Novi Sad J. Math.
Vol. 42, No. 2, 2012, 75-79

# LOCALLY FINSLER $A$-MODULES OVER LOCALLY $C^*$-ALGEBRAS

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**Abstract.** Suppose $A$ is a Frechet locally $C^*$-algebra and $B$ is a commutative locally $C^*$-algebra. In this paper, we study the notion of locally Finsler module and we show that if $E$ is both a full locally Finsler module over $A$ and $B$ such that there is a map $\varphi : A \to B$ with the closed range in such a way that $ax = \varphi(a)x$ and $\varphi(\rho_A(x)) = \rho_B(x)$, then $\varphi$ is a continuous and bijective $*$-homomorphism. Moreover, we show that if $\varphi(A)$ is of the second category in $B$, or the locally Finsler seminorms on $E$ are symmetric, then $\varphi$ is $*$-isomorphism of locally $C^*$-algebras.

*AMS Mathematics Subject Classification* (2000): 46L08 (should be 2010)

*Key words and phrases:* $*$-isomorphism; locally Finsler $A$-module; locally $C^*$-algebra

## 1. Introduction

In the basic paper [5] M. S. Moslehian shows the following result:

"Let $E$ be both full Hilbert $C^*$-module on $C^*$-algebras $A$ and $B$ and let $\varphi : A \to B$ be a map such that $ax = \varphi(a)x$ and $\varphi(\langle x, y\rangle_A) = \langle x, y\rangle_B$, then $\varphi$ is $*$-isomorphism of $C^*$-algebras."

A similar result was proved by M. Amyari and A. Niknam in [1] for a full Finsler $C^*$-module $E$ which is both an $A$-module and a $B$- module. In 1988, Phillips introduced the Hilbert locally $C^*$-modules as a generalization of Hilbert $C^*$-modules, when the inner product takes values in a locally $C^*$-algebra rather than a $C^*$-algebra, where a locally $C^*$-algebra is a complete Hausdorff $*$-algebra whose topology is determined by a family of $C^*$-seminorms.

We recall that a family $\mathcal{P}$ of submultiplicative seminorms on a $*$-algebra $A$ is called $C^*$-family, if

$(i)$ $p(xx^*) = p(x)^2$ *for each* $p \in \mathcal{P}$.

$(ii)$ $p(x) = p(x^*)$ *for each* $p \in \mathcal{P}$.

In 2007, M. Joita extended the above mentioned result for full Hilbert locally $C^*$-modules $E$ and $F$ on locally $C^*$-algebras $A$ and $B$, respectively. In this paper we introduce the concept of locally Finsler $A$-module and obtain the same results for a full locally Finsler module such as $E$.

Through this section $A$, $\mathcal{P}$ and $E$ denote an arbitrary locally $C^*$-algebra, separating family of $C^*$-seminorms on $A$ generating the topology on $A$ and a left $A$-module, respectively.

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**Definition 1.1.** The $A$-valued map $\rho_A : E \to A^+$ is called locally Finsler seminorm if
$(i)$ For each $p \in \mathcal{P}$, $\bar{p} : E \to \mathbb{R}^+$ with $\bar{p}(x) = p(\rho_A(x))^{1/2}$ is a seminorm on $E$.
$(ii)$ $\rho_A(ax) = a\rho_A(x)a^*$ for each $a \in A$ and $x \in E$.

**Definition 1.2.** If $E$ possesses a locally Finsler seminorm, then $E$ is called a pre-Finsler $A$-module. Moreover, if $E$ is complete with respect to the topology of the family of Finsler seminorms $\mathcal{P}_{\mathcal{E}} = \{\bar{p}\}_{p\in\mathcal{P}}$, then $E$ is called a locally Finsler $A$-module.

**Definition 1.3.** A locally Finsler seminorm $\rho_A$ on $E$ is called symmetric if for each $a \in A$ and $x \in E$, $\rho_A(ax) = \rho_A(a^*x)$ .

It is clear that if $A$ is commutative, then every locally Finsler seminorm is symmetric.

**Lemma 1.4.** *Let $E$ be a Finsler $A$-module and $\bar{p}(x) = p(< x, x >)^{1/2}$, for each $x \in E$ and $p \in \mathcal{P}$. Then $\bar{p}(ax) \leq p(a)\bar{p}(x)$.*

*Proof.* Let $p \in \mathcal{P}$. Then we have

$$\bar{p}(ax)^2 = p(\rho_A(ax)) = p(a\rho_A(x)a^*) \leq p(a)p(\rho_A(x))p(a^*) = p(a)^2\bar{p}(x)^2.$$

It follows that $\bar{p}(ax) \leq p(a)\bar{p}(x)$. □

## 2. Full locally Finsler $A$-module

In this section we study the full locally Finsler $A$-modules. Through this section we suppose that $\{p_j\}_{j\in J}$ is a family of separating $C^*$- seminorms on $A$ and $U_j = \{a \in A;\ p_j(a) \leq 1\}$.

**Definition 2.1.** A locally Finsler $A$-module $E$ is called full if the linear span of the set $\{\rho_A(x); x \in E\}$ is dense in $A$.

**Definition 2.2.** A locally $C^*$-algebra $A$ is called Frechet locally $C^*$-algebra, if its topology is generated by a countable separating family of $C^*$-seminorms.

**Lemma 2.3.** *Let $E$ be a full locally Finsler $A$-module on a locally $C^*$-algebra $A$ and $a \in A$. If $\rho_A(ax) = 0$ for each $x \in E$, then $a = 0$.*

*Proof.* Let $\rho_A(ax) = 0$ or equivalently $a\rho_A(x)a^* = 0$ for all $x \in E$. For $b \in A$, the fullness of $E$ implies that there exists a net $\{b_\alpha\}_\alpha$ in $A$ such that $b = \lim_\alpha b_\alpha$ and each $b_\alpha$ is of the following form

$$\sum_{i=1}^{k_\alpha} \lambda_{i,\alpha}\rho_A(x_{i,\alpha}) \quad (x_{i,\alpha} \in E \text{ and } \lambda_{i,\alpha} \in \mathbb{C}).$$

Now we have

$$aba^* = \lim_\alpha ab_\alpha a^* = \lim_\alpha (a\sum_{i=1}^{k_\alpha} \lambda_{i,\alpha}\rho_A(x_{i,\alpha})a^*) = \lim \sum_{i=1}^{k_\alpha} \lambda_{i,\alpha}a\rho_A(x_{i,\alpha})a^* = 0.$$

Hence for $b = a^*a$ we have $p_j(a)^4 = p_j(aa^*)^2 = p_j(aa^*(aa^*)^*) = 0$, which implies $a = 0$. □

As an immediate consequence of this lemma, we have the next corollary:

**Corollary 2.4.** *Let $E$ be a full locally Finsler $A$-module on a locally $C^*$-algebra $A$ and $a \in A$. Then $a = 0$ iff $ax = 0$ for all $x \in E$.*

The following result is due to Phillips [6, Proposition 5.2].

**Lemma 2.5.** *Let $\varphi : A \to B$ be a $*$-homomorphism from a Frechet locally $C^*$-algebra $A$ into a locally $C^*$-algebra $B$. Then $\varphi$ is continuous.*

Now we are ready to prove the main result of this paper.

**Theorem 2.6.** *Let $E$ be both a full Finsler module on a Frechet locally $C^*$-algebra $A$ and on a commutative locally $C^*$-algebra $B$ . Let $\varphi : A \to B$ be a map with closed range such that $ax = \varphi(a)x$ and $\varphi(\rho_A(x)) = \rho_B(x)$, where $a \in A$, $x \in E$. Then $\varphi$ is a bijective continuous $*$-homomorphism.*

*Proof.* Since $(\varphi(ab) - \varphi(a)\varphi(b))x = ((ab)x - a(bx)) = 0$, by Corollary 2.4 $\varphi(ab) = \varphi(a)\varphi(b)$. Similarly, $\varphi(\lambda a + b) = \lambda\varphi(a) + \varphi(b)$. Therefore $\varphi$ is a homomorphism. To show that $\varphi$ preserves involution, suppose $a \in A$. For $x \in E$ we have, $\rho_A(ax) = a\rho_A(x)a^*$. So

$$\rho_B(ax) = \varphi(\rho_A(ax)) = \varphi(a\rho_A(x)a^*) = \varphi(a)\rho_B(x)\varphi(a^*).$$

But $\rho_B(ax) = \rho_B(\varphi(a)x) = \varphi(a)\rho_B(x)\varphi(a)^*$. Therefore, we have

$$\varphi(a)\rho_B(x)(\varphi(a)^* - \varphi(a^*)) = 0.$$

A similar argument for $\rho_A(a^*x)$ implies that

$$\varphi(a^*)\rho_B(x)(\varphi(a) - \varphi(a^*)^*) = 0.$$

Since the family of $C^*$-seminorms on $B$ is separating, we have

$$(\varphi(a)^* - \varphi(a^*))\rho_B(x)\varphi(a^*)^* = (\varphi(a^*)\rho_B(x)(\varphi(a) - \varphi(a^*)^*))^* = 0.$$

Now commutativity of $B$ implies that

$$\varphi(a^*)^*\rho_B(x)(\varphi(a)^* - \varphi(a^*)) = 0.$$

Therefore

$$\begin{aligned}\rho_B((\varphi(a) - \varphi(a^*)^*)x) &= (\varphi(a) - \varphi(a^*)^*)\rho_B(x)(\varphi(a)^* - \varphi(a^*)) \\ &= \varphi(a)\rho_B(x)(\varphi(a)^* - \varphi(a^*)) - \\ &\quad \varphi(a^*)^*\rho_B(x)(\varphi(a)^* - \varphi(a^*)) = 0.\end{aligned}$$

Using Lemma 2.3, $\varphi(a^*) = \varphi(a)^*$. Hence $\varphi$ preserves $*$ and so it is continuous by Lemma 2.5. Now let $\varphi(a) = 0$, then $\varphi(a)x = 0$ for every $x \in E$ which implies that $ax = 0$ for each $x \in E$. Corollary 2.4 shows that $a = 0$ and so $\varphi$ is one to one. Suppose that $b \in B$. There is a net $\{b_\alpha\}_\alpha$ in $B$ such that $b_\alpha \longrightarrow b$ and each $b_\alpha$ is of the form $\sum_{i=1}^{k_\alpha} \lambda_{i,\alpha}\rho_B(x_{i,\alpha})$, in which $\lambda_{i,\alpha} \in \mathbb{C}$ and $x_{i,\alpha} \in E$. Consider $a_\alpha = \sum_{i=1}^{k_\alpha} \lambda_{i,\alpha}\rho_A(x_{i,\alpha})$. Clearly, $\varphi(a_\alpha) = b_\alpha$, so $\varphi(a_\alpha) \longrightarrow b$. But the range of $\varphi$ is closed, hence $\varphi$ is surjective. Therefore $\varphi$ is a bijective continuous $*$-homomorphism. □

**Corollary 2.7.** *Let $E$, $A$, $B$ and $\varphi$ be as in Theorem 2.6 such that $\varphi(A)$ is of the second category in $B$. Then $\varphi$ is a $*$-isomorphism of locally $C^*$-algebras. In particular, if $B$ is a Frechet locally $C^*$-algebra, then $\varphi$ is a $*$-isomorphism.*

**Proposition 2.8.** *Let $E$ be both full locally Finsler module on Frechet locally $C^*$-algebras $A$ and $B$ such that $\rho_B$, $\rho_A$ are symmetric. Then $\varphi$ is a $*$-isomorphism of locally $C^*$-algebras.*

*Proof.* We only prove the $*$-preserving of $\varphi$. Let $a \in A$ and $x \in E$. As in Theorem 2.6, we have

$$\varphi(a)\rho_B(x)(\varphi(a)^* - \varphi(a^*)) = 0 = \varphi(a^*)\rho_B(x)(\varphi(a) - \varphi(a^*)^*).$$

But,

$$\begin{aligned}\varphi(a^*)\rho_B(x)(\varphi(a) - \varphi(a^*)^*) &= \varphi(\rho_A(a^*x)) - \rho_B(\varphi(a^*)x) \\ &= \varphi(\rho_A(ax)) - \rho_B(\varphi(a^*)^*x) \\ &= (\varphi(a) - \varphi(a^*)^*)\rho_B(x)\varphi(a^*).\end{aligned}$$

Hence, $(\varphi(a) - \varphi(a^*)^*)\rho_B(x)\varphi(a^*) = 0$. It follows that

$$\varphi(a^*)^*\rho_B(x)(\varphi(a)^* - \varphi(a^*)) = 0.$$

Thus, $\rho_B((\varphi(a) - \varphi(a^*)^*)x) = 0$. Now we have $\varphi(a)^* = \varphi(a^*)$ by Lemma 2.3. □

In [1], an example is given to show that we can not drop the fullness of $E$ in Theorem 2.6.

*Remark* 2.9. Theorem 2.6 shows that, we can substitute the condition $\|\rho_A(x)\| = \|\rho_B(x)\|$ of the main Theorem in [1] by commutativity of $B$ (or symmetry of $\rho_A$, $\rho_B$) and obtain the same result.

**Proposition 2.10.** *Let $E$, $A$, $B$ and $\varphi$ be as in Theorem 2.6. Then the topology of the family of seminorms $\mathcal{Q}_\mathcal{E}$ on $E$ is weaker than the topology of the family of seminorms $\mathcal{P}_\mathcal{E}$ on $E$.*

Proof. We show that the identity map from $(E, \mathcal{P}_\mathcal{E})$ onto $(E, \mathcal{Q}_\mathcal{E})$ is continuous. Let $q \in \mathcal{Q}_\mathcal{E}$. Since $\varphi : A \to B$ is continuous, there is $p_q \in \mathcal{P}_\mathcal{E}$ such that

$$q(\varphi(a)) \leq p_q(a) \qquad (a \in A).$$

Let $x \in E$. Then we have

$$\bar{q}(x)^2 = q(\rho_B(x)) = q(\varphi(\rho_A(x))) \leq p_q(\rho_A(x)) = \bar{p_q}(x).$$

So, $\bar{q}(x) \leq \bar{p_q}(x)$ and this completes the proof.

**Corollary 2.11.** *Let $E$, $A$, $B$ and $\varphi$ be as in Theorem 2.6. Moreover, if $B$ is Frechet, then $\mathcal{P}_\mathcal{E}$ and $\mathcal{Q}_\mathcal{E}$ induce the same topology on $E$.*

Proof. It is enough to show that the identity map from $(E, \mathcal{P}_\mathcal{E})$ onto $(E, \mathcal{Q}_\mathcal{E})$ is a homeomorphism. But this is obvious by the previous proposition.

## Acknowledgement

The authors would like to thank the referee for valuable suggestions and comments.

## References

[1] Amyari, M., Niknam, A., A note on Finsler modules. Bull. Iran Math. Soc. 29 (2003), 77-81.

[2] Bakić, D., Guljaš, B., On a class of module maps of Hilbert $C^*$-modules. Math. Commun. 7 (2002), 177-192.

[3] Inoue, A., Locally $C^*$-algebra. Mem. Fac. Kyushu. Univ. Ser. A 25 (1971), 197-235.

[4] Joiţa, M., A note about full Hilbert modules over Frechet locally $C^*$-algebras. Novi Sad J. Math. 37 (2007), 27-32.

[5] Moslehian, M. S., On full Hilbert $C^*$-modules. Bull. Malays. Math. Sci. Soc. 24 (2001), 45-47.

[6] Phillips, N. C., Inverse limits of $C^*$-algebras. J. Operator Theory 19 (1988), 159-195.

[7] Tagavih, A., Modules with seminorms which take values in a $C^*$-algebra. Int. J. Math. Anal. 3 (2009), 1917-1922.

[8] Wegge-Olsen, N. E., $K$-theory and $C^*$-algebras. OUP, 1993.

*Received by the editors August 8, 2011*