\bar{X}, \bar{Y}) \in U_p$, such that $H'(X, Y)$ attains an absolute maximum value.

From (3.2), we have

$$K(\bar{X}, \bar{Y}) + K(\bar{X}, J_i\bar{Y}) \leq \frac{1}{4}H'(X, Y), \quad i = 1, 2, 3. \tag{3.9}$$

On the other hand, for each unit tangent vector $X \in D'(p)$, it is easy to see that every holomorphic sectional curvature $H(X)$ of a submanifold M in a quaternion space form $\tilde{M}(c)$ satisfies

$$H(X) \leq c. \tag{3.10}$$

In view of (3.9) and (3.10), we get

$$K(\bar{X}, \bar{Y}) + K(\bar{X}, J_i\bar{Y}) \leq \frac{c}{2}, \quad i = 1, 2, 3, \tag{3.11}$$

which implies (3.5).

Now, if the equality case in (3.5) holds identically on M, then

$$K(X, Y) + K(X, J_iY) \geq 2\ \inf K^r = \frac{c}{2}. \tag{3.12}$$

Therefore, from (3.3) and (3.12), we have

$$h(X, Y) = 0, \tag{3.13}$$

for all $X, Y \in D'(p)$ with $g(X, Y) = g(X, J_iY) = 0, \quad i = 1, 2, 3$.

It follows that

$$h(X + J_iY, X - J_iY) = 0, \quad h(X + Y, X - Y) = 0, \quad i = 1, 2, 3, \tag{3.14}$$

this implies that $h(X, X) = 0$.

Since every tangent vector must lie in a totally real plane section of T_pM, we will have

$$h(X,Y)=0, \text{ for all } X,Y\in T_pM. \tag{3.15}$$

Consequently the equality of (3.5) implies that p must be a totally geodesic point.

The converse is straightforward. □

Theorem 3.3. *Let M be an n-dimensional $(n\geq 2)$ invariant submanifold in a $4m$-dimensional quaternion space form $\tilde{M}(c)$. Then*

(i) for each $k\in(-\infty,4]$, $\delta_k^r(p)$ satisfies

$$\delta_k^r(p)\leq \frac{1}{2}(n^2-n+12d)\frac{c}{4}-k\frac{c}{4},\quad p\in M \tag{3.16}$$

(ii)

$$\delta_k^r(p)= \frac{1}{2}(n^2-n+12d)\frac{c}{4}-k\frac{c}{4},\quad p\in M, \tag{3.17}$$

holds for some $k\in(-\infty,4)$ if and only if p is a totally geodesic point.

(iii) The invariant submanifold M satisfies

$$\delta_4^r(p)= \frac{1}{2}(n^2-n+12d)\frac{c}{4}-c,\quad p\in M, \tag{3.18}$$

if and only if there exists an orthonormal basis $\{e_1,.....,e_d,\bar{e}_1=J_ie_1,.....,\bar{e}_d=J_ie_d\}$, $i=1,2,3$ for T_pM and an orthonormal basis $\{\alpha_1,.....,\alpha_{(m-d)},\bar{\alpha}_1=J_i\alpha_1,.....,\bar{\alpha}_{(m-d)}=J_i\alpha_{(m-d)}\}$,
$i=1,2,3$ for $T_p^{\perp}M$ such that the shape operator of M takes the following forms:

$$A_{\alpha_r}=\begin{pmatrix} A'_{\alpha_r} & A''_{\alpha_r} & 0\\ A''_{\alpha_r} & -A'_{\alpha_r} & 0\\ 0&0&0\end{pmatrix},\quad A_{\bar{\alpha}_r}=\begin{pmatrix} -A''_{\alpha_r} & A'_{\alpha_r} & 0\\ A'_{\alpha_r} & A''_{\alpha_r} & 0\\ 0&0&0\end{pmatrix} \tag{3.19}$$

$$A'_{\alpha_r}=\begin{pmatrix} h_{11}^r & h_{12}^r & 0\\ h_{12}^r & -h_{11}^r & 0\\ 0&0&0\end{pmatrix},\quad A''_{\alpha_r}=\begin{pmatrix} \bar{h}_{11}^r & \bar{h}_{12}^r & 0\\ \bar{h}_{12} & -\bar{h}_{11}^r & 0\\ 0&0&0\end{pmatrix}, \tag{3.20}$$

where $r\in\{n+1,.....,4m\}$.

Proof. Since an invariant submanifold $M(n\geq 2)$ of a quaternion space form $\tilde{M}(c)$ is minimal, from the Gauss equation (2.13) and (2.2), the scalar curvature τ and the second fundamental form h at p satisfies

$$2\tau(p)=(n^2-n+12d)\frac{c}{4}-\|h\|^2, \tag{3.21}$$

which implies

$$\tau(p) \leq \frac{1}{2}(n^2 - n + 12d)\frac{c}{4} \tag{3.22}$$

with the equality holding if and only if p is a totally geodesic point.

Now, we suppose that $\pi \subset T_pM$ is a given totally real plane section. We choose an orthonormal basis $\{e_1, ..., e_d, \bar{e}_1 = J_ie_1, ..., \bar{e}_d = J_ie_d\}$, $i = 1, 2, 3$ for T_pM and an orthonormal basis $\{\alpha_1, ..., \alpha_{(m-d)}, \bar{\alpha}_1 = J_i\alpha_1, ..., \bar{\alpha}_{(m-d)} = J_i\alpha_{(m-d)}\}$, $i = 1, 2, 3$ for $T_p^{\perp}M$ such that $\pi = span\{e_1, e_2\}$.

With respect to such a basis, we have

$$A_{\alpha_r} = \begin{pmatrix} A'_{\alpha_r} & A''_{\alpha_r} & 0 \\ A''_{\alpha_r} & -A'_{\alpha_r} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\bar{\alpha}_r} = \begin{pmatrix} -A''_{\alpha_r} & A'_{\alpha_r} & 0 \\ A'_{\alpha_r} & A''_{\alpha_r} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ r \in \{n+1, ..., 4m\}, \tag{3.23}$$

where A'_{α_r} and A''_{α_r} are $d \times d$ matrices. By (3.21) and (3.23), we have

$$\begin{aligned} & -2\tau(p) + (n^2 - n + 12d)\frac{c}{4} \\ \geq & \ 4 \sum_{r=n+1}^{4m} \{(h_{11}^r)^2 + (h_{22}^r)^2 + 2(h_{12}^r)^2 + (\bar{h}_{11}^r)^2 + (\bar{h}_{22}^r)^2 + 2(\bar{h}_{12}^r)^2\} \\ \geq & \ -8 \sum_{r=n+1}^{4m} \{h_{11}^r h_{22}^r - (h_{12}^r)^2 + \bar{h}_{11}^r \bar{h}_{22}^r - (\bar{h}_{12}^r)^2\} \\ = & \ -8\{g(h(e_1, e_1), h(e_2, e_2)) - \| h(e_1, e_2) \|^2\} \\ = & \ -8(K(\pi) - \frac{c}{4}). \end{aligned}$$

It gives

$$\tau(p) - 4K(\pi) \leq \frac{1}{2}(n^2 - n + 12d)\frac{c}{4} - c, \tag{3.24}$$

with equality holding if

$$h_{11}^r + h_{22}^r = 0, h_{1j}^r = h_{2j}^r = h_{ij}^r = 0, r \in \{n+1, ..., 4m\}, i, j \in \{3, ..., n\}. \tag{3.25}$$

Since the inequality (3.24) holds for any totally real plane section, we get

$$\delta_4^r(p) = \tau(p) - 4 \ inf \ K^r(p) \leq \frac{1}{2}(n^2 - n + 12d)\frac{c}{4} - c. \tag{3.26}$$

For $\lambda \in (0, \infty)$, from (3.22), we get

$$\lambda\tau(p) \leq \frac{\lambda}{2}(n^2 - n + 12d)\frac{c}{4}, \tag{3.27}$$

which, together with (3.26), proves that the inequality (3.16) is satisfied when $k \in (0, 4)$. In fact, (3.22) and (3.26) are special cases of $k = 0$ and $k = 4$

respectively. The inequality (3.16) with $k \in (-\infty, 0)$ follows from (3.5) and (3.22).

Now, if the equality (3.17) holds at p for some $k \in (-\infty, 4)$, then we have three cases:

A. $k = 0$, which gives

$$\delta_0^r(p) = \tau(p) = \frac{1}{2}(n^2 - n + 12d)\frac{c}{4}, \tag{3.28}$$

then (3.22) implies that p is a totally geodesic point.

B. $k \in (0, 4)$, then applying (3.26) and the definition of $\delta_k^r(p)$, we have

$$\begin{aligned}\delta_k^r(p) =& \tau(p) - k\ inf\ K^r(p)\\ =&(1 - \frac{k}{4})\delta_0^r(p) + \frac{k}{4}\delta_4^r(p)\\ \leq&\frac{1}{2}(n^2 - n + 12d)\frac{c}{4} - k\frac{c}{4},\end{aligned} \tag{3.29}$$

which implies, in particular that

$$\tau(p) = \delta_0^r(p) = \frac{1}{2}(n^2 - n + 12d)\frac{c}{4}.$$

Therefore, p is a totally geodesic point.

C. $k \in (-\infty, 0)$, then (3.22) together with definition of $\delta_k^r(p)$ and (3.5) yield

$$\begin{aligned}\delta_k^r(p) =& \tau(p) - k \inf K^r(p)\\ \leq&\frac{1}{2}(n^2 - n + 12d)\frac{c}{4} - k\frac{c}{4}.\end{aligned} \tag{3.30}$$

In particular, this gives $\delta_0^r = \frac{1}{2}(n^2 - n + 12d)\frac{c}{4}$. Hence, p must be a totally geodesic point. Conversely, if p is a totally geodesic point, then applying (3.5) and (3.22), we have (3.17).

Now, we assume M is an invariant submanifold of $\tilde{M}(c)$ which satisfies (3.18). Then, the inequality (3.24) becomes equality which yields (3.25). From this we conclude that the shape operators of M at p takes the form as in (3.19) with respect to some orthonormal basis

$$\{e_1, ..., e_d, J_ie_1, ..., J_ie_d, \alpha_1, ..., \alpha_{(m-d)}, J_i\alpha_1, ..., J_i\alpha_{(m-d)}\}, i = 1, 2, 3 \tag{3.31}$$

for $T_p\tilde{M}(c)$.

Conversely, suppose that the shape operator of M at p takes the form as in (3.19) with respect to an orthonormal basis (3.31), then the inequality (3.24) becomes an equality, which, together with (3.16), gives

$$\frac{1}{2}(n^2 - n + 12d)\frac{c}{4} - c \geq \delta_4^r(p) \geq \tau(p) - 4K(\pi) = \frac{1}{2}(n^2 - n + 12d)\frac{c}{4} - c, \tag{3.32}$$

which gives (3.18).

Hence, the proof of the theorem is completed. □

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References

[1] Barros, M., Chen, B.Y., Urbano, F., Quaternion CR-submanifolds of quaternion manifolds. Kodai Math. J. 4 (1981), 399-417.

[2] Chen, B.Y., Some pinching and classification theorems for minimal submanifolds. Arch. Math. (Basel) 60 (1993), 568-578.

[3] Chen, B.Y., A series of Kaehlerian invariants and their applications to Kaehlerian geometry. Beitrage Algebra Geom. 42(2001), 165-178.

[4] Ishihara, S., Quaternion Kaehlerian manifolds. J. Differential Geom. 9 (1974), 483-500.

[5] Li, X., Huang, G., Xu, J., Some inequalities for submanifolds in locally conformal almost cosymplectic manifolds. Soochow J. Math. 31 (3) (2005), 309-319.

[6] Mihai, I., Al-Solamy, F., Shahid, M.H., On Ricci curvature of a quaternion CR-submanifold of a quaternion space form. Radovi Mathematicki 12 (2003), 91-98.

[7] Vilcu, G.E., B.Y. Chen inequalities for slant submanifolds in quaternionic space forms. Turk. J. Math. 34 (2010), 115-128.

[8] Yoon, D.W., A basic inequality for submanifolds in quaternion space forms. Balkan J. Geom. and Its Appl. 9 (2) (2004), 92-102.

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