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ON A SPECIAL TYPE OF RIEMANNIAN MANIFOLD ADMITTING A TYPE OF SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The objective of the present paper is to study a type of semi-symmetric metric connection on a special Riemannian manifold.

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1. Introduction

H. A. Hayden [12] introducted semi-symmetric linear connection on a Riemannian manifold and this was further developed by K. Yano [16], M. Prvanović [13], M. C. Chaki and A. Konar [5], J. A. Schouten [14], U. C. De and S. C Biswas [9], U. C. De ([6], [7]), U. C. De and B. K. De [8], T. Q. Binh [2] and many others.

Let M be an n-dimensional Riemannian manifold of class C^∞ endowed with the Riemannian metric g and D be the Levi-Civita connection on (M^n, g).

A linear connection ∇ defined on (M^n, g) is said to be semi-symmetric [11] if its torsion tensor T is of the form

$$T(X,Y) = \pi(Y)X - \pi(X)Y \tag{1.1}$$

where π is a 1-form and ρ is a vector field given by

$$\pi(X) = g(X, \rho), \tag{1.2}$$

for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on M^n.

A semi-symmetric connection ∇ is called a semi-symmetric metric connection [12] if it further satisfies

$$\nabla g = 0. \tag{1.3}$$

A relation between the semi-symmetric metric connection ∇ and the Levi-Civita connection D on (M^n, g) has been obtained by K. Yano [16], which is given by

$$\nabla_X Y = D_X Y + \pi(Y)X - g(X,Y)\rho. \tag{1.4}$$

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We also have

$$(\nabla_X \pi)Y = (D_X \pi)Y - \pi(X)\pi(Y) + \pi(\rho)g(X,Y). \tag{1.5}$$

Further, a relation between the curvature tensor R of the semi-symmetric metric connection ∇ and the curvature tensor K of the Levi-Civita connection D is given by [16]

$$R(X,Y)Z = K(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X + g(X,Z)QY - g(Y,Z)QX, \tag{1.6}$$

where α is a tensor field of type (0,2) and Q is a tensor field of type (1,1) given by

$$\alpha(Y,Z) = g(QY,Z) = (D_Y \pi)(Z) - \pi(Y)\pi(Z) + \frac{1}{2}\pi(\rho)g(Y,Z). \tag{1.7}$$

From (1.6) and (1.7), we obtain

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = \tilde{K}(X,Y,Z,W) - \alpha(Y,Z)g(X,W) +\\ \alpha(X,Z)g(Y,W) - g(Y,Z)\alpha(X,W) +\\ g(X,Z)\alpha(Y,W),\end{aligned} \tag{1.8}$$

where

$$\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W), \quad \tilde{K}(X,Y,Z,W) = g(K(X,Y)Z,W). \tag{1.9}$$

In 1967, R. N. Sen and M. C. Chaki [15] studied certain curvature restrictions on a certain kind of conformally flat space of class one and they obtained the following expression of the covariant derivative of the curvature tensor:

$$\begin{aligned}K^h_{ijk,l} = 2\lambda_l K^h_{ijk} + \lambda_i K^h_{ljk} + \lambda_j K^h_{ilk} +\\ \lambda_k K^h_{ijl} + \lambda^h K_{lijk},\end{aligned} \tag{1.10}$$

where K^h_{ijk} are the components of the curvature tensor K,

$$K_{ijkl} = g_{hl} K^h_{ijk},$$

λ_i is a non-zero covarient vector and ","denotes covarient differentiation with respect to the metric tensor g_{ij}.

Later in 1987, M. C. Chaki [4] called a manifold whose curvature tensor satisfies (1.10), as a pseudo symmetric manifold. In the index-free notation this can be stated as follows: A non-flat Riemannian manifold (M^n, g), $n \geq 2$ is said to be a pseudo symmetric manifold [4] if its curvature tensor K satisfies the condition

$$\begin{aligned}(D_X K)(Y,Z)W = 2A(X)K(Y,Z)W + A(Y)K(X,Z)W +\\ A(Z)K(Y,X)W + A(W)K(Y,Z)X +\\ g(K(Y,Z)W,X)U,\end{aligned} \tag{1.11}$$

where A is a non-zero 1-form and U is a vector field defined by

$$A(X) = g(X, U), \quad for\ all\ \ X, \tag{1.12}$$

and D denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of E. Cartan [3]. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$.

In a recent paper, U. C. De and A. K. Gazi [10] introduced a type of non-flat Riemannian manifold (M^n, g), $n \geq 2$ whose curvature tensor K of type (1,3) satisfies the condition

$$\begin{aligned}(D_X K)(Y, Z)W = [A(X) + B(X)]K(Y, Z)W + A(Y)K(X, Z)W +\\ A(Z)K(Y, X)W + A(W)K(Y, Z)X +\\ g(K(Y, Z)W, X)U,\end{aligned} \tag{1.13}$$

where A, U and D have the meaning already mentioned and B is a non-zero 1-form and V is a vector field defined by

$$B(X) = g(X, V), \quad for\ all\ \ X. \tag{1.14}$$

Such a manifold was called an almost pseudo-symmetric manifold and it was denoted by $(APS)_n$.

If $B = A$, then from the definitions it follows that $(APS)_n$ reduces to a $(PS)_n$. In the same paper, the authors constructed two non-trivial examples of $(APS)_n$.

A non-flat Riemannian manifold (M^n, g), $(n > 3)$, is called an almost pseudo-Ricci symmetric monifold [5] if its Ricci tensor $\tilde{S}$ of type $(0, 2)$ is not identically zero and satisfies the condition

$$(D_X\tilde{S})(Y, Z) = [A(X)+B(X)]\tilde{S}(Y, Z)+A(Y)\tilde{S}(X, Z)+A(Z)\tilde{S}(Y, X), \tag{1.15}$$

where A and B are two 1-forms and D denotes the operator of covariant differentiation with respect to the metric tensor g. In such a case A and B are called the associated 1-form and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$.

In 1981, M. C. Chaki and A. Konar [5] studied a Riemannian manifold which admits a type of semi-symmetric metric connection whose curvature tensor R vanishes and torsion tensor T is recurrent with respect to ∇, that is,

$$R(X, Y)Z = 0, \tag{1.16}$$

and

$$(\nabla_X T)(Y, Z) = \pi(X)T(Y, Z), \tag{1.17}$$

where π is a non-zero 1-form.

In the present paper we prove the following:

Theorem 2.1. If an almost pseudo-symmetric manifold admits a semi-symmetric metric connection ∇ whose curvature tensor R vanishes and torsion tensor T is recurrent, then

(i) the 1-form π is closed,

(ii) the vector field ρ is irrotational,

(iii) the integral curves of the vector field ρ are geodesic provided ρ is a unit vector field,

(iv) the manifold is an almost pseudo-Ricci symmetric manifold provided the 1-form A and 1-form π are equal.

Theorem 2.2. If an almost pseudo-symmetric manifold of dimension $(n > 3)$ admits a semi-symmetric metric connection ∇ whose curvature tensor R vanishes, the torsion tensor T is recurrent and 1-form π and 1-form A are equal, then the vector field ρ is a proper concircular vector field.

Theorem 2.3. If an almost pseudo-symmetric manifold admits a semi-symmetric metric connection ∇ whose curvature tensor R vanishes, the torsion tensor T is recurrent and 1-form π and 1-form A are equal, then the manifold is a subprojective manifold in the sense of Adati.

2. Proof of the main result

Theorem 2.1. *If an almost pseudo-symmetric manifold admits a semi-symmetric metric connection ∇ whose curvature tensor R vanishes and torsion tensor T is recurrent, then*

(i) the 1-form π is closed,

(ii) the vector field ρ is irrotational,

(iii) the integral curves of the vector field ρ are geodesic provided ρ is a unit vector field,

(iv) the manifold is an almost pseudo-Ricci symmetric manifold provided the 1-form A and 1-form π are equal.

Proof. From (1.1), we have

$$(C_1^1 T)(Y) = (n-1)\pi(Y), \tag{2.1}$$

where C_1^1 denotes the operator of contraction.

From (2.1), it follows that

$$(\nabla_X C_1^1 T)(Y) = (n-1)(\nabla_X \pi)(Y). \tag{2.2}$$

From (1.17), we obtain

$$(\nabla_X C_1^1 T)(Y) = \pi(X)(C_1^1 T)(Y). \tag{2.3}$$

From (2.1) and (2.3), we get

$$(\nabla_X C_1^1 T)(Y) = (n-1)\pi(X)\pi(Y). \tag{2.4}$$

From (2.2) and (2.4), we have

$$(\nabla_X \pi)(Y) = \pi(X)\pi(Y). \tag{2.5}$$

Combining (1.5) and (2.5), it follows that

$$(D_X\pi)(Y) = 2\pi(X)\pi(Y) - \pi(\rho)g(X,Y). \tag{2.6}$$

Therefore,

$$(D_X\pi)(Y) - (D_Y\pi)(X) = 0, \tag{2.7}$$

which implies that π is closed.

From (2.7), we obtain

$$g(Y, \nabla_X\rho) = g(X, \nabla_Y\rho), \tag{2.8}$$

which implies that the vector field corresponding to the 1-form π of the semi-symmetric metric connection is irrotational.

From (2.8) it follows that if ρ is a unit vector field, then

$$\nabla_\rho\rho = 0,$$

which implies that the integral curves of the vector field ρ are geodesic.

Substituting (2.6) in (1.7), we get

$$\alpha(X,Y) = \pi(X)\pi(Y) - \frac{1}{2}\pi(\rho)g(X,Y). \tag{2.9}$$

So,

$$QX = \pi(X)\rho - \frac{1}{2}\pi(\rho)X. \tag{2.10}$$

Therefore,

$$\begin{aligned} R(X,Y)Z = K(X,Y)Z + \pi(X)[\pi(Z)Y - g(Y,Z)\rho] - \\ \pi(Y)[\pi(Z)X - g(X,Z)\rho] + \\ \pi(\rho)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.11}$$

Since $R = 0$ by hypothesis, we have

$$\begin{aligned} K(X,Y)Z = \pi(X)[g(Y,Z)\rho) - \pi(Z)Y] + \\ \pi(Y)[\pi(Z)X - g(X,Z)\rho] - \\ \pi(\rho)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.12}$$

Contracting X, it follows that

$$\tilde{S}(Y,Z) = -(n-2)\pi(\rho)g(Y,Z) + (n-2)\pi(Y)\pi(Z). \tag{2.13}$$

In (2.13) we put $Y = Z = e_i$,where $\{e_i\}$,$1 \leq i \leq n$ is an orthonormal basis of the tangent space at any point of the manifold M^n and then summing over i, we obtain

$$\tilde{r} = -(n-1)(n-2)\pi(\rho). \tag{2.14}$$

Putting $Z = \rho$ in (2.13), we obtain

$$\tilde{S}(Y, \rho) = 0. \tag{2.15}$$

Again from (2.12), we get

$$K(X, Y)\rho = 0. \tag{2.16}$$

Therefore,

$$\tilde{K}(X, Y, \rho, Z) = 0.$$

That is,

$$\tilde{K}(X, Y, Z, \rho) = 0. \tag{2.17}$$

or,

$$\pi(K(X, Y)Z) = 0. \tag{2.18}$$

Suppose the 1-form A and 1-form π are equal, that is,

$$A(X) = \pi(X). \tag{2.19}$$

From (1.13), we obtain by contraction

$$\begin{aligned}(D_X\tilde{S})(Y, Z) = [A(X) + B(X)]\tilde{S}(Y, Z) + A(Y)\tilde{S}(X, Z) + \\ A(Z)\tilde{S}(Y, X) + A(K(X, Y)Z) + A(K(X, Z)Y).\end{aligned} \tag{2.20}$$

From (2.18) and (2.19), it follows that $A(K(X, Y)Z) = 0$.
Therefore (2.20) becomes

$$(D_X\tilde{S})(Y, Z) = [A(X)+B(X)]\tilde{S}(Y, Z)+A(Y)\tilde{S}(X, Z)+A(Z)\tilde{S}(Y, X). \tag{2.21}$$

Hence the manifold under consideration is an almost pseudo Ricci-symmetric manifold.

This completes the proof. □

It is known that [16] if a Riemannian manifold (M^n, g) admits a semi-symmetric metric connection ∇ whose curvature tensor vanishes, then the manifold is conformally flat.

Hence we can state the following corollary:

Corollary 2.1. *If an almost pseudo-symmetric manifold of dimension* $(n > 3)$ *admits a semi-symmetric metric connection* ∇ *whose curvature tensor* R *vanishes and torsion tensor* T *is recurrent, then the almost pseudo-symmetric manifold is conformally flat.*

Theorem 2.2. *If an almost pseudo-symmetric manifold of dimension* $(n > 3)$ *admits a semi-symmetric metric connection* ∇ *whose curvature tensor* R *vanishes, the torsion tensor* T *is recurrent and* 1*-form* π *and* 1*-form* A *are equal, then the vector field* ρ *is a proper concircular vector field.*

Proof. Let $\tilde{L}$ denotes the symmetric endomorphism of the tangent space at each point of almost pseudo-symmetric manifold corresponding to the Ricci tensor, that is,

$$\tilde{S}(X,Y) = g(\tilde{L}X,Y), \tag{2.22}$$

for every vector fields X and Y.

Then from (2.13), we get

$$\tilde{L}(X) = -(n-2)\pi(\rho)X + (n-2)\pi(X)\rho. \tag{2.23}$$

Now,

$$(D_X\tilde{S})(Y,Z) = D_X\tilde{S}(Y,Z) - \tilde{S}(D_XY,Z) - \tilde{S}(Y,D_XZ). \tag{2.24}$$

Putting $Z=\rho$ and combining relations (2.15) and (2.24), we get

$$(D_X\tilde{S})(Y,\rho) = D_X\tilde{S}(Y,\rho) - \tilde{S}(D_XY,\rho) - \tilde{S}(Y,D_X\rho) = -\tilde{S}(Y,D_X\rho). \tag{2.25}$$

Now putting $Z=\rho$ and $A=\pi$ in (2.21) and substituting (2.15), we obtain

$$(D_X\tilde{S})(Y,\rho) = \pi(\rho)\tilde{S}(Y,X). \tag{2.26}$$

Combining (2.25) and (2.26), it follows that

$$-\tilde{S}(Y,D_X\rho) = \pi(\rho)\tilde{S}(Y,X). \tag{2.27}$$

Combining relations (2.22) in (2.27), we get

$$-\tilde{L}(D_X\rho) = \pi(\rho)\tilde{L}(X). \tag{2.28}$$

Now from (2.23) and (2.28), we have

$$\pi(\rho)(D_X\rho) - \pi(D_X\rho)\rho = -[\pi(\rho)]^2X + \pi(\rho)\pi(X)\rho. \tag{2.29}$$

Or,

$$D_X\rho = -\pi(\rho)X + \omega(X)\rho, \tag{2.30}$$

where

$$\omega(X) = \pi(X) + \frac{\pi(D_X\rho)}{\pi(\rho)},$$

from which it follows that π is closed. Hence ω is closed.

From (2.30), we conclude that ρ is a proper concircular vector field [15].

Hence the proof of Theorem is completed. □

Theorem 2.3. *If an almost pseudo-symmetric manifold admits a semi-symmetric metric connection ∇ whose curvature tensor R vanishes, the torsion tensor T is recurrent and 1-form π and 1-form A are equal, then the manifold is a subprojective manifold in the sense of Adati.*

Proof. It is known that [16] if conformally flat manifold $(M^n,g)(n>3)$ admits a proper concircular vector field, then the manifold is a subprojective manifold in the sense of Adati [1].

Since almost pseudo-symmetric manifold under consideration is conformally flat and admits a proper concircular vector field ρ. □

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