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δ-FIBONACCI NUMBERS, PART II

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Abstract. This is a continuation of paper [6]. We study fundamental properties and applications of the, so called, δ-Fibonacci numbers $a_n(\delta)$ and $b_n(\delta)$. For these numbers, many special identities and interesting relations can be generated. Also the formulas connecting the numbers $a_n(\delta)$ and $b_n(\delta)$ with Fibonacci and Lucas numbers are presented. Moreover, some polynomials generated by $a_n(\delta)$ and $b_n(\delta)$ are discussed.

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1. Introduction

Let $\xi = \exp(i2\pi/5)$. Then, the following formulas hold true [6]:

$$\big(1 + \delta(\xi + \xi^4)\big)^n = a_n(\delta) + b_n(\delta)(\xi + \xi^4) \tag{1.1}$$

and

$$\big(1 + \delta(\xi^2 + \xi^3)\big)^n = a_n(\delta) + b_n(\delta)(\xi^2 + \xi^3) \tag{1.2}$$

for $\delta \in \mathbb{C}$, $n \in \mathbb{N}$. Hence, the below recurrent relations can easily be generated [6]:

$$a_0(\delta) = a_1(\delta) = 1, \qquad b_0(\delta) = 0, \qquad b_1(\delta) = \delta,$$

$$a_{n+1}(\delta) = a_n(\delta) + \delta b_n(\delta), \tag{1.3}$$

$$b_{n+1}(\delta) = \delta a_n(\delta) + (1-\delta) b_n(\delta). \tag{1.4}$$

We note that $a_n(\delta), b_n(\delta) \in \mathbb{Z}[\delta]$, for every $n \in \mathbb{N} \cup \{0\}$. Both $a_n(\delta)$ and $b_n(\delta)$, for $n \in \mathbb{N}$, are called the $\delta-$Fibonacci numbers. It should be emphasized that these numbers are the simplest form of more general classes of numbers, introduced recently in the papers [3, 4, 5, 7, 8].

Additionally, in paper [6] the following Binet's formulas for $a_n(\delta)$ and $b_n(\delta)$ are presented

$$a_n(\delta) = \frac{5+\sqrt{5}}{10}\left(\frac{2-\delta+\sqrt{5}\,\delta}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{2-\delta-\sqrt{5}\,\delta}{2}\right)^n \tag{1.5}$$

and

$$b_n(\delta) = \frac{\sqrt{5}}{5}\left(\frac{2-\delta+\sqrt{5}\,\delta}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{2-\delta-\sqrt{5}\,\delta}{2}\right)^n, \tag{1.6}$$

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for every $n = 1, 2, \ldots$, as well as the formulas connecting $a_n(\delta)$ and $b_n(\delta)$ with Fibonacci and Lucas numbers (denoted by F_n and L_n, respectively):

$$a_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} F_{k-1}(-\delta)^k, \tag{1.7}$$

$$b_n(\delta) = \sum_{k=1}^{n} \binom{n}{k} F_k(-1)^{k-1}\delta^k. \tag{1.8}$$

We note that (by (1.3) and (1.4)) the following equalities are true (see also Remark 2.1):

$$\begin{aligned} &a_n(1) = F_{n+1}, && a_n(-1) = F_{2n-1}, && a_n(-2) = F_{3n-1},\\ &b_n(1) = F_n, && b_n(-1) = -F_{2n}, && b_n(-2) = -F_{3n}. \end{aligned} \tag{1.9}$$

The next lemma consists of some other basic technical facts which will turn out to be useful in the current paper.

Lemma 1.1. *a) We have*

$$\begin{aligned} -\beta &:= \xi + \xi^4 = 2\cos\tfrac{2\pi}{5} = \tfrac{\sqrt{5}-1}{2},\\ -\alpha &:= \xi^2 + \xi^3 = -2\cos\tfrac{\pi}{5} = -\tfrac{\sqrt{5}+1}{2}. \end{aligned} \tag{1.10}$$

b) Any two of the numbers 1, α, β, are linearly independent over $\mathbb{Q}$. Moreover, we have

$$\alpha + \beta = \alpha\,\beta = -1, \qquad \alpha^2 = \alpha + 1 \qquad \textit{and} \qquad \beta^2 = \beta + 1.$$

c) Let $f_k \in \mathbb{Q}[\delta]$ and $g_k \in \mathbb{Q}[\delta]$, $k = 1, 2$. Then, for any $a, b \in \mathbb{R}$ linearly independent over $\mathbb{Q}$, if

$$f_1(\delta)a + g_1(\delta)b = f_2(\delta)a + g_2(\delta)b, \quad \text{for} \quad \delta \in \mathbb{Q},$$

then

$$f_1(\delta) = f_2(\delta) \quad \text{and} \quad g_1(\delta) = g_2(\delta), \quad \text{for} \quad \delta \in \mathbb{C}.$$

2. Reduction formulas

This section concerns the main reduction formulas for the indices of δ–Fibonacci numbers. It is a supplement of Section 6 from the paper [6].

At the beginning we get

$$\begin{aligned} \big(1 + \delta\,(\xi + \xi^4)\big)^{kn} &= \big(a_k(\delta) + b_k(\delta)\,(\xi + \xi^4)\big)^n\\ &= a_k^n(\delta)\,\Big(1 + \frac{b_k(\delta)}{a_k(\delta)}\,(\xi + \xi^4)\Big)^n\\ &= a_k^n(\delta)\,\Big(a_n\Big(\frac{b_k(\delta)}{a_k(\delta)}\Big) + b_n\Big(\frac{b_k(\delta)}{a_k(\delta)}\Big)\,(\xi + \xi^4)\Big). \end{aligned} \tag{2.1}$$

On the other hand, we have

$$\left(1+\delta\left(\xi+\xi^4\right)\right)^{kn} = a_{kn}(\delta) + b_{kn}(\delta)\left(\xi+\xi^4\right). \tag{2.2}$$

By comparing (2.1) with (2.2) and using Lemma 1.1 *b*) we obtain the following basic reduction identities (formulas (6.4) and (6.5) from [6]):

$$a_{kn}(\delta) = a_k^n(\delta)\, a_n\Big(\frac{b_k(\delta)}{a_k(\delta)}\Big) \tag{2.3}$$

and

$$b_{kn}(\delta) = a_k^n(\delta)\, b_n\Big(\frac{b_k(\delta)}{a_k(\delta)}\Big). \tag{2.4}$$

It appears that the above formulas carry important reduction properties, which are especially noticeable in the background of the following summation formulas:

$$\sum_{k=1}^{N} a_{k^{n+1}}(\delta) = \sum_{k=1}^{N} a_k^{k^n}(\delta)\, a_{k^n}\Big(\frac{b_k(\delta)}{a_k(\delta)}\Big) \tag{2.5}$$

and

$$\sum_{k=1}^{N} b_{k^{n+1}}(\delta) = \sum_{k=1}^{N} a_k^{k^n}(\delta)\, b_{k^n}\Big(\frac{b_k(\delta)}{a_k(\delta)}\Big). \tag{2.6}$$

Hence, for $\delta = 1$ we have

$$\sum_{k=1}^{N} F_{k^{n+1}+1} = \sum_{k=1}^{N} F_{k+1}^{k^n}\, a_{k^n}\Big(\frac{F_k}{F_{k+1}}\Big), \tag{2.7}$$

$$\sum_{k=1}^{N} F_{k^{n+1}} = \sum_{k=1}^{N} F_{k+1}^{k^n}\, b_{k^n}\Big(\frac{F_k}{F_{k+1}}\Big), \tag{2.8}$$

and next for $n = 1$ we get

$$\sum_{k=1}^{N} F_{k^2+1} = \sum_{k=1}^{N} F_{k+1}^{k}\, a_k\Big(\frac{F_k}{F_{k+1}}\Big), \tag{2.9}$$

$$\sum_{k=1}^{N} F_{k^2} = \sum_{k=1}^{N} F_{k+1}^{k}\, b_k\Big(\frac{F_k}{F_{k+1}}\Big). \tag{2.10}$$

Another example of using formulas (2.3) and (2.4) is the following

$$F_{25\cdot 36} = F_{37}^{25}\, b_{25}\Big(\frac{F_{36}}{F_{37}}\Big) = F_{37}^{25}\, a_5^5\Big(\frac{F_{36}}{F_{37}}\Big)\, b_5\left(\frac{b_5\Big(\frac{F_{36}}{F_{37}}\Big)}{a_5\Big(\frac{F_{36}}{F_{37}}\Big)}\right),$$

$$F_{37} = F_{36} + F_{35} = F_7^6\, b_6\Big(\frac{F_6}{F_7}\Big) + F_8^5\, a_5\Big(\frac{F_7}{F_8}\Big)$$

and

$$b_6\left(\frac{F_6}{F_7}\right) = a_3^2\left(\frac{F_6}{F_7}\right) b_2\left(\frac{b_3\left(\frac{F_6}{F_7}\right)}{a_3\left(\frac{F_6}{F_7}\right)}\right).$$

We note that, by (1.3) and (1.4), we get

$$\begin{aligned} a_5\left(\frac{F_{36}}{F_{37}}\right) &= a_4\left(\frac{F_{36}}{F_{37}}\right) + \frac{F_{36}}{F_{37}}\, b_4\left(\frac{F_{36}}{F_{37}}\right) \\ &= a_2^2\left(\frac{F_{36}}{F_{37}}\right)\left[a_2\left(\frac{b_2\left(\frac{F_{36}}{F_{37}}\right)}{a_2\left(\frac{F_{36}}{F_{37}}\right)}\right) + \frac{F_{36}}{F_{37}}\, b_2\left(\frac{b_2\left(\frac{F_{36}}{F_{37}}\right)}{a_2\left(\frac{F_{36}}{F_{37}}\right)}\right)\right], \end{aligned}$$

$$\begin{aligned} b_5\left(\frac{F_{36}}{F_{37}}\right) &= \frac{1}{F_{37}}\left(F_{36}\, a_4\left(\frac{F_{36}}{F_{37}}\right) + F_{35}\, b_4\left(\frac{F_{36}}{F_{37}}\right)\right) \\ &= \frac{F_{36}}{F_{37}}\, a_2^2\left(\frac{F_{36}}{F_{37}}\right)\left[a_2\left(\frac{b_2\left(\frac{F_{36}}{F_{37}}\right)}{a_2\left(\frac{F_{36}}{F_{37}}\right)}\right) + \frac{F_{35}}{F_{36}}\, b_2\left(\frac{b_2\left(\frac{F_{36}}{F_{37}}\right)}{a_2\left(\frac{F_{36}}{F_{37}}\right)}\right)\right]. \end{aligned}$$

Remark 2.1. We note that the relations

$$F_{k+1}^n a_n\left(\frac{F_k}{F_{k+1}}\right) = F_{kn+1} \quad \textit{and} \quad F_{k+1}^n b_n\left(\frac{F_k}{F_{k+1}}\right) = F_{kn}$$

hold for every $k \in \mathbb{Z} \setminus \{-1\}$, $n \in \mathbb{N}$.

For example, we get

$$(F_{-2})^n b_n\left(\frac{F_{-3}}{F_{-2}}\right) = F_{-3n},$$

i.e.

$$\begin{aligned} (-1)^n b_n(-2) &= (-1)^{n-1} F_{3n}, \\ b_n(-2) &= -F_{3n}, \end{aligned}$$

and similarly

$$a_n(-2) = +F_{3n-1}.$$

Corollary 2.2. *If* $\delta \in \mathbb{C}$, $k \in \mathbb{N}$ *and* $a_k(\delta) = b_k(\delta)$, *then*

$$a_k^n(\delta)\, F_{n+1} = a_{kn}(\delta), \tag{2.11}$$

$$a_k^n(\delta)\, F_n = b_{kn}(\delta). \tag{2.12}$$

For example:

$$a_2(\delta) = b_2(\delta) \;\Leftrightarrow\; \delta = \frac{\sqrt{2}}{2} e^{\pm i\pi/4}.$$

Hence we get

$$a_{2n}\left(\frac{\sqrt{2}}{2}e^{\pm i\pi/4}\right)=F_{n+1}\Big(1\pm\frac{1}{2}\,i\Big)^n,$$
$$b_{2n}\left(\frac{\sqrt{2}}{2}e^{\pm i\pi/4}\right)=F_n\Big(1\pm\frac{1}{2}\,i\Big)^n,$$

for every $n\in\mathbb{N}$. Moreover, if $\delta\in\mathbb{C}$, $k\in\mathbb{N}$ and $a_k(\delta)=b_k(\delta)$, then

$$\lim_{n\to\infty} dist\Big(\alpha\, a_k(\delta),\ \sqrt[n]{a_{kn}(\delta)}\Big)=0,$$

where the symbol $\sqrt[n]{a_{kn}(\delta)}$ denotes the set of all complex roots of $n-$th order of $a_{kn}(\delta)$, for every $n\in\mathbb{N}$. If additionally $\delta\in\mathbb{R}$, then

$$\lim_{n\to\infty}\sqrt[2n+1]{a_{k(2n+1)}(\delta)}=\alpha\, a_k(\delta),$$

where the symbol $\sqrt[2n+1]{a_{k(2n+1)}(\delta)}$ denotes now the only real root of $a_{k(2n+1)}(\delta)$.

At the end of this section we present one more result of the reductive nature.

Theorem 2.3. *We have*

$$\begin{aligned}(2.13)\quad & a_k\big(a_n(\delta)+b_n(\delta)\,(\xi+\xi^4)\big)+b_k\big(a_n(\delta)+b_n(\delta)\,(\xi+\xi^4)\big)\,(\xi+\xi^4)\\ &=\sum_{\substack{r,s\geq 0\\ r+s\leq k}}(-1)^s\,\frac{k!}{r!\,s!}\,\delta^{-r-s}\,\big(a_{(r+s)n+r}(\delta)+b_{(r+s)n+r}(\delta)\,(\xi+\xi^4)\big).\end{aligned}$$

Proof. We have

$$\begin{aligned}&\Big(1+\big(1+\delta\,(\xi+\xi^4)\big)^n(\xi+\xi^4)\Big)^k\\ &\quad=a_k\big(a_n(\delta)+b_n(\delta)\,(\xi+\xi^4)\big)+b_k\big(a_n(\delta)+b_n(\delta)\,(\xi+\xi^4)\big)\,(\xi+\xi^4).\end{aligned}$$

On the other hand, we get

$$\begin{aligned}&\Big(1+\big(1+\delta\,(\xi+\xi^4)\big)^n(\xi+\xi^4)\Big)^k\\ &\quad=\Big(1\backslash\delta\,\big(1+\delta\,(\xi+\xi^4)\big)^{n+1}-1\backslash\delta\,\big(1+\delta\,(\xi+\xi^4)\big)^n+1\Big)^k\\ &\quad=\sum_{\substack{r,s\geq 0\\ r+s\leq k}}(-1)^s\,\frac{k!}{r!\,s!}\,\delta^{-r-s}\,\big(1+\delta\,(\xi+\xi^4)\big)^{(r+s)n+r}.\ \square\end{aligned}$$

3. Sums of some series

According to formula (1.1), two following formulas can be easily deduced

$$(3.1)\qquad \sum_{n=1}^{\infty}\big(a_n(\delta)+2\,b_n(\delta)\,\cos(\tfrac{2\pi}{5})\big)^{-1}=\big(2\,\delta\,\cos(\tfrac{2\pi}{5})\big)^{-1},$$

whenever $\left|1+2\,\delta\,\cos(2\,\pi\backslash 5)\right|>1$, i.e., $\left|\delta+(\sqrt{5}+1)\backslash 2\right|>(\sqrt{5}+1)\backslash 2$, and

$$\text{(3.2)}\quad \sum_{n=1}^{\infty}\big(a_n(\delta)+2\,b_n(\delta)\,\cos(\tfrac{2\,\pi}{5})\big)=-\Big(2\,\delta\,\cos(\tfrac{2\,\pi}{5})\,\big(1+2\,\delta\,\cos(\tfrac{2\,\pi}{5})\big)\Big)^{-1},$$

whenever $\left|\delta+(\sqrt{5}+1)\backslash 2\right|<(\sqrt{5}+1)\backslash 2$. Next, from equation (1.2) we get

$$\text{(3.3)}\quad \sum_{n=1}^{\infty}\big(a_n(\delta)-2\,b_n(\delta)\,\cos(\tfrac{\pi}{5})\big)^{-1}=-\big(2\,\delta\,\cos(\tfrac{\pi}{5})\big)^{-1},$$

whenever $\left|1-2\,\delta\,\cos(\pi\backslash 5)\right|>1$, i.e., $\left|\delta-(\sqrt{5}-1)\backslash 2\right|>(\sqrt{5}-1)\backslash 2$, and

$$\text{(3.4)}\quad \sum_{n=1}^{\infty}\big(a_n(\delta)-2\,b_n(\delta)\,\cos(\tfrac{\pi}{5})\big)=\frac{1-2\,\delta\,\cos(\frac{\pi}{5})}{2\,\delta\,\cos(\frac{\pi}{5})},$$

whenever $\left|\delta-(\sqrt{5}-1)\backslash 2\right|<(\sqrt{5}-1)\backslash 2$.

Note that all the above formulas can be easily presented in a more general form. We think that the case $\delta=1$ (for Fibonacci numbers) is the most representative and the appropriate formulas are listed below:

a)

$$\text{(3.5)}\quad \sum_{n=0}^{\infty}\big(F_{kn+1}+2\,F_{kn}\cos(\tfrac{2\,\pi}{5})\big)^{-s}=\begin{cases}\dfrac{F_{sk+1}+2\,F_{sk}\,\cos(\frac{2\,\pi}{5})}{F_{sk+1}+2\,F_{sk}\,\cos(\frac{2\,\pi}{5})-1}, & \text{for any } k,s\in\mathbb{N},\\[2ex] \dfrac{\big(2\,\cos(\frac{\pi}{5})\big)^{sk}}{\big(2\,\cos(\frac{\pi}{5})\big)^{sk}-1}, & \text{for any } k\in\mathbb{N} \text{ and } s\in\mathbb{R}_+;\end{cases}$$

b)

$$\text{(3.6)}\quad \sum_{n=0}^{\infty}\big(F_{kn+1}+2\,F_{kn}\,\cos(\tfrac{2\,\pi}{5})\big)^{-s}\big(F_{ln+1}+2\,F_{ln}\,\cos(\tfrac{2\,\pi}{5})\big)^{-r}=\begin{cases}\dfrac{F_{sk+lr+1}+2\,F_{sk+lr}\,\cos(\frac{2\,\pi}{5})}{F_{sk+lr+1}+2\,F_{sk+lr}\,\cos(\frac{2\,\pi}{5})-1}, & \text{for any } s,r\in\mathbb{N},\\[2ex] \dfrac{\big(2\,\cos(\frac{\pi}{5})\big)^{sk+lr}}{\big(2\,\cos(\frac{\pi}{5})\big)^{sk+lr}-1}, & \text{for any } k,l\in\mathbb{N} \text{ and } s,r\in\mathbb{R}_+;\end{cases}$$

c)

$$\text{(3.7)}\quad \sum_{n=0}^{\infty}\big(F_{kn+1}-2\,F_{kn}\,\cos(\tfrac{\pi}{5})\big)^{s}=\big(1-F_{sk+1}+2\,F_{sk}\,\cos(\tfrac{\pi}{5})\big)^{-1},$$

for every $s\in\mathbb{N}$;

d)

$$\sum_{n=0}^{\infty}\Big((-1)^{kn}\big(2\,F_{kn}\,\cos(\tfrac{\pi}{5})-F_{kn+1}\big)\Big)^{s} \tag{3.8}$$
$$=\begin{cases}\Big(1-(-1)^{ks}\,\big(F_{ks+1}-2\,F_{ks}\,\cos(\tfrac{\pi}{5})\big)\Big)^{-1}, & \text{for any } k,s\in\mathbb{N},\\ \Big(1-\big((-1)^{k}\big(F_{k+1}-2\,F_{k}\cos(\tfrac{\pi}{5})\big)\big)^{s}\Big)^{-1} & \\ =\Big(1-\big(2\,\cos(\tfrac{2\,\pi}{5})\big)^{ks}\Big)^{-1}, & \text{for } k\in\mathbb{N} \text{ and } s\in\mathbb{R}_{+};\end{cases}$$

e)

$$\sum_{n=0}^{\infty}\Big((-1)^{kn}\,\big(F_{kn+1}-2\,F_{kn}\,\cos(\tfrac{\pi}{5})\big)\Big)^{s}\,\Big((-1)^{ln}\,\big(F_{ln+1}-2\,F_{ln}\,\cos(\tfrac{\pi}{5})\big)\Big)^{r} \tag{3.9}$$
$$=\begin{cases}\Big(1-(-1)^{ks+lr}\,\big(F_{sk+lr+1}-2\,F_{sk+lr}\,\cos(\tfrac{\pi}{5})\big)\Big)^{-1}, \text{ for any } k,l,s,r\in\mathbb{N},\\ \Big(1-\big((-1)^{k}\,\big(F_{k+1}-2\,F_{k}\,\cos(\tfrac{\pi}{5})\big)\big)^{s}\,\big((-1)^{l}\,\big(F_{l+1}-2\,F_{l}\,\cos(\tfrac{\pi}{5})\big)\big)^{r}\Big)^{-1}\\ =\Big(1-\big(2\,\cos(\tfrac{2\,\pi}{5})\big)^{ks+lr}\Big)^{-1}, \text{ for any } k,l\in\mathbb{N} \text{ and } s,r\in\mathbb{R}_{+}.\end{cases}$$

4. Two-parameter convolution formulas

Results presented in the current paper refer to Theorems 5.5 and 5.6 from [6]. In particular, the identities (4.3) and (4.4) generalize identities (5.19) and (5.20) from the paper [6].

Theorem 4.1. *We have*

$$\sum_{k=0}^{n}\big(a_{n-k}(\delta)\,b_k(\varepsilon)-a_k(\varepsilon)\,b_{n-k}(\delta)\big) \tag{4.1}$$
$$=2^{-n+1}\sum_{k=0}^{\lfloor n/2\rfloor}5^k\binom{n}{2k+1}(4-\varepsilon-\delta)^{n-2k-1}\,(\varepsilon-\delta)^{2k+1},$$

$$\sum_{k=0}^{n}\big(2\,a_k(\varepsilon)\,a_{n-k}(\delta)-2\,b_k(\varepsilon)\,b_{n-k}(\delta)-a_k(\varepsilon)\,b_{n-k}(\delta)-a_{n-k}(\delta)\,b_k(\varepsilon)\big) \tag{4.2}$$
$$=2^{-n+1}\sum_{k=0}^{\lfloor n/2\rfloor}5^k\binom{n}{2k}(4-\varepsilon-\delta)^{n-2k}\,(\varepsilon-\delta)^{2k},$$

$$
\begin{aligned}
&2\Big(\frac{2}{1-\varepsilon\,\delta}\Big)^n \big(a_n(\delta)\,b_n(\varepsilon)-a_n(\varepsilon)\,b_n(\delta)\big)\\
&=\sum_{k=0}^{n}\Big(a_{n-k}\Big(\frac{2\,\delta}{1-\varepsilon\,\delta}\Big)\,b_k\Big(\frac{2\,\varepsilon}{1-\varepsilon\,\delta}\Big)-a_k\Big(\frac{2\,\varepsilon}{1-\varepsilon\,\delta}\Big)\,b_{n-k}\Big(\frac{2\,\delta}{1-\varepsilon\,\delta}\Big)\Big)\\
&=2\sum_{k=0}^{\lfloor n/2\rfloor}5^k\binom{n}{2k+1}\Big(2-\frac{\varepsilon+\delta}{1-\varepsilon\,\delta}\Big)^{n-2k-1}\Big(\frac{\varepsilon-\delta}{1-\varepsilon\,\delta}\Big)^{2k+1}
\end{aligned}\tag{4.3}
$$

and

$$
\begin{aligned}
&\Big(\frac{2}{1-\varepsilon\,\delta}\Big)^n\big(2\,a_n(\varepsilon)\,a_n(\delta)-2\,b_n(\varepsilon)\,b_n(\delta)-a_n(\varepsilon)\,b_n(\delta)-a_n(\delta)\,b_n(\varepsilon)\big)\\
&=\sum_{k=0}^{n}\Big(2\,a_k\Big(\frac{2\,\varepsilon}{1-\varepsilon\,\delta}\Big)\,a_{n-k}\Big(\frac{2\,\delta}{1-\varepsilon\,\delta}\Big)-2\,b_k\Big(\frac{2\,\varepsilon}{1-\varepsilon\,\delta}\Big)\,b_{n-k}\Big(\frac{2\,\delta}{1-\varepsilon\,\delta}\Big)\\
&\quad-a_n\Big(\frac{2\,\varepsilon}{1-\varepsilon\,\delta}\Big)\,b_n\Big(\frac{2\,\delta}{1-\varepsilon\,\delta}\Big)-a_n\Big(\frac{2\,\delta}{1-\varepsilon\,\delta}\Big)\,b_n\Big(\frac{2\,\varepsilon}{1-\varepsilon\,\delta}\Big)\Big)\\
&=2\sum_{k=0}^{\lfloor n/2\rfloor}5^k\binom{n}{2k}\Big(2-\frac{\varepsilon+\delta}{1-\varepsilon\,\delta}\Big)^{n-2k}\Big(\frac{\varepsilon-\delta}{1-\varepsilon\,\delta}\Big)^{2k},
\end{aligned}\tag{4.4}
$$

for every $\varepsilon,\delta\in\mathbb{C}$, $n\in\mathbb{N}$.

Proof. Let $\xi=\exp(2\,\pi\,i/5)$. We have

$$
\begin{aligned}
&\big(2+\varepsilon(\xi+\xi^4)+\delta\,(\xi^2+\xi^3)\big)^n\\
&\quad=\big(\big(1+\varepsilon\,(\xi+\xi^4)\big)+\big(1+\delta\,(\xi^2+\xi^3)\big)\big)^n\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}\big(1+\varepsilon\,(\xi+\xi^4)\big)^k\,\big(1+\delta\,(\xi^2+\xi^3)\big)^{n-k}\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}\big(a_k(\varepsilon)+b_k(\varepsilon)\,(\xi+\xi^4)\big)\,\big(a_{n-k}(\delta)+b_{n-k}(\delta)\,(\xi^2+\xi^3)\big).
\end{aligned}
$$

By equalities (1.10) we obtain

$$
\begin{aligned}
\big(a_k(\varepsilon)+b_k(\varepsilon)(\xi+\xi^4)\big)&\,\big(a_{n-k}(\delta)+b_{n-k}(\delta)\,(\xi^2+\xi^3)\big)\\
&=\big(a_k(\varepsilon)+b_k(\varepsilon)\,\tfrac{\sqrt5-1}{2}\big)\,\big(a_{n-k}(\delta)-b_{n-k}(\delta)\,\tfrac{\sqrt5+1}{2}\big)\\
&=a_k(\varepsilon)\,a_{n-k}(\delta)-b_k(\varepsilon)\,b_{n-k}(\delta)\\
&\quad-\frac12\,\big(a_k(\varepsilon)\,b_{n-k}(\delta)+a_{n-k}(\delta)\,b_k(\varepsilon)\big)\\
&\quad+\frac{\sqrt5}{2}\,\big(a_{n-k}(\delta)\,b_k(\varepsilon)-a_k(\varepsilon)\,b_{n-k}(\delta)\big).
\end{aligned}
$$

Furthermore, we get

$$
\begin{aligned}
\big(2+\varepsilon\,(\xi+\xi^4)+\delta\,(\xi^2+\xi^3)\big)^n&=\Big(2-\frac12\,(\varepsilon+\delta)+\frac{\sqrt5}{2}\,(\varepsilon-\delta)\Big)^n\\
&=\sum_{k=0}^{\lfloor n/2\rfloor}5^k\binom{n}{2k}a^{n-2k}\,b^{2k}+\sqrt5\sum_{k=0}^{\lfloor n/2\rfloor}5^k\binom{n}{2k+1}a^{n-2k-1}\,b^{2k+1},
\end{aligned}
$$

where

$$2\,a := 4 - \varepsilon - \delta \qquad \text{and} \qquad 2\,b := \sqrt{5}\,(\varepsilon - \delta).$$

Moreover, we have

$$\begin{aligned}\big(1 + \varepsilon\,(\xi + \xi^4)\big)\,\big(1 + \delta\,(\xi^2 + \xi^3)\big) &= 1 - \varepsilon\,\delta + \varepsilon\,(\xi + \xi^4) + \delta\,(\xi^2 + \xi^3)\\ &= \frac{1 - \varepsilon\,\delta}{2}\Big(2 + \frac{2\,\varepsilon}{1 - \varepsilon\,\delta}\,(\xi + \xi^4) + \frac{2\,\delta}{1 - \varepsilon\,\delta}\,(\xi^2 + \xi^3)\Big)\end{aligned}$$

and

$$\begin{aligned}&\big(1 + \varepsilon(\xi + \xi^4)\big)^n\,\big(1 + \delta\,(\xi^2 + \xi^3)\big)^n\\ &\quad = \big(a_n(\varepsilon) + b_n(\varepsilon)\,(\xi + \xi^4)\big)\,\big(a_n(\delta) + b_n(\delta)\,(\xi^2 + \xi^3)\big)\\ &\quad = a_n(\varepsilon)\,a_n(\delta) - b_n(\varepsilon)\,b_n(\delta) + a_n(\varepsilon)\,b_n(\delta)\,(\xi^2 + \xi^3) + a_n(\delta)\,b_n(\varepsilon)\,(\xi + \xi^4)\\ &\quad = a_n(\varepsilon)\,a_n(\delta) - b_n(\varepsilon)\,b_n(\delta) - \frac{1}{2}\,\big(a_n(\varepsilon)\,b_n(\delta) + a_n(\delta)\,b_n(\varepsilon)\big)\\ &\quad + \frac{\sqrt{5}}{2}\,\big(a_n(\delta)\,b_n(\varepsilon) - a_n(\varepsilon)\,b_n(\delta)\big). \square\end{aligned}$$

5. Inner products of δ-Fibonacci vectors

The aim of this section is to determine the compact analytical description of the following three sums:

$$S_{1,N}(\delta) := \sum_{k=0}^{N-1} a_k(\delta)\,b_k(\delta),$$

$$S_{2,N}(\delta) := \sum_{k=0}^{N-1} a_k^2(\delta), \qquad S_{3,N}(\delta) := \sum_{k=0}^{N-1} b_k^2(\delta).$$

We achieve the assumed goal in two independent ways.

The first way

Let us start with the identity written below (see formula (2.8) in [6]):

$$\begin{aligned}\big(1 - \delta - \delta^2\big)^k &= \big(1 + \delta\,(\xi + \xi^4)\big)^k\big(1 + \delta\,(\xi^2 + \xi^3)\big)^k\\ &= \big(a_k(\delta) + b_k(\delta)\,(\xi + \xi^4)\big)\big(a_k(\delta) + b_k(\delta)\,(\xi^2 + \xi^3)\big)\\ &= a_k^2(\delta) - a_k(\delta)\,b_k(\delta) - b_k^2(\delta).\end{aligned} \tag{5.1}$$

Hence, we obtain

$$S_{2,N}(\delta) - S_{1,N}(\delta) - S_{3,N}(\delta) = \sum_{k=0}^{N-1}\big(1 - \delta - \delta^2\big)^k \tag{5.2}$$

$$= \frac{1 - \big(1 - \delta - \delta^2\big)^N}{1 - \big(1 - \delta - \delta^2\big)} = \frac{1}{\delta + \delta^2}\Big(1 - \big(1 - \delta - \delta^2\big)^N\Big)$$

$$= \sum_{k=0}^{N-1}(-1)^k\binom{N}{k}\,(\delta + \delta^2)^k.$$

Remark 5.1. If $1-\delta-\delta^2=1$, i.e. $\delta=0$ or $\delta=-1$, then we set

$$S_{2,N}(\delta)-S_{1,N}(\delta)-S_{3,N}(\delta):=\lim_{\substack{\delta\to 0\\ or\\ \delta\to -1}}\frac{1}{\delta+\delta^2}\Big(1-\big(1-\delta-\delta^2\big)^N\Big)$$
$$\overset{H}{=}\lim_{\substack{\delta\to 0\\ or\\ \delta\to -1}} N\big(1-\delta-\delta^2\big)^{N-1}=N.$$

On the other hand, we have

$$\big(1+\delta\,(\xi+\xi^4)\big)^{2k}=\big(a_k(\delta)+b_k(\delta)\,(\xi+\xi^4)\big)^2$$
$$=a_k^2(\delta)+b_k^2(\delta)+\big(2\,a_k(\delta)\,b_k(\delta)-b_k^2(\delta)\big)(\xi+\xi^4),$$

which implies the identity

$$S_{2,N}(\delta)+S_{3,N}(\delta)+(\xi+\xi^4)\big(2\,S_{1,N}(\delta)-S_{3,N}(\delta)\big) \tag{5.3}$$
$$=\sum_{k=0}^{N-1}\big(1+\delta\,(\xi+\xi^4)\big)^{2k}=\frac{1-\big(1+\delta\,(\xi+\xi^4)\big)^{2N}}{1-\big(1+\delta\,(\xi+\xi^4)\big)^2}$$
$$=\frac{1-a_{2N}(\delta)-b_{2N}(\delta)(\xi+\xi^4)}{1-a_2(\delta)-b_2(\delta)(\xi+\xi^4)}$$
$$=\frac{\begin{vmatrix} b_2(\delta) & 1-a_{2N}(\delta)\\ 1-a_2(\delta) & a_{2N}(\delta)+b_{2N}(\delta)-1\end{vmatrix}+\begin{vmatrix} 1-a_{2N}(\delta) & -b_{2N}(\delta)\\ 1-a_2(\delta) & -b_2(\delta)\end{vmatrix}(\xi+\xi^4)}{\begin{vmatrix} b_2(\delta) & 1-a_2(\delta)+b_2(\delta)\\ 1-a_2(\delta) & b_2(\delta)\end{vmatrix}}.$$

Hence, we obtain two new identities

$$S_{2,N}(\delta)+S_{3,N}(\delta)=\frac{\begin{vmatrix} b_2(\delta) & 1-a_{2N}(\delta)\\ 1-a_2(\delta) & a_{2N}(\delta)+b_{2N}(\delta)-1\end{vmatrix}}{\begin{vmatrix} b_2(\delta) & 1-a_2(\delta)+b_2(\delta)\\ 1-a_2(\delta) & b_2(\delta)\end{vmatrix}},$$

i.e. ($\delta\neq 0,-1\pm\sqrt{5}$):

(5.4)
$$\delta\big(5-(\delta+1)^2\big)\big(S_{2,N}(\delta)+S_{3,N}(\delta)\big)=2\,(1-\delta)\,(a_{2N}(\delta)-1)+(2-\delta)\,b_{2N}(\delta),$$

and

$$2\,S_{1,N}(\delta)-S_{3,N}(\delta)=\frac{\begin{vmatrix} 1-a_{2N}(\delta) & -b_{2N}(\delta)\\ 1-a_2(\delta) & -b_2(\delta)\end{vmatrix}}{\begin{vmatrix} b_2(\delta) & 1-a_2(\delta)+b_2(\delta)\\ 1-a_2(\delta) & b_2(\delta)\end{vmatrix}},$$

i.e. ($\delta\neq 0,-1\pm\sqrt{5}$):

(5.5)
$$\delta\big(5-(\delta+1)^2\big)\big(2\,S_{1,N}(\delta)-S_{3,N}(\delta)\big)=(2-\delta)\,(a_{2N}(\delta)-1)-\delta\,b_{2N}(\delta).$$

Finally, from the identities (5.2), (5.4) and (5.5), we get

$$
\begin{aligned}
(5.6)\quad 5\,S_{2,N}(\delta) &= 2\big(S_{2,N}(\delta) - S_{1,N}(\delta) - S_{3,N}(\delta)\big) + 3\big(S_{2,N}(\delta) + S_{3,N}(\delta)\big)\\
&\quad + 2\,S_{1,N}(\delta) - S_{3,N}(\delta) = \frac{2}{\delta+\delta^2}\big(1-(1-\delta-\delta^2)^N\big)\\
&\quad + \frac{1}{\delta\,(5-(\delta+1)^2)}\big((8-7\,\delta)(a_{2N}(\delta)-1) + (6-4\,\delta)\,b_{2N}(\delta)\big),
\end{aligned}
$$

$$
\begin{aligned}
(5.7)\quad 5\,S_{3,N}(\delta) &= \frac{2}{\delta+\delta^2}\big((1-\delta-\delta^2)^N - 1\big)\\
&\quad + \frac{1}{\delta\,(5-(\delta+1)^2)}\big((2-3\,\delta)(a_{2N}(\delta)-1) + (4-\delta)\,b_{2N}(\delta)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
(5.8)\quad 5\,S_{1,N}(\delta) &= \frac{1}{\delta+\delta^2}\big((1-\delta-\delta^2)^N - 1\big)\\
&\quad + \frac{1}{\delta\,(5-(\delta+1)^2)}\big((6-4\,\delta)(a_{2N}(\delta)-1) + (2-3\,\delta)\,b_{2N}(\delta)\big).
\end{aligned}
$$

For example, from (5.6) for $N = 2$ we obtain

$$
(5.9)\quad 2\,\delta\,(\delta^2+\delta+3)(4-2\,\delta-\delta^2) = (8-7\,\delta)(a_4(\delta)-1) + (6-4\,\delta)\,b_4(\delta).
$$

Corollary 5.2. *Using (1.9) we get (see also [1, 2]):*

$$
(5.10)\quad 5\,S_{3,N}(1) = 5\sum_{k=0}^{N-1} F_k^2 = (-1)^N + 3\,F_{2N} - F_{2N+1} = L_{2N-1} + (-1)^N;
$$

$$
\begin{aligned}
5\,S_{2,N}(1) &= 5\sum_{k=1}^{N} F_{k+1}^2 = F_{2N+1} + 2\,F_{2N} - (-1)^N\\
(5.11)\quad &= F_{2N+2} + F_{2N} - (-1)^N = L_{2N+1} - (-1)^N;
\end{aligned}
$$

$$
(5.12)\quad 5\,S_{1,N}(1) = 5\sum_{k=0}^{N-1} F_k\,F_{k+1} = L_{2N} + \tfrac{1}{2}\big((-1)^N - 5\big);
$$

$$
(5.13)\quad 5\,S_{3,N}(-1) = 5\sum_{k=0}^{N-1} F_{2k}^2 = 1 - 2\,N + F_{4N-2};
$$

(5.14)

$$
5\,S_{2,N+1}(-1) = 5\sum_{k=0}^{N} F_{2k-1}^2 = 2\,N + 5 + F_{4N};
$$

(5.15)

$$
-5\,S_{1,N+1}(-1) = 5\sum_{k=0}^{N} F_{2k-1}\,F_{2k} = N - 1 + F_{4N+1}.
$$

Moreover, from (5.15) the following formula can be derived

$$-5\,S_{1,N+1}(-1) = \frac{5}{2}\sum_{k=0}^{N}(L_{2k} - F_{2k})\,F_{2k} = \frac{5}{2}\sum_{k=0}^{N} L_{2k}\,F_{2k} - \frac{5}{2}\sum_{k=0}^{N} F_{2k}^2.$$

Hence, by (5.13) and (5.15), we get

$$2\,N - 2 + 2\,F_{4N+1} = 5\sum_{k=0}^{N} L_{2k}\,F_{2k} - \Big(F_{4N+2} - 2\,N - 1\Big),$$

i.e

$$5\sum_{k=0}^{N} L_{2k}\,F_{2k} = L_{4N+2} - 3. \tag{5.16}$$

The second way

First, we prove (in two ways) two auxiliary formulas:

$$\delta\sum_{k=0}^{N-1} b_k(\delta) = a_N(\delta) - 1 \tag{5.17}$$

and

$$\delta\sum_{k=0}^{N-1} a_k(\delta) = a_N(\delta) + b_N(\delta) - 1. \tag{5.18}$$

Proof. Immediately from (1.3) we derive

$$\delta\sum_{k=0}^{N-1} b_k(\delta) = \sum_{k=0}^{N-1}\big(a_{k+1}(\delta) - a_k(\delta)\big) = a_N(\delta) - a_0(\delta) = a_N(\delta) - 1.$$

Next, from (1.4) we obtain

$$\begin{aligned}\delta^2\sum_{k=0}^{N-1} a_k(\delta) &= \delta\sum_{k=0}^{N-1} b_{k+1}(\delta) + (\delta - 1)\,\delta\sum_{k=0}^{N-1} b_k(\delta)\\ &= \big(a_{N+1}(\delta) - 1\big) + (\delta - 1)\big(a_N(\delta) - 1\big)\\ &= a_{N+1}(\delta) + (\delta - 1)\,a_N(\delta) - \delta.\end{aligned}$$

Hence, by (1.3) again, we get

$$\delta\sum_{k=0}^{N-1} a_k(\delta) = a_N(\delta) + b_N(\delta) - 1.\ \square$$

Now, by the Binet's formulas (1.5) and (1.6), we obtain

$$\sum_{k=1}^{N} a_{2k}(\delta) = \frac{2\,\delta+1+\sqrt{5}}{2\,\sqrt{5}} 2\,\sqrt{5}\Big(\frac{2-\delta+\sqrt{5}\,\delta}{2}\Big) \cdot \frac{\big(\frac{2-\delta+\sqrt{5}\,\delta}{2}\big)^{2N}-1}{\big(\frac{2-\delta+\sqrt{5}\,\delta}{2}\big)^{2}-1}$$
$$- \frac{2\,\delta+1-\sqrt{5}}{2\,\sqrt{5}} \cdot \left(\frac{2-\delta-\sqrt{5}\,\delta}{2}\right) \cdot \frac{\big(\frac{2-\delta-\sqrt{5}\,\delta}{2}\big)^{2N}-1}{\big(\frac{2-\delta-\sqrt{5}\,\delta}{2}\big)^{2}-1}$$
$$= \frac{1}{\sqrt{5}\,\delta} \cdot \frac{2\,\delta+1+\sqrt{5}}{(3-\sqrt{5})\delta+2(\sqrt{5}-1)} \cdot \left(\frac{2-\delta+\sqrt{5}\,\delta}{2}\right)^{2N+1}$$
$$- \frac{1}{\sqrt{5}\,\delta} \cdot \frac{2\,\delta+1+\sqrt{5}}{(3-\sqrt{5})\delta+2\,(\sqrt{5}-1)} \cdot \left(\frac{2-\delta+\sqrt{5}\,\delta}{2}\right)$$
$$- \frac{1}{\sqrt{5}\,\delta} \cdot \frac{2\,\delta+1-\sqrt{5}}{(3+\sqrt{5})\,\delta-2\,(\sqrt{5}+1)} \cdot \left(\frac{2-\delta-\sqrt{5}\,\delta}{2}\right)^{2N+1}$$
$$+ \frac{1}{\sqrt{5}\,\delta} \cdot \frac{2\,\delta+1-\sqrt{5}}{(3+\sqrt{5})\,\delta-2\,(\sqrt{5}+1)} \cdot \left(\frac{2-\delta-\sqrt{5}\,\delta}{2}\right)$$
$$= \frac{1}{\delta(\delta^2+2\,\delta-4)}\Big(2\,(\delta-1)\,a_{2N+2}(\delta) + (\delta-2)\,b_{2N+2}(\delta) - \delta^3 - 2\,\delta^2 + 2\,\delta + 2\Big),$$

i.e. the following identity holds

$$\delta\,(\delta^2+2\,\delta-4) \sum_{k=0}^{N-1} a_{2k}(\delta) = 2\,(\delta-1)\,\big(a_{2N}(\delta)-1\big) + (\delta-2)\,b_{2N}(\delta). \tag{5.19}$$

Next, from (1.3) it follows that

$$\begin{aligned}\delta \sum_{k=1}^{N-1} b_{2k}(\delta) &= \sum_{k=1}^{N-1} a_{2k+1}(\delta) - \sum_{k=1}^{N-1} a_{2k}(\delta)\\ &= \sum_{k=0}^{2N-1} a_k(\delta) - 2 \sum_{k=1}^{N-1} a_{2k}(\delta) - a_0(\delta) - a_1(\delta).\end{aligned}$$

Hence, by (5.18) and (5.19) we obtain

$$\begin{aligned}&\delta^2\,(\delta^2+2\,\delta-4) \sum_{k=1}^{N-1} b_{2k}(\delta)\\ &\quad= (\delta^2+2\,\delta-4)\big(a_{2N}(\delta)+b_{2N}(\delta)-1\big) - 2\,\big(2\,(\delta-1)\,a_{2N}(\delta)\\ &\quad\quad + (\delta-2)\,b_{2N}(\delta) - \delta^3 - 2\,\delta^2 + 2\,\delta + 2\big) - 2\,(\delta^3+2\,\delta^2-4\,\delta)\\ &\quad= \delta\,(\delta-2)\,\big(a_{2N}(\delta)-1\big) + \delta^2\,b_{2N}(\delta),\end{aligned}$$

i.e.

$$\delta\,(\delta^2+2\,\delta-4) \sum_{k=1}^{N-1} b_{2k}(\delta) = (\delta-2)\,\big(a_{2N}(\delta)-1\big) + \delta\,b_{2N}(\delta). \tag{5.20}$$

Next, form (1.5) and (1.6) we obtain

$$5\,S_{1,N}(\delta) = \sum_{k=0}^{N-1} a_{2k}(\delta) + 2\sum_{k=0}^{N-1} b_{2k}(\delta) - \sum_{k=0}^{N-1}(1-\delta-\delta^2)^k.$$

Thus, by (5.19) and (5.20) we derive

$$\begin{aligned}
&5\,\delta\,(\delta^2+2\,\delta-4)\,S_{1,N}(\delta) = 2\,(\delta-1)\big(a_{2N}(\delta)-1\big) + (\delta-2)\,b_{2N}(\delta)\\
&+2\,(\delta-2)\big(a_{2N}(\delta)-1\big) + 2\,\delta\,b_{2N}(\delta) + \frac{1}{\delta+\delta^2}\big((1-\delta-\delta^2)^N-1\big)\\
&= (4\,\delta-6)\big(a_{2N}(\delta)-1\big) + (3\,\delta-2)b_{2N}(\delta) + \frac{5(\delta^2+2\delta-4)}{\delta+1}\big((1-\delta-\delta^2)^N-1\big),
\end{aligned}$$

which is exactly the formula (5.8).

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