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TWO GENERAL FIXED POINT THEOREMS FOR PAIRS OF WEAKLY COMPATIBLE MAPPINGS IN G-METRIC SPACES

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Abstract. The purpose of this paper is to prove two general fixed point theorems in G-metric spaces for weakly compatible mappings satisfying implicit relations which generalize some results from [1], [7], [8], [9].

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1. Introduction

Let (X, d) be a metric space and $S, T : (X, d) \to (X, d)$ be the mappings. In 1994, Pant [15] introduced the notion of pointwise R-weakly compatible mappings. It is proved in [16] that the notion of R-weakly commuting is equivalent to the commutativity at coincidence points. Jungck [6] defined S and T to be weakly compatible if $Sx = Tx$ implies $STx = TSx$. Thus, S and T are weakly compatible if and only if S and T are pointwise R-weakly commuting.

In [4] and [5], Dhage introduced a new class of generalized metric space, named D-metric space. Mustafa and Sims [12, 13] proved that most of the claims of concerning the fundamental topological structures on D-metric spaces are incorrect and introduce appropriate notion of generalized metric space, named G-metric space. In fact, Mustafa and other authors studied many fixed point results for self-mappings in G-metric spaces under certain condition [1–3, 7, 9, 21].

In [17, 18] and in other papers, the first author studied fixed points for mappings satisfying implicit relations.

Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, compact metric spaces, paracompact metric spaces, Tychonoff spaces, reflexive spaces, D-metric spaces, probabilistic metric spaces, in two or three metric spaces for single valued mappings, hybrid mappings and set valued mappings.

Quite recently, the method has been used in the study of fixed points for mappings satisfying contractive conditions of integral type and in the fuzzy metric spaces. There exists a vast literature on this topic.

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The mentioned method unified different types of contractive and extensive contractions. The proofs of some fixed point theorems are more simple.

Also, this method allows the study of local and global properties of fixed point structures.

Recently, the authors initiated the study of fixed points in G-metric spaces using implicit relations in [19] and in [20].

This paper differs from the two mentioned papers by the types of implicit relations used and by the types of studied functions.

2. Preliminaries

Definition 2.1 ([13]). Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

$(G_1) : G(x, y, z) = 0$ if $x = y = z$,

$(G_2) : 0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

$(G_3) : G(x, x, y) \leq G(x, y, z)$ for all $x, y \in X$ with $z \neq y$,

$(G_4) : G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables),

$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a G-metric on X, and the pair (X, G) is called a G-metric space.

Note that $G(x, y, z) = 0$, then $x = y = z$.

Definition 2.2 ([13]). Let (X, G) be a metric space. A sequence (x_n) in X is said to be

a) G-convergent if for $\varepsilon > 0$, there is an $x \in X$ and $k \in \mathbb{N}$ such that for all $m, n \geq k$, $G(x, x_n, x_m) < \varepsilon$.

b) G-Cauchy if for each $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that for all $n, m, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$.

Lemma 2.1 ([13]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

1) (x_n) is G-convergent to x;

2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;

3) $G(x_n, x, x) \to 0$ as $n \to \infty$;

4) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Lemma 2.2 ([13]). *If (X, G) is a G-metric space, then the following are equivalent:*

1) The sequence (x_n) is G-Cauchy.

2) For every $\varepsilon > 0$, there is $k \in \mathbb{N}^*$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m > k$.

Definition 2.3 ([13]). Let (X, G) and (X', G') be two G-metric spaces and let $f : (X, G) \to (X', G')$ be a function. Then, f is said to be continuous at a point $x \in X$ if for $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ and $G(a, x, y) < \delta$, then $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous if f is G-continuous at each $a \in X$.

Lemma 2.3 ([13]). *Let (X, G) and (X', G') be two G-metric spaces. Then, a function $f : (X, G) \to (X', G')$ is G-continuous at a point $x \in X$ if and only if it is sequentially continuous, that is, whenever (x_n) is G-continuous to x we have that $f(x_n)$ is G-convergent to $f(x)$.*

Lemma 2.4 ([13]). *Let (X, G) be a G-metric space, then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

3. Implicit relations

Definition 3.1. Let $\mathfrak{F}_1$ be the set of all continuous functions $F(t_1, \ldots, t_4) : \mathbb{R}_+^4 \to \mathbb{R}$ such that

(F_1) : There exists $h \in [0, 1)$ such that for $u, v \geq 0$ and $F(u, v, v, u) \leq 0$, then $u \leq hv$.

(F_2) : $F(t, t, 0, 0) \leq 0$ implies $t = 0$.

Example 3.1. $F(t_1, \ldots, t_4) = t_1 - at_2 - bt_3 - ct_4$, where $a, b, c \geq 0$ and $a + b + c < 1$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u - av - bv - cu \leq 0$. Then, $u \leq hv$, where $0 \leq h_1 = \dfrac{a+b}{1-c} < 1$.

(F_2) : $F(t, t, 0, 0) = t(1 - a) \leq 0$, which implies $t = 0$.

Example 3.2. $F(t_1, \ldots, t_4) = t_1 - at_2 - k(t_3 + 2t_4)$, where $a + 3k < 1$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u - av - k(v + 2u) \leq 0$. Then, $u \leq hv$, where $0 \leq h_1 = \dfrac{a+k}{1-2k} < 1$.

(F_2) : $F(t, t, 0, 0) = t(1 - a) \leq 0$, which implies $t = 0$.

Example 3.3. $F(t_1, \ldots, t_4) = t_1 - at_2 - b\max\{t_3, t_4\}$, where $a, b \geq 0$ and $a + b < 1$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u - av - b\max\{u, v\} \leq 0$. If $u > v$ then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k < 1$.

(F_2) : $F(t, t, 0, 0) = t(1 - a) \leq 0$, which implies $t = 0$.

Example 3.4. $F(t_1, \ldots, t_4) = t_1 - k\max\{t_2, t_3, t_4\}$, where $k \in [0, 1)$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u - k\max\{u, v\} \leq 0$. If $u > v$ then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$ and $u \leq hv$, where $0 \leq h = k < 1$.

(F_2) : $F(t, t, 0, 0) = t(1 - k) \leq 0$, which implies $t = 0$.

Example 3.5. $F(t_1, \ldots, t_4) = t_1 - a\max\left\{t_2, \dfrac{t_3 + 2t_4}{3}, \dfrac{t_4 + 2t_3}{3}\right\} \leq 0$, where $0 \leq a < 1$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u - a\max\left\{v, \dfrac{v + 2u}{3}, \dfrac{u + 2v}{3}\right\} \leq 0$. If $u > v$, then $u(1 - a) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a < 1$.

(F_2) : $F(t, t, 0, 0) = t(1 - a) \leq 0$, which implies $t = 0$.

Example 3.6. $F(t_1, \ldots, t_4) = t_1^2 - t_1(at_2 + bt_3 + ct_4) \leq 0$, where $a, b, c \geq 0$ and $0 \leq a + b + c < 1$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u^2 - a(av + bv + cu) \leq 0$. If $u > 0$, then $u - av - bv - cu \leq 0$ and $u \leq hv$, where $0 \leq h = \dfrac{a+b}{1-c} < 1$. If $u = 0$, then $u \leq hv$.

(F_2) : $F(t, t, 0, 0) = t^2(1 - a) \leq 0$, which implies $t = 0$.

Example 3.7. $F(t_1, \ldots, t_4) = t_1 - k \max\left\{t_2, \dfrac{t_3 + t_4}{2}\right\}$, where $k \in [0, 1)$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u - k \max\left\{v, \dfrac{u+v}{2}\right\} \leq 0$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$ and $u \leq hv$, where $0 \leq h = k < 1$.

(F_2) : $F(t, t, 0, 0) = t^2(1 - a) \leq 0$, which implies $t = 0$.

Example 3.8. $F(t_1, \ldots, t_4) = t_1 - c \max\left\{t_2, \sqrt{t_3 t_4}\right\}$, where $c \in [0, 1)$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u - c \max\{v, \sqrt{uv}\} \leq 0$. If $u > v$, then $u(1 - c) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $0 \leq h = c < 1$.

(F_2) : $F(t, t, 0, 0) = t(1 - c) \leq 0$, which implies $t = 0$.

Example 3.9. $F(t_1, \ldots, t_4) = t_1 - a \max\{t_1, t_2\} - b \max\{t_3, t_4\}$, where $a, b \geq 0$ and $a + b < 1$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u - a \max\{u, v\} - b \max\{u, v\} \leq 0$. If $u > v$, then $u(1 - (a + b)) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $0 \leq h = a + b < 1$.

(F_2) : $F(t, t, 0, 0) = t(1 - (a + b)) \leq 0$, which implies $t = 0$.

Example 3.10. $F(t_1, \ldots, t_4) = t_1^3 - c\dfrac{t_3^2 t_4^2}{1 + t_2}$, where $0 \leq c < 1$.

(F_1) : Let $u, v \geq 0$ be and $F(u, v, v, u) = u^3 - c\dfrac{v^2 u^2}{1 + v} \leq 0$. If $u > 0$, then $u \leq cv\dfrac{v}{1+v} \leq cv$. Hence, $u \leq hv$, where $0 \leq c < 1$. If $u = 0$, then $u \leq hv$.

(F_2) : $F(t, t, 0, 0) = t^3 \leq 0$ which implies $t = 0$.

Definition 3.2. Let $\mathfrak{F}_2$ be the set of all continuous functions $F(t_1, \ldots, t_4) : \mathbb{R}_+^4 \to \mathbb{R}$ such that

(F_1) : F is nonincreasing in variable t_3,

(F_2) : There exists $h_1 \in [0, 1)$ such that $F(u, v, u+v, 0) \leq 0$ implies $u \leq h_1 v$,

(F_3) : There is $h_2 \in [0, 1)$ such that for $t, t' > 0$ with $F(t, t, t, t') \leq 0$,
then $t \leq h_2 t'$.

Example 3.11. $F(t_1, \ldots, t_4) = t_1 - at_2 - bt_3 - ct_4$, where $a, b, c \geq 0$ and $0 \leq a + 2b + c < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, u + v, 0) = u - av - b(u + v) \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{a+b}{1-b} < 1$.

(F_3) : Let $t, t' \geq 0$ be and $F(t, t, t, t') = t - at - bt - ct' \leq 0$, which implies $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{c}{1-(a+b)} < 1$.

Example 3.12. $F(t_1, \ldots, t_4) = t_1 - at_2 - b(t_3 + 2t_4)$, where $a, b \geq 0$ and $0 \leq a + 3b < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, u+v, 0) = u - av - b(u+v) \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{a+b}{1-b} < 1$.

(F_3) : Let $t, t' \geq 0$ be and $F(t, t, t, t') = t - at - b(t + 3t') \leq 0$, which implies $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{2b}{1-(a+b)} < 1$.

Example 3.13. $F(t_1, \ldots, t_4) = t_1 - at_2 - b\max\{t_3, t_4\}$, where $a, b \geq 0$ and $0 \leq a + 2b < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, u+v, 0) = u - av - b(u+v) \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{a+b}{1-b} < 1$.

(F_3) : Let $t, t' \geq 0$ be and $F(t, t, t, t') = t - at - b\max\{t, t'\} \leq 0$. If $t > t'$ then $t(1 - (a+b)) \leq 0$, a contradiction. Hence $t \leq t'$ and $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{b}{1-a} < 1$.

Example 3.14. $F(t_1, \ldots, t_4) = t_1 - k\max\{t_2, t_3, t_4\}$, where $k \in \left[0, \dfrac{1}{2}\right)$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, u+v, 0) = u - k(u+v) \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{k}{1-k} < 1$.

(F_3) : Let $t, t' \geq 0$ be and $F(t, t, t, t') = t - k\max\{t, t'\} \leq 0$. If $t > t'$ then $t(1-k) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq h_2 t'$, where $0 \leq h_2 = k < 1$.

Example 3.15. $F(t_1, \ldots, t_4) = t_1 - a\max\left\{t_2, \dfrac{t_3 + 2t_4}{3}, \dfrac{t_4 + 2t_3}{3}\right\}$, where $0 \leq a < \dfrac{3}{4}$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, u+v, 0) = u - a\max\left\{v, \dfrac{u+v}{3}, \dfrac{2(u+v)}{3}\right\} \leq 0$. If $u > v$, then $u\left(1 - \dfrac{4a}{3}\right) \leq 0$, a contradiction, hence $u \leq v$ which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{4a}{3} < 1$.

(F_3) : Let $t, t' \geq 0$ be and $F(t, t, t, t') = t - a\max\left\{t, \dfrac{t + 2t'}{3}, \dfrac{t' + 2t}{3}\right\} \leq 0$. If $t > t'$ then $t(1-a) \leq 0$, a contradiction, hence $t \leq t'$ which implies $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{4a}{3} < 1$.

Example 3.16. $F(t_1,\dots,t_4) = t_1^2 - t_1(at_2 + bt_3 + ct_4)$, where $a,b,c \ge 0$ and $0 \le a + 2b + c < 1$.

(F_1) : Obviously.

(F_2) : Let $u,v \ge 0$ be and $F(u,v,u+v,0) = u^2 - u(ab + b(u+v)) \le 0$. If $u > 0$, then $u - av - bu - bv \le 0$, which implies $u \le h_1 v$, where $0 \le h_1 = \dfrac{a+b}{1-b} < 1$. If $u = 0$ then $u \le h_1 v$.

(F_3) : Let $t,t' \ge 0$ be and $F(t,t,t,t') = t^2 - t(at + bt + ct') \le 0$, which implies $t - at - bt - ct' \le 0$ and $t \le h_2 t'$, where $0 \le h_2 = \dfrac{c}{1-(a+b)} < 1$.

Example 3.17. $F(t_1,\dots,t_4) = t_1 - k\max\left\{t_2, \dfrac{t_3+t_4}{2}\right\}$, where $k \in [0,1)$.

(F_1) : Obviously.

(F_2) : Let $u,v \ge 0$ be and $F(u,v,u+v,0) = u - k\max\left\{v, \dfrac{u+v}{2}\right\} \le 0$. If $u > v$, then $u(1-k) \le 0$, a contradiction. Hence, $u \le v$, which implies $u \le h_1 v$, where $0 \le h_1 = k < 1$.

(F_3) : Let $t,t' \ge 0$ be and $F(t,t,t,t') = t - k\max\left\{t, \dfrac{t+t'}{2}\right\} \le 0$. If $t > t'$ then $t(1-k) \le 0$, a contradiction, hence $t \le t'$, which implies $t \le h_2 t'$, where $0 \le h_2 = k < 1$.

Example 3.18. $F(t_1,\dots,t_4) = t_1 - c\max\{t_2, \sqrt{t_3 t_4}\}$, where $c \in [0,1)$.

(F_1) : Obviously.

(F_2) : Let $u,v \ge 0$ be and $F(u,v,u+v,0) = u - cv \le 0$ which implies $u \le h_1 v$, where $0 \le h_1 = c < 1$.

(F_3) : Let $t,t' \ge 0$ be and $F(t,t,t,t') = t - c\max\{t,t'\} \le 0$. If $t > t'$ then $t(1-c) \le 0$, a contradiction, hence $t \le t'$, which implies $t \le h_2 t'$, where $0 \le h_2 = c < 1$.

Example 3.19. $F(t_1,\dots,t_4) = t_1 - a\max\{t_1,t_2\} - b\max\{t_3,t_4\}$, where $a,b \ge 0$ and $a + 2b < 1$.

(F_1) : Obviously.

(F_2) : Let $u,v \ge 0$ be and $F(u,v,u+v,0) = u - a\max\{u,v\} - b(u+v) \le 0$. If $u \ge v$ then $u(1-(a+b)) \le 0$, a contradiction, hence $u \le v$, which implies $u \le h_1 v$, where $0 \le h_1 = \dfrac{b}{1-(a+b)} < 1$.

(F_3) : Let $t,t' \ge 0$ be and $F(t,t,t,t') = t - at - b\max\{t,t'\} \le 0$. If $t < t'$ then $t(1-(a+b)) \le 0$, a contradiction, hence $t \le t'$, which implies $t \le h_2 t'$, where $0 \le h_2 = \dfrac{b}{1-a} < 1$.

4. Main results

Definition 4.1. Let f and g be self-maps of a nonempty set X. If $w = fx = gx$ for some $x \in X$, then w is said to be a point of coincidence of f and g.

Lemma 4.1 (Abbas and Rhoades [1]). *Let f, g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Lemma 4.2. *Let (X,G) be a G-metric space and $f,g:(X,G)\to(X,G)$ such that*

$$F(G(fx,fy,fy),G(gx,gy,gy),G(gx,fx,fx),G(gy,fy,fy))\le 0 \tag{4.1}$$

for all $x,y\in X$ and F satisfying property (F_2). Then, f and g have at most one point of coincidence.

Proof. Suppose that $u=fp=gp$ and $v=fq=gq$. Then by (4.1) we have

$$F(G(fq,fp,fp),G(gq,gp,gp),G(gq,fq,fq),G(gp,fp,fp))\le 0,$$

so

$$F(G(gq,gp,gp),G(gq,gp,gp),0,0)\le 0.$$

By (F_2), it follows that $G(gq,gp,gp)=0$ which implies $gq=gp$. Hence $u=fp=gp=gq=fq=v$. Hence, u is the unique point of coincidence. □

Theorem 4.1. *Let (X,G) be a G-metric space and $f,g:(X,G)\to(X,G)$ satisfying inequality (4.1) for all $x,y\in X$, where $F\in\mathfrak{F}_1$. If $f(X)\subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of X. Choose a point $x_1\in X$ such that $fx_0=gx_1$. This can be done since $f(X)\subset g(X)$. Continuing this process, having chosen $x_n\in X$, we obtain $x_{n+1}\in X$ such that $fx_n=gx_{n+1}$. Then, by (4.1) we obtain successively

$$\begin{gathered}F(G(fx_{n-1},fx_n,fx_n),G(gx_{n-1},gx_n,gx_n),\\ G(gx_{n-1},fx_{n-1},fx_{n-1}),G(gx_n,fx_n,fx_n))\le 0,\end{gathered}$$

$$\begin{gathered}F(G(gx_n,gx_{n+1},gx_{n+1}),G(gx_{n-1},gx_n,gx_n),\\ G(gx_{n-1},gx_n,gx_n),G(gx_n,gx_{n+1},gx_{n+1}))\le 0.\end{gathered}$$

By (F_1) we obtain

$$G(gx_n,gx_{n+1},gx_{n+1})\le hG(gx_{n-1},gx_n,gx_n) \tag{4.2}$$

Continuing the above process we obtain

$$G(gx_n,gx_{n+1},gx_{n+1})\le h^nG(x_0,x_1,x_1). \tag{4.3}$$

Then for $m>n$ by (G_5) we obtain

$$\begin{aligned}G(x_n,x_m,x_m)&\le G(gx_n,gx_{n+1},gx_{n+1})+G(gx_{n+1},gx_{n+2},gx_{n+2})+\\ &\quad+\ldots+G(x_{m-1},x_m,x_m)\\ &\le (h^n+h^{n+1}+\ldots+h^{m-1})G(x_0,x_1,x_1)\\ &\le \frac{h^n}{h-1}G(x_0,x_1,x_1)\end{aligned}$$

which implies that $G(gx_n, gx_m, gx_m) \to 0$ as $m, n \to \infty$. Hence, (gx_n) is a G-Cauchy sequence in x. Since $g(X)$ is G-complete, there exists a point q in $g(X)$ such that $gx_n \to q$ as $n \to \infty$. Consequently, we can find a point $p \in X$ such that $gp = q$. We prove that p is a coincidence point of f and g, i.e. $fp = gp$. By (4.1) we have successively

$$F(G(gx_n, fp, fp), G(gx_{n-1}, gp, gp), G(gx_{n-1}, fx_{n-1}, fx_{n-1}), G(gp, fp, fp)) \leq 0,$$

$$F(G(gx_n, fp, fp), G(gx_{n-1}, gp, gp), G(gx_{n-1}, gx_n, gx_n), G(gp, fp, fp)) \leq 0.$$

Letting n tend to infinity we obtain

$$F(G(gp, fp, fp), 0, 0, G(gp, fp, fp)) \leq 0.$$

By (F_1) we have $G(gp, fp, fp) = 0$, i.e. $gp = fp$. Hence $u = fp = gp$ is a point of coincidence of f and g. Moreover, if f and g are weakly compatible, by Lemma 4.1, u is the unique common fixed point of f and g. □

Corollary 4.1 ([1, Th. 2.3])**.** *Let (X, G) be a G-metric space and $f, g : (X, G) \to (X, G)$. Suppose that*

$$G(fx, fy, fz) \leq aG(gx, gy, gz)+bG(gx, fx, fx)+cG(gy, fy, fy)+dG(gz, fz, fz)$$

for all $x, y, z \in X$, where $a + b + c + d < 1$. If $f(X) \subset g(X)$ and $g(X)$ is a G-complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. If $z = y$, then

$$G(fx, fy, fy) \leq aG(gx, gy, gy) + bG(gx, fx, fx) + (c + d)G(gy, fy, fy)$$

and the proof follows by Theorem 4.1 and Example 3.1. □

Remark 4.1.

1) By Theorem 4.1 and Example 3.1, with $a = q$, $b = c = 0$, we obtain Theorem 2.1 [7].

2) If $g = Id$, by Theorem 4.1 and Example 3.1, with $a = b = c$, we obtain Theorem 2.1 [9].

3) If $g = Id$, by Theorem 4.1 and Example 3.2, with $k = \beta$, we obtain the results from Theorem 2.2 [9].

4) If $g = Id$, by Theorem 4.1 and Example 3.3, we obtain the result from Theorem 2.3 [9].

Corollary 4.2. *Let (X, G) be a G-metric space and $f, g : (X, G) \to (X, G)$ such that*

$$G(fx, fy, fy) \leq k \max\{G(gx, gy, gy), G(gx, fx, fx), G(gy, fy, fy)\}$$

for all $x, y \in X$ and $k \in [0, 1)$. If $f(X) \subset g(X)$ and $g(X)$ is a G-complete subspace of (X, G), then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. The proof follows by Theorem 4.1 and Example 3.4. □

Remark 4.2.

1) Let

$$G(fx,fy,fy) \leq k \max\{G(gx,fx,fx), G(gy,fy,fy), G(gz,fz,fz)\}$$

for all $x,y,z \in X$. If $y = z$ then

$$\begin{aligned} G(fx,fy,fy) &\leq k \max\{G(gx,fx,fx), G(gy,fy,fy)\} \\ &\leq k \max\{G(gx,fx,fx), G(gx,gy,gy), G(gy,fy,fy)\}. \end{aligned}$$

By Corollary 4.2 we obtain Theorem 2.4 [1].

2) If $g = Id$, by Corollary 4.2 we obtain a result from Theorem 2.3 [8].

Lemma 4.3. *Let (X,G) be a G-metric space and $f,g:(X,G) \to (X,G)$ such that*

$$F(G(fx,fy,fy), G(gx,gy,gy), G(gx,fy,fy), G(gy,fx,fx)) \leq 0 \tag{4.4}$$

for all $x,y \in X$ and F satisfying property (F_3) of $\mathfrak{F}_2$. Then f and g have at most one unique point of coincidence.

Proof. Suppose that $u = fp = gp$ and $v = fq = gq$ with $u \neq v$. Then by (4.4) we have successively

$$F(G(gq,gp,gp), G(gq,gp,gp), G(gq,fp,fp), G(gq,fq,fq)) \leq 0$$

$$F(G(gq,gp,gp), G(gq,gp,gp), G(gq,gp,gp), G(gq,gq,gq)) \leq 0$$

which implies by (F_3) that

$$G(gq,gp,gp) \leq h_2 G(gp,gq,gq).$$

Similarly, we obtain

$$G(gp,gq,gq) \leq h_2 G(gq,gp,gp),$$

which implies $G(gq,gp,gp)(1-h_2^2) \leq 0$. Hence $G(gq,gp,gp) = 0$, i.e. $gq = gp$. Therefore, $u = fp = gp = gq = fq = v$. □

Theorem 4.2. *Let (X,G) be a G-metric space and $f,g:(X,G) \to (X,G)$ satisfying inequality (4.4) for all $x,y \in X$ and $F \in \mathfrak{F}_2$. If $f(X) \subset g(X)$ and $g(X)$ is a G-complete subspace of (X,G), then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. Let $x_0 \in X$ and $fx_n = gx_{n+1}$ as in Theorem 4.1. Then by (4.4) we obtain

$$\begin{gathered} F(G(fx_{n-1},fx_n,fx_n), G(gx_{n-1},gx_n,gx_n), \\ G(gx_{n-1},fx_n,fx_n), G(gx_n,fx_{n-1},fx_{n-1})) \leq 0 \end{gathered}$$

$$F(G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_{n-1}, gx_{n+1}, gx_{n+1}), 0) \leq 0.$$

By (F_1) and (G_5) we obtain

$$F(G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_{n+1}, gx_{n+1}), 0) \leq 0.$$

By (F_2) we obtain

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq h_1 G(gx_{n-1}, gx_n, gx_n). \tag{4.5}$$

Continuing this process, we obtain

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq h_1^n G(gx_0, gx_1, gx_1).$$

As in Theorem 4.1, (gx_n) is a G-Cauchy sequence. Since $g(X)$ is G - complete, there exists q in $g(X)$ such that $gx_n \to q$ as $n \to \infty$. Consequently, we can find a point $p \in X$ such that $g(p) = q$.

By (4.4) we have successively

$$F(G(fx_{n-1}, fp, fp), G(gx_{n-1}, gp, gp), G(gx_{n-1}, fp, fp), G(gp, fx_{n-1}, fx_{n-1})) \leq 0$$

$$F(G(gx_n, fp, fp), G(gx_{n-1}, gp, gp), G(gx_{n-1}, fp, fp), G(gp, gx_n, gx_n)) \leq 0.$$

Letting n tend to infinity we obtain

$$F(G(gp, fp, fp), 0, G(gp, fp, fp), 0) \leq 0$$

which implies $gp = fp$.

Hence, $w = gp = fp$ is a point of coincidence of f and g. By Lemma 4.3, w is the unique point of coincidence of f and g. Moreover, if f and g are weakly compatible, by Lemma 4.1, w is the unique common fixed point of f and g. □

Corollary 4.3 ([1, Th. 2.6])**.** *Let (X, G) be a G-metric space and $f, g : (X, G) \to (X, G)$ such that*

$$G(fx, fy, fy) \leq a[G(gx, fy, fy) + G(gy, fx, fx)],$$

for all $x, y \in X$, where $a \in \left[0, \frac{1}{2}\right)$. If $f(X) \subset g(X)$ and $g(X)$ is a G-complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. The proof follows by Theorem 4.2 and Example 3.11 with $a = 0$ and $b = c$. □

If $g = Id$, by Theorem 4.2 we obtain

Corollary 4.4. *Let (X, G) be a G-metric space and $f : (X, G) \to (X, G)$ such that*

$$F(G(fx, fy, fy), G(x, y, y), G(x, fy, fy), G(y, fx, fx)) \leq 0$$

for all $x, y \in X$ and $F \in \mathfrak{F}_2$. Then f has a unique fixed point.

Remark 4.3. By Theorem 4.1 and 4.2 and Examples 3.1-3.19 we obtain other fixed point theorems in G-metric spaces.

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