

DECOMPOSITION OF LOCALLY ASSOCIATIVE Γ -AG-GROUPOIDS ¹

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Abstract. It is well known that power associativity and congruences play an important role in the decomposition of semigroups and Γ -semigroups. In this study, we discuss these notions for a non-associative and non-commutative algebraic structure, known as Γ -AG-groupoids. Specifically, we show that for a locally associative Γ -AG-groupoid S with a left identity, S/ρ is the maximal weakly separative homomorphic image of S , where ρ is a relation on S defined by: $a\rho b$ if and only if $a\gamma b^n = b^{n+1}$ and $b\gamma a^n = a^{n+1}$ for some positive integer n and for all $\gamma \in \Gamma$, where $a, b \in S$.

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1. Introduction

An Abel-Grassmann's groupoid, abbreviated as AG -groupoid, is a groupoid whose elements satisfy the left invertive law: $(ab)c = (cb)a$ (See [8, 14]). Though Kazim and Naseerudin [3] called a groupoid which satisfies left invertive law, left almost semigroup, Holgate [2] named the same structure the left invertive groupoid. Later, Mushtaq and others have investigated the structure further and added many useful results to the theory of LA-semigroups in [5, 4]. In this study we prefer to use the term AG-groupoid instead of left almost semigroup. AG-groupoids are in fact a generalization of commutative semigroups.

In 1981, Sen ([9],[10]), introduced the notion of the Γ -semigroup, which is in fact a generalization of the notion of the semigroup. Following [9], a non-empty set M is called a Γ -semigroup if there exists a non empty set Γ such that the mapping $M \times \Gamma \times M \rightarrow M$, written (a, α, c) by $a\alpha c$, satisfies the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

Recently [11], we presented the concept of Γ -AG-groupoids and discussed Γ -ideals in Γ -AG-groupoids. Moreover, in continuation, we have studied M-systems in Γ -AG-groupoids (see [12]). To make this structure more applicable,

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in [13], Shah, Rehman and Khan stepped into the fuzzy concept and discussed the fuzzy Γ -ideals in Γ -AG-groupoids.

The notion of locally associative LA-semigroups was introduced by Mushtaq and Yusuf [6]. An LA-semigroup S is said to be locally associative if $(aa)a = a(aa)$ for all $a \in S$. Later in [4], Mushtaq and Iqbal extensively studied on locally associative LA-semigroups and have extended the results of Hewitt and Zuckerman [1], to locally associative LA-semigroups.

We can find many useful books regarding the study of semigroups, but here for readers, we refer to the book by Attila Nagy, Special Classes of semigroups, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001 (see [7]). In this book we can find a lot of useful results about the least left separative congruences, separative congruences and weakly separative congruences of semigroups.

Inspired by the book [7], we introduce the notion of locally associative Γ -AG-groupoids and generalize some results for Γ -AG-groupoids. We show that for a locally associative Γ -AG-groupoid S , S/ρ is the maximal weakly separative homomorphic image of S .

2. Preliminaries

In this section we refer to [11] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more detail we refer to the papers in the references.

Definition 2.1. [11] *Let S and Γ be non-empty sets. We call S to be a Γ -AG-groupoid if there exists a mapping $S \times \Gamma \times S \rightarrow S$, written as (a, γ, c) by $a\gamma c$, such that S satisfies the identity $(a\gamma b)\mu c = (c\gamma b)\mu a$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.*

Example 2.2. (a)[11] *Let S be an arbitrary AG-groupoid and Γ any non-empty set. Define a mapping $S \times \Gamma \times S \rightarrow S$, by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that S is a Γ -AG-groupoid.*

(b)[11] *Let $\Gamma = \{1, 2, 3\}$. Define a mapping $\mathbb{Z} \times \Gamma \times \mathbb{Z} \rightarrow \mathbb{Z}$ by $a\gamma b = b - \gamma - a$ for all $a, b \in \mathbb{Z}$ and $\gamma \in \Gamma$, where “ $-$ ” is a usual subtraction of integers. Then \mathbb{Z} is a Γ -AG-groupoid.*

Definition 2.3. [11] *Let G and Γ be non-empty sets. If there exists a mapping $G \times \Gamma \times G \rightarrow G$, written as (x, γ, y) by $x\gamma y$, G is called Γ -medial if it satisfies the identity $(x\alpha y)\beta(l\gamma m) = (x\alpha l)\beta(y\gamma m)$ for all $x, y, l, m \in G$ and $\alpha, \beta, \gamma \in \Gamma$.*

Lemma 2.4. *Every Γ -AG-groupoid is Γ -medial.*

Proof. Let S be a Γ -AG-groupoid and for all $x, y, l, m \in S$ and $\alpha, \beta, \gamma \in \Gamma$, using definition of Γ -AG-groupoid repeatedly we have

$$\begin{aligned} (x\alpha y)\beta(l\gamma m) &= [(l\gamma m)\alpha y]\beta x \\ &= [(y\gamma m)\alpha l]\beta x \\ &= (x\alpha l)\beta(y\gamma m). \end{aligned}$$

□

Definition 2.5. [11] An element $e \in S$ is called a left identity of a Γ -AG-groupoid if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.

Lemma 2.6. Let S be a Γ -AG-groupoid with a left identity e , then $a\alpha(b\beta c) = b\alpha(a\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Proof. For all $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$, consider

$$\begin{aligned} a\alpha(b\beta c) &= (e\gamma a)\alpha(b\beta c) \\ &= (e\gamma b)\alpha(a\beta c), \text{ by Lemma 2.4} \\ &= b\alpha(a\beta c). \end{aligned}$$

□

Definition 2.7. A Γ -AG-groupoid S is called a locally associative Γ -AG-groupoid if for all $a \in S$ and $\alpha, \beta \in \Gamma$, $(a\alpha a)\beta a = a\alpha(a\beta a)$.

Example 2.8. Let $S = \{a, b, c\}$ be a locally associative AG-groupoid defined by the following Cayley table.

\cdot	a	b	c
a	c	c	b
b	b	b	b
c	b	b	b

Let for all $a, b \in S$ and $\alpha \in \Gamma$, define a mapping $S \times \Gamma \times S \rightarrow S$ by $a\alpha b = a \cdot b$. Then it is easy to verify that S is a Γ -AG-groupoid and also the identity $(a\alpha a)\beta a = a\alpha(a\beta a)$ holds for all $a \in S$ and all $\alpha, \beta \in \Gamma$.

Remark 2.9. Let S be a locally associative Γ -AG-groupoid. Then for all $\alpha \in \Gamma$, the power of an element of S is defined as:

$$a_\alpha^1 = a, \quad a_\alpha^n \alpha a = a_\alpha^{n+1}, \quad \text{where } n \text{ is any positive integer.}$$

Remark 2.10. If S is a locally associative Γ -AG-groupoid with left identity e , then the powers are associative, that is, $a_\alpha^{(n+m)+k} = a_\alpha^{n+(m+k)}$.

3. A Locally associative Γ -AG-groupoid with a left identity element

Lemma 3.1. If a is an arbitrary element of a locally associative Γ -AG-groupoid with a left identity element e then, for every $\alpha, \beta \in \Gamma$ and every positive integer n , $a_\alpha^n = a_\beta^n$

Proof. $a_\alpha^2 = (e\beta a)_\alpha^2 = (e\beta a)\alpha(e\beta a) = (e\beta e)\alpha(a\beta a) = a_\beta^2$.

Assume that $a_\alpha^k = a_\beta^k$. Then

$$a_\alpha^{k+1} = a_\alpha^k \alpha a = e\beta a_\beta^k \alpha(e\beta a) = (e\beta e)\alpha(a_\beta^k \beta a) = a_\beta^{k+1}. \quad \square$$

From Lemma 3.1 it follows that the powers of an element a of a locally associative Γ -AG-groupoid with a left identity does not depend on the choice of $\alpha \in \Gamma$. Thus the powers of a will be denoted by a, a^2, a^3, \dots .

Before we proceed next, we give an example of a locally associative Γ -AG-groupoid with left identity which is not a Γ -semigroup i.e., not associative. Let $S = \{a, b, c, d\}$ be a locally associative Γ -AG-groupoid defined as follows:

\cdot	a	b	c	d
a	d	d	b	d
b	d	d	a	d
c	a	b	c	d
d	d	d	d	d

Define a mapping $S \times \Gamma \times S \rightarrow S$ by $a\alpha b = a \cdot b$, for all $a, b \in S$ and $\alpha \in \Gamma$. Then S is a locally associative Γ -AG-groupoid with left identity c . Also, we can see that S is not a Γ -semigroup. For instance, $(a\alpha c)\beta c = a \neq b = a\alpha(c\beta c)$.

Proposition 3.2. *In a locally associative Γ -AG-groupoid S with a left identity, the following hold:*

- (1) $a^m \alpha a^n = a^{m+n}$
- (2) $(a^m)^n = a^{mn}$
- (3) $(aab)^n = a^n \alpha b^n$, for all $a \in S, \alpha \in \Gamma$ and positive integers m and n .

Proof. (1) and (2) are straightforward and directly follow from Remark 2.10. We prove (3) by induction. For $n = 1$, the proof is obvious. If $n = 2$ then $(aab)^2 = (aab)\alpha(aab) = (a\alpha a)\alpha(b\alpha b) = a^2 \alpha b^2$. Assume that the result is true for $n = k$, i.e., $(aab)^k = a^k \alpha b^k$. Then $(aab)^{k+1} = (aab)^k \alpha (aab) = (a^k \alpha b^k) \alpha (aab) = (a^k \alpha a) \alpha (b^k \alpha b) = a^{k+1} \alpha b^{k+1}$. Hence the result is true for all positive integers. \square

Before the following proposition, we define K_α as: $K_\alpha = \{a \in S \text{ such that } a\gamma e = a, \text{ where } \gamma \in \Gamma\}$.

Proposition 3.3. *In a locally associative Γ -AG-groupoid S with a left identity element e , the following hold:*

- (1) K_α is a commutative sub Γ -semigroup of S with identity e .
- (2) For all $a \in S, a^n \in K_\alpha$, where $n \geq 2$.

Proof. (1) Let $a, b \in K_\alpha$ and $\alpha \in \Gamma$. Then by definition $a\alpha e = a, b\alpha e = b$. Now consider $(aab)\alpha e = (aab)\alpha(e\alpha e)$. Using Lemma 2.4, we have $(aab)\alpha e = (aab)\alpha(e\alpha e) = (a\alpha e)\alpha(b\alpha e) = a\alpha b$. Hence $a\alpha b \in K_\alpha$. Also, $a\alpha b = (a\alpha e)\alpha b = (b\alpha e)\alpha a = b\alpha a$. Let $a, b, c \in K_\alpha$. Then by Lemma 2.4 we have $a\alpha(b\alpha c) = (a\alpha e)\alpha(b\alpha c) = (a\alpha b)\alpha(e\alpha c) = (a\alpha b)\alpha c$. Hence K_α is a commutative sub Γ -semigroup of S with identity e .

- (2) Trivially holds. \square

Lemma 3.4. *In a locally associative Γ -AG-groupoid S with a left identity, if $a\gamma b^m = b^{m+1}$ and $b\gamma a^n = a^{n+1}$ hold for some positive integers m and n , then $a\alpha b$.*

Proof. Let $m < n$. Then, by Proposition 3.2 and Lemma 2.6,

$$a\gamma b^n = a\gamma(b^{n-m}\gamma b^m) = b^{n-m}\gamma(a\gamma b^m) = b^{n-m}\gamma b^{m+1} = b^{n+1}.$$

□

On a locally associative Γ -AG-groupoid S , a relation σ is called weakly separative if $a^2\sigma(a\alpha b)\sigma b^2$ implies that $a\sigma b$ for every $\alpha \in \Gamma$.

Theorem 3.5. *Let ρ and σ be weakly separative congruences on S . If $\rho \cap (K_\alpha \times K_\alpha) \subseteq \sigma \cap (K_\alpha \times K_\alpha)$, then $\rho \subseteq \sigma$.*

Proof. Assume that $a\rho b$. Then, by definition, $(a^2\alpha(a\alpha b))^2\rho(a^2\alpha(a\alpha b))\alpha(a^2\alpha b^2)$ and $(a^2\alpha(a\alpha b))\alpha(a^2\alpha b^2)\rho(a^2\alpha b^2)^2$. By Proposition 3.3, it follows that $(a^2\alpha(a\alpha b))^2, (a^2\alpha b^2)^2 \in K_\alpha$. Also, by Lemmas 2.4, 2.6 and by Propositions 3.2, 3.3 we have

$$\begin{aligned} (a^2\alpha(a\alpha b))\alpha(a^2\alpha b^2) &= (a^2\alpha a^2)\alpha((a\alpha b)\alpha b^2) \\ &= a^4\alpha(b^3\alpha a) = b^3\alpha(a^4\alpha a) = b^3\alpha a^5 \in K_\alpha. \end{aligned}$$

Consequently

$$(a^2\alpha(a\alpha b))^2\sigma(a^2\alpha(a\alpha b))\alpha(a^2\alpha b^2) \text{ and } (a^2\alpha(a\alpha b))\alpha(a^2\alpha b^2)\sigma(a^2\alpha b^2)^2.$$

It follows that $(a^2\alpha(a\alpha b))\sigma(a^2\alpha b^2)$. Now, as $(a^2\alpha b^2)\rho a^2$ and by Proposition 3.3, $a^2\alpha b^2, a^2 \in K_\alpha$. Then it follows that $a^2\alpha b^2\sigma a^4$. Also by Proposition 3.2, we can say $a^2\alpha b^2 = (a\alpha b)^2$. Then $(a^2)^2\sigma(a^2\alpha(a\alpha b))\sigma(a\alpha b)^2$. Thus we have $a^2\sigma(a\alpha b)$. Finally, $a^2\rho b^2$ and $a^2, b^2 \in K_\alpha$ and hence $a^2\sigma(a\alpha b)\sigma b^2$, and consequently $a\sigma b$, which completes the proof. □

Let S be a locally associative Γ -AG-groupoid. A relation ρ on S is defined as follows: $a\rho b$ if and only if $a\gamma b^n = b^{n+1}$ and $b\gamma a^n = a^{n+1}$ for some positive integer n and for all $\gamma \in \Gamma$, where $a, b \in S$.

Lemma 3.6. *In a locally associative Γ -AG-groupoid S with a left identity element e , if $a\gamma b^m = b^{m+1}$ and $b\gamma a^n = a^{n+1}$ then $a\rho b$ for some positive integers m and n .*

Proof. Let $m < n$. Then $b^{n-m}\alpha(a\gamma b^m) = b^{n-m}\alpha b^{m+1} = b^{n+1}$, by Proposition 3.2. Using Lemma 2.6, it follows that $a\alpha(b^{n-m}\gamma b^m) = b^{n+1}$. Consequently, $a\gamma b^n = b^{n+1}$ and also by hypothesis $b\gamma a^n = a^{n+1}$. Hence $a\rho b$. □

Lemma 3.7. *In a locally associative Γ -AG-groupoid S with a left identity, the relation ρ is a congruence relation.*

Proof. The relation ρ is obviously reflexive and symmetric. To prove the transitivity, let $a\rho b$ and $b\rho c$. Then $a\gamma b^n = b^{n+1}$, $b\gamma a^n = a^{n+1}$ and $b\gamma c^m = c^{m+1}$, $c\gamma b^m = b^{m+1}$ for some positive integers m and n and for all $\gamma \in \Gamma$. Assume that

$k = (n + 1)(m + 1) - 1$ i.e., $k = n(m + 1) + m$. Using Proposition 3.2 and Lemma 2.6,

$$\begin{aligned}
a\gamma c^k &= a\gamma c^{n(m+1)+m} = a\gamma(c^{n(m+1)}\gamma c^m) = a\gamma\{(c^{m+1})^n\gamma c^m\} \\
&= a\gamma\{(b\gamma c^m)^n\gamma c^m\} = a\gamma\{(b^n\gamma c^{mn})\gamma c^m\} = a\gamma\{(c^m\gamma c^{mn})\gamma b^n\} \\
&= a\gamma(c^{m(n+1)}\gamma b^n) = c^{m(n+1)}\gamma(a\gamma b^n) = c^{m(n+1)}\gamma b^{n+1} \\
&= \dots \\
&= \dots \\
&= \dots c^{k+1}.
\end{aligned}$$

Likewise, it can be proved that $c\gamma a^k = a^{k+1}$. Hence ρ is transitive. Consequently, ρ is an equivalence relation on S . For compatibility, let $a\gamma b^n = b^{n+1}$ and $b\gamma a^n = a^{n+1}$ for some positive integer n and all $\gamma \in \Gamma$. Then for every $c \in S$ and $\beta, \gamma \in \Gamma$,

$$\begin{aligned}
(a\gamma c)\beta(b\gamma c)^n &= (a\gamma c)\beta(b^n\gamma c^n) = (a\gamma b^n)\beta(c\gamma c^n) \\
&= b^{n+1}\beta c^{n+1} = (b\beta c)^{n+1} = (b\gamma c)^{n+1}
\end{aligned}$$

Similarly, $(b\gamma c)\beta(a\gamma c)^n = (a\gamma c)^{n+1}$. Hence ρ is a congruence relation on S . \square

Remark 3.8. The congruence relation ρ on a locally associative Γ -AG-groupoid S with a left identity is weakly separative.

Theorem 3.9. *If a locally associative Γ -AG-groupoid S with a left identity e is weakly separative, then it is a commutative Γ -semigroup with identity e .*

Proof. Using Proposition 3.3, it is sufficient to show that $S = K_\alpha$ (for some $\alpha \in \Gamma$). Let $a \in S$ and $\alpha, \beta \in \Gamma$ be arbitrary. To get $aae = a$, it is sufficient to show that $(a\alpha e)^2 = (a\alpha e)\beta a = a^2$, because S is weakly separative. By (3) of Proposition 3.2 and (2) of Proposition 3.3,

$$(a\alpha e)^2 = a^2\alpha e = a^2.$$

As

$$(a^2)^2 = a^2\beta a^2 = (a\alpha e)^2\beta a^2 = ((a\alpha e)\beta a)^2$$

and (using also Lemma 2.4 and Lemma 2.6)

$$a^2\gamma((a\alpha e)\beta a) = (a\alpha e)\gamma(a^2\beta a) = (a\alpha a^2)\gamma(e\beta a) = (a^2)^2,$$

we can conclude that $a^2 = (a\alpha e)\beta a$. Thus $(a\alpha e)^2 = (a\alpha e)\beta a = a^2$, indeed. Thus the theorem is proved. \square

Theorem 3.10. *Let S be a locally associative Γ -AG-groupoid with a left identity e . Then S/ρ is the maximal weakly separative homomorphic image of S .*

Proof. By Remark 3.8, ρ is weakly separative, that is, S/ρ is weakly separative. Let σ be a weakly separative congruence of S . We show that $\rho \subseteq \sigma$. Assume $a\rho b$ for some $a, b \in S$, that is, $a\gamma b^n = b^{n+1}$ and $b\gamma a^n = a^{n+1}$ for some positive integer n and all $\gamma \in \Gamma$. Then $a\gamma b^n \sigma b^{n+1}$ and $b\gamma a^n \sigma a^{n+1}$. If $n = 1$ then $a\gamma b \sigma b^2$ and $b\gamma a = a^2$. By Theorem 3.9, $(a\gamma b)\sigma(b\gamma a)$ and so $a^2\sigma(a\gamma b)b^2$, from which we get $a\sigma b$. Assume $n \geq 2$. Then

$$\begin{aligned} (a\gamma b^{n-1})^2 &= (a\gamma b^{n-1})\gamma(a\gamma b^{n-1}) = (a\gamma a)\gamma(b^{n-1}\gamma b^{n-1}) \\ &= (a\gamma b^{n-2})\gamma(a\gamma b^n)\sigma(a\gamma b^{n-2})\gamma b^{n+1} = (b^{n+1}\gamma b^{n-2})\gamma a \\ &= b^{2n-1}\gamma a. \end{aligned}$$

As $b^{2n-1}\gamma a = (b^n\gamma b^{n-1})\gamma a = (a\gamma b^{n-1})\gamma b^n$ and

$$b^{2n-1}\gamma a = (b^{n-1}\gamma b^n)\gamma a = (a\gamma b^n)\gamma b^{n-1}\sigma b^{n+1}\gamma b^{n-1} = b^{2n} = (b^n)^2,$$

we have

$$(a\gamma b^{n-1})^2\sigma(a\gamma b^{n-1})\alpha b^n \sigma (b^n)^2.$$

As σ is weakly separative, we can conclude

$$a\gamma b^{n-1}\sigma b^n.$$

In a similar way we can prove that

$$b\gamma a^{n-1}\sigma a^n.$$

Continuing on this procedure, we get

$$b^2\sigma a\gamma b \sigma b\gamma a \sigma a^2.$$

From this we can get $a\sigma b$. Hence, $\rho \subseteq \sigma$ and S/ρ is the maximal weakly separative homomorphic image of S . \square

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