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A NEW CHARACTERIZATION OF S_8

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Abstract. Let G be a group and $\pi_e(G)$ be the set of element orders of G. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Set nse(G):=$\{m_k | k \in \pi_e(G)\}$. In this work we prove if G is a group such that nse(G)=nse(S_8), then $G \cong S_8$.

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1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. Also, the set of element orders of G is denoted by $\pi_e(G)$. A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. Set m_i=$m_i(G)$=$|\{g \in G|$ the order of g is $i\}|$ and nse(G):=$\{m_i | i \in \pi_e(G)\}$.

For the set nse(G), the most important problem is related to Thompson's problem. In 1987, J. G. Thompson put forward the following problem. For each finite group G and each integer $d \geq 1$, let $G(d) = \{x \in G | x^d = 1\}$. Defining G_1 and G_2 is of the same order type if, and only if, $|G_1(d)| = |G_2(d)|$, $d = 1, 2, 3, \cdots$. Suppose G_1 and G_2 are of the same order type. If G_1 is solvable, is G_2 necessarily solvable?

W. J. Shi in [10] made the above problem public in 1989. Unfortunately, no one could solve it or give a counterexample until now, and it remains open. The influence of nse(G) on the structure of finite groups was studied by some authors (see [1, 9, 7, 6]). In this paper we continue this work and show that the symmetric group S_8 is characterizable by nse(G). In fact, the main theorem of our paper is as follows:

Main Theorem: Let G be a group such that nse(G)=nse(S_8). Then $G \cong S_8$.

We note that there are finite groups which are not characterizable by nse(G) and $|G|$. In 1987, Thompson gave an example as follows:
Let G_1= $(C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be the maximal subgroups of M_{23}, where $\rtimes$ is a semidirect product symbol. Then nse(G_1) = nse(G_2) and $|G_1|$=$|G_2|$, but $G_1 \ncong G_2$. Throughout this paper, we denote by ϕ the Euler totient function. If G is a finite group, then we denote by P_q a Sylow

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$q-$subgroup of G and $n_q(G)$ is the number of Sylow $q-$subgroup of G, that is, $n_q(G)$=$|\mathrm{Syl}_q(G)|$. We use $a \mid b$ to mean that a divides b, if p is a prime, then $p^n \mid\mid b$ means $p^n \mid b$ but $p^{n+1} \nmid b$. All other notations are standard and we refer to [8], for example.

2. Preliminary Results

In this section, for the proof of the main theorem we need the following Lemmas:

Lemma 2.1. [2] Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.

Lemma 2.2. [9] Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = sup\{m_k | k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2 - 1)$.

Lemma 2.3. [3] Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow $p-$subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.

Lemma 2.4. [4] Let G be a finite solvable group and $|G| = m \cdot n$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, ..., p_r\}$ and h_m be the number of $\pi-$Hall subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, ..., s\}$:

1. $q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some p_j.
2. The order of some chief factor of G is divisible by $q_i^{\beta_i}$.

Lemma 2.5. [1] Let G be a finite group, $P \in \mathrm{Syl}_p(G)$, where $p \in \pi(G)$. Let G have a normal series $K \trianglelefteq L \trianglelefteq G$. If $P \leq L$ and $p \nmid |K|$, then the following hold:
(1) $N_{G/K}(PK/K) = N_G(P)K/K$;
(2) $|G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;
(3) $|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(L/K)t = n_p(G) = n_p(L)$ for some positive integer t, and $|N_K(P)|t = |K|$.

Lemma 2.6. [5] If G is a simple K_3- group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$.

Lemma 2.7. [11] Let G be a simple group of order $2^a \cdot 3^b \cdot 5 \cdot p^c$, where $p \neq 2$, 3, 5 is a prime, and $abc \neq 0$. Then G is isomorphic to one of the following groups: A_7, A_8, A_9; M_{11}, M_{12}; $L_2(q)$, $q = 11$, 16, 19, 31, 81; $L_3(4)$, $L_4(3)$, $S_6(2)$, $U_4(3)$ or $U_5(2)$. In particular, if $p = 11$, then $G \cong M_{11}$, M_{12}, $L_2(11)$ or $U_5(2)$; if $p = 7$, then $G \cong A_7$, A_8, A_9, A_{10}, $L_2(49)$, $L_3(4)$, $S_4(7)$, $S_6(2)$, $U_3(5)$, $U_4(3)$, J_2, or $O_8^+(2)$.

Let G be a group such that nse(G)=nse(S_8). By Lemma 2.2, we can assume that G is finite. Let m_n be the number of elements of order n. We note that $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. Also, we note that if $n > 2$, then $\phi(n)$ is even. If $n \in \pi_e(G)$, then by Lemma 2.1 and the above notation, we have:

$$\begin{cases} \phi(n) \mid m_n \\ n \mid \sum_{d|n} m_d \end{cases} \qquad (*)$$

In the proof of the main theorem, we often apply $(*)$ and the above comments.

3. Proof of the Main Theorem.

Let G be a group such that nse(G)=nse(S_8)={1, 763, 1232, 1344, 2688, 3360, 4032, 5040, 5460, 5760, 10640}. First we prove that $\pi(G) \subseteq \{2,3,5,7\}$. Since 763 $\in$nse(G), it follows that $2 \in \pi(G)$ and $m_2 = 763$. Let $2 \neq p \in \pi(G)$. By $(*)$ we have $p \in \{3, 5, 7, 43, 37, 71, 2689, 3361\}$.

We will show $43 \notin \pi(G)$. Suppose $43 \in \pi(G)$. By $(*)$, $m_{43} = 5460$. If $43^2 \in \pi_e(G)$, then $\phi(43^2) \mid m_n$ where $m_n \in$ nse(G), a contradiction. Hence $\exp(P_{43}) = 43$. Thus by Lemma 2.1, $|P_{43}| \mid (1 + m_{43}) = 5461$, so $|P_{43}| = 43$. We prove $86 \notin \pi_e(G)$.

Suppose $86 \in \pi_e(G)$, we know that if P and Q are Sylow 43−subgroups of G, then P and Q are conjugate, which implies that $C_G(P)$ and $C_G(Q)$ are conjugate. Therefore $m_{86} = \phi(86) \cdot n_{43} \cdot k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_{43})$. Since $n_{43} = m_{43}/\phi(43) = 130$, $5460 \mid m_{86}$. Therefore $m_{86} = 5460$. On the other hand, $86 \mid (1+m_2+m_{43}+m_{86}) = 11684$, which is a contradiction.

Hence $86 \notin \pi_e(G)$. Then the group P_{43} acts fixed point freely on the set of elements of order 2. Hence $|P_{43}| \mid m_2 = 763$, a contradiction. Arguing as above, we can prove 37, 71, 2689 and $3361 \notin \pi(G)$. Hence $\pi(G) \subseteq \{2, 3, 5, 7\}$.

If $3, 5, 7 \in \pi(G)$, then $m_3 \in \{1232, 10640\}$, $m_5 = 1344$ and $m_7 = 5760$ by $(*)$. It is clear that G does not contain any element of order 81, 25, 512 and 343 by $(*)$. If $49 \in \pi_e(G)$, then $m_{49} \in \{1344, 5460\}$. Hence by Lemma 2.1, $|P_7| \mid (1+m_7+m_{49}) = 7105$ or 11221, so $|P_7| = 49$. Therefore $n_7 = m_{49}/\phi(49) = 32$ or 130. Since $n_7 = 1+7k$ for some k, we get a contradiction. Thus $49 \notin \pi_e(G)$.

We conclude if $5, 7 \in \pi(G)$, then $\exp(P_5) = 5$ and $\exp(P_7) = 7$, also by Lemma 2.1 $|P_5| = 5$ and $|P_7| = 7$. Hence $n_5 = m_5/\phi(5) = 2^4 \times 7 \times 3$ and $n_7 = m_7/\phi(7) = 2^6 \times 3 \times 5$. Thus if $5 \in \pi(G)$, then $3, 7 \in \pi(G)$ and if $7 \in \pi(G)$, then $3, 5 \in \pi(G)$.

So if we show that $\pi(G)$ could not be the sets $\{2\}$, $\{2, 3\}$, then $\pi(G)$ must be equal to $\{2, 3, 5, 7\}$. We consider the following cases:

<u>Case a.</u> Suppose that $\pi(G) = \{2\}$. Hence $\pi_e(G) \subseteq \{1, 2, 4, 8, 16, 32, 64, 128, 256\}$. Since nse($G$) have eleven elements, we get a contradiction.

<u>Case b.</u> Suppose that $\pi(G) = \{2, 3\}$. Since $81 \notin \pi_e(G)$, $\exp(P_3) = 3$, 9 or 27. Let $\exp(P_3) = 3$. Thus $|P_3| \mid (1 + m_3) = 1233$ or 10641 by Lemma 2.1. Hence $|P_3| \mid 9$. If $|P_3| = 3$, then $n_3 = m_3/\phi(3) = 616$ or 5320. Because $7 \notin \pi(G)$, we get a contradiction.

Let $|P_3| = 9$. Since $\exp(P_3) = 3$ and $2^8 \times 3 \notin \pi_e(G)$, $\pi_e(G) \subseteq \{1, 2, 3, 2^2, \ldots, 2^8\} \cup \{2 \times 3, 2^2 \times 3, \ldots, 2^7 \times 3\}$. Hence $|\pi_e(G)| \leq 17$. Therefore $40320 + 1232k_1 + 1344k_2 + 2688k_3 + 3360k_4 + 4032k_5 + 5040k_6 + 5460k_7 +$

$5760k_8 + 10640k_9 = 2^m \times 9$ where m, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8 and k_9 are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \leq 6$.

We know that $40320 \leq 2^m \times 9 \leq 40320 + 10640 \times 6$, hence $m = 13$ and so $1232k_1 + 1344k_2 + 2688k_3 + 3360k_4 + 4032k_5 + 5040k_6 + 5460k_7 + 5760k_8 + 10640k_9 = 33408$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \leq 6$. By simple computer calculation, it is easy to see this equation has no solution.

Let $\exp(P_3) = 9$. By $(*)$ we have $m_9 \in \{4032, 5040, 5760\}$. Assume $m_3 = 10640$. Then by $(*)$, $9 \mid (1 + m_3 + m_9)$. Since $m_9 \in \{4032, 5040, 5760\}$, we get a contradiction. Hence $m_3 = 1232$ and hence by Lemma 2.1, $|P_3| \mid 81$.

If $|P_3| = 9$, then $n_3 = m_9/\phi(9) \in \{672, 840, 960\}$, which is a contradiction by $5, 7 \notin \pi(G)$.

Assume $|P_3| = 27$. Since $\exp(P_3) = 9$ and $2^8 \times 3, 2^8 \times 9 \notin \pi_e(G)$, $\pi_e(G) \subseteq \{1, 2, 3, 2^2, \ldots, 2^8\} \cup \{2 \times 3, 2^2 \times 3, \ldots, 2^7 \times 3\} \cup \{2 \times 9, 2^2 \times 9, \ldots, 2^7 \times 9\}$. On the other hand, if $2^8 \in \pi_e(G)$ since $2^8 \times 3 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 256. Hence $|P_3| \mid m_{256} = 5760$, a contradiction. Thus $2^8 \notin \pi_e(G)$ and $|\pi_e(G)| \leq 24$. Therefore $40320 + 1232k_1 + 1344k_2 + 2688k_3 + 3360k_4 + 4032k_5 + 5040k_6 + 5460k_7 + 5760k_8 + 10640k_9 = 2^m \times 27$ where m, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8 and k_9 are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \leq 13$. We have $40320 \leq 2^m \times 27 \leq 40320 + 10640 \times 14$, so $m = 11$ or 12.

If $m = 11$, then $1232k_1 + 1344k_2 + 2688k_3 + 3360k_4 + 4032k_5 + 5040k_6 + 5460k_7 + 5760k_8 + 10640k_9 = 14976$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \leq 13$. By computer calculation, it is easy to see this equation has no solution.

If $m = 12$, then $1232k_1 + 1344k_2 + 2688k_3 + 3360k_4 + 4032k_5 + 5040k_6 + 5460k_7 + 5760k_8 + 10640k_9 = 70272$, where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \leq 13$.

If $2^7 \times 9 \in \pi_e(G)$, then $|\pi_e(G)| = 24$. In this case the equation have 31 solutions. For example $(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9) = (0, 0, 0, 7, 1, 1, 0, 1, 3)$ is one of the solutions. We show this is impossible. Since $2^7 \times 9$, $2^7 \times 3 \in \pi_e(G)$, $m_{2^7\times 9} = 2688$ or 5760 and $m_{2^7\times 3} = 2688$ or 5760. We know $2^8 \notin \pi_e(G)$, thus $\exp(P_2) = 2, 4, 8, 16, 32, 64$ or 128. Hence if $\exp(P_2) = 2^i$ where $1 \leq i \leq 7$, then $|P_2| \mid (1 + m_2 + \ldots + m_{2^i})$, by Lemma 2.1. In fact, $|P_2| \mid (1 + 763 + 1232t_1 + 1344t_2 + 2688t_3 + 3360t_4 + 4032t_5 + 5040t_6 + 5460t_7 + 5760t_8 + 10640t_9)$, where t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8 and t_9 are non-negative integers and $0 \leq t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9 \leq 6$. Because $k_1 = 0$ and $m_3 = 1232$, $m_{2^i} \neq 1232$ for $1 \leq i \leq 7$, $t_1 = 0$. Since $k_8 = 1$ and $m_{2^7\times 9} = 2688$ or 5760 and also $m_{2^7\times 3} = 2688$ or 5760, $0 \leq t_8 \leq 1$. Also $k_2 = 0$ and $k_3 = 0$, thus $0 \leq t_2 \leq 1$ and $0 \leq t_3 \leq 1$. Also we have $0 \leq t_5 \leq 2$, $0 \leq t_6 \leq 2$, $0 \leq t_7 \leq 1$ and $0 \leq t_9 \leq 4$. By an easy computer calculation, $|P_2| \mid 2^9$, a contradiction. Arguing as above for other solutions we get a contradiction.

If $2^7 \times 9 \notin \pi_e(G)$, then $|\pi_e(G)| \leq 23$ and the above equation where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \leq 12$, have 25 solutions. For example $(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9) = (0, 0, 0, 1, 1, 5, 0, 1, 3)$ is one of the solutions. We show that this is impossible. Arguing as above, $t_1 = 0$, $0 \leq t_2 \leq 1$, $0 \leq t_3 \leq 1$, $0 \leq t_4 \leq 2$, $0 \leq t_5 \leq 2$, $0 \leq t_6 \leq 6$, $0 \leq t_7 \leq 1$, $0 \leq t_8 \leq 2$ and $0 \leq t_9 \leq 4$. By an easy computer calculation, $|P_2| \mid 2^{10}$, a contradiction.

If $|P_3| = 81$, then $40320 + 1232k_1 + 1344k_2 + 2688k_3 + 3360k_4 + 4032k_5 + 5040k_6 + 5460k_7 + 5760k_8 + 10640k_9 = 2^m \times 81$, where m, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8 and k_9 are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \leq 13$. We know that $40320 \leq 2^m \times 81 \leq 40320 + 10640 \times 14$, hence $m = 9$, 10 or 11. Arguing as above we get a contradiction. If $\exp(P_3) = 27$, then $|P_3| \mid (1 + m_3 + m_9 + m_{27})$ by Lemma 2.1. It is clear that $|P_3| = 27$ or 3^n where $n \gneqq 3$. Hence if $|P_3| = 27$, then $n_3 = m_{27}/\phi(27) \in \{224, 280, 320\}$. Since 5, $7 \notin \pi(G)$, we get a contradiction.

If $|P_3| = 3^n$ where $n \gneqq 3$, then by Lemma 2.3, m_{27} is a multiple of 27, a contradiction.

Therefore $\pi(G) = \{2, 3, 5, 7\}$. We prove that $21 \notin \pi_e(G)$. Suppose that $21 \in \pi_e(G)$, then $m_{21} = \phi(21) \cdot n_7 \cdot k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_7)$. Since $n_7 = m_7/\phi(7) = 960$, $5760 \mid m_{21}$ and $m_{21} = 5760$. On the other hand, by $(*)$ $21 \mid (1 + m_3 + m_7 + m_{21}) = 12753$ or 22161, which is a contradiction. Thus $21 \notin \pi_e(G)$. Arguing as above, we can prove that $14 \notin \pi_e(G)$. Since $21 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 7. Hence $|P_3| \mid m_7 = 5760$, and hence $|P_3| = 3$ or 9. Also, since $14 \notin \pi_e(G)$, the group P_2 acts fixed point freely on the set of elements of order 7. Hence $|P_2| \mid m_7$, then $|P_2| \mid 2^7$. On the other hand, $40320 \leq |G|$ then $|G| = 2^7 \times 3^2 \times 5 \times 7 = |S_8|$.

Now we claim that G is a nonsolvable group. Suppose that G is a solvable group. Since $n_7 = 960$ by Lemma 2.4, $3 \equiv 1 \pmod 7$, which is a contradiction. Hence G is a nonsolvable group and $p \mid\mid |G|$, where $p \in \{5, 7\}$. Therefore G has a normal series

$$1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$$

such that N is a maximal solvable normal subgroup of G and H/N is a nonsolvable minimal normal subgroup of G/N. Then, H/N is a non-abelian simple K_3-group or simple K_4-group.

If H/N be simple K_3- group, then by Lemma 2.6, H/N is isomorphic to one of the groups: A_5 , A_6, $L_2(7)$ or $L_2(8)$.

Suppose that $H/N \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5N/N \in \mathrm{Syl}_5(H/N)$, $n_5(H/N)t = n_5(G)$ for some positive integer t and $5 \nmid t$, by Lemma 2.5. Since $n_5(A_5) = 6$, $n_5(G) = 6t$. Then $m_5 = n_5(G) \times 4 = 24t = 1344$ and $t = 56$. So, by Lemma 2.5, $56 \times |N_N(P_5)| = |N|$. Since $|N| \mid 2^6 \times 3 \times 7$, $n_7(N) = 1$, 8 or 64. So, the number of elements of order 7 in G is 6, 48 or 384, which is a contradiction.

Suppose that $H/N \cong A_6$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5N/N \in \mathrm{Syl}_5(H/N)$, $n_5(H/N)t = n_5(G)$ for some positive integer t and $5 \nmid t$, by Lemma 2.5. Since $n_5(A_6) = 36$, $n_5(G) = 36t$. Then $m_5 = n_5(G) \times 4 = 144t = 1344$ and $t = 28/3$, which is a contradiction.

Suppose that $H/N \cong L_2(7)$. If $P_7 \in \mathrm{Syl}_7(G)$, then $P_7N/N \in \mathrm{Syl}_7(H/N)$, $n_7(H/N)t = n_7(G)$ for some positive integer t and $7 \nmid t$, by Lemma 2.5. Since $n_7(L_2(7)) = 8$, $n_7(G) = 8t$. Thus $m_7 = n_7(G) \times 6 = 48t = 5760$ and $t = 120$. So, by Lemma 2.5, $120 \times |N_N(P_7)| = |N|$. Since $|N| \mid 2^4 \times 3^2 \times 5$, $n_5(N) = 1$ or 6. So, the number of elements of order 5 in G is 4 or 24, which is a contradiction.

Suppose that $H/N \cong L_2(8)$. If $P_7 \in \mathrm{Syl}_7(G)$, then $P_7N/N \in \mathrm{Syl}_7(H/N)$, $n_7(H/N)t = n_7(G)$ for some positive integer t and $7 \nmid t$, by Lemma 2.5. Since

$n_7(L_2(7)) = 36$, $n_7(G) = 36t$. Thus $m_7 = n_7(G) \times 6 = 216t = 5760$ and $t = 80/3$, a contradiction.

Hence H/N is simple K_4-group. By Lemma 2.7, H/N is isomorphic to A_7, A_8 or $L_3(4)$.

Suppose that $H/N \cong A_7$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5N/N \in \mathrm{Syl}_5(H/N)$, $n_5(H/N)t = n_5(G)$ for some positive integer t and $5 \nmid t$, by Lemma 2.5. Since $n_5(A_7) = 126$, $n_5(G) = 126t$. Thus $m_5 = n_5(G) \times 4 = 504t = 1344$ and $t = 8/3$, a contradiction.

Suppose that $H/N \cong L_3(4)$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5N/N \in \mathrm{Syl}_5(H/N)$, $n_5(H/N)t = n_5(G)$ for some positive integer t and $5 \nmid t$, by Lemma 2.5. Since $n_5(L_3(4)) = 2016$, $n_5(G) = 2016t$. Thus $m_5 = n_5(G) \times 4 = 8064t = 1344$, a contradiction.

Hence $H/N \cong A_8$. Now set $\overline{H} := H/N \cong A_8$ and $\overline{G} := G/N$. On the other hand, we have

$$A_8 \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

Let $K = \{x \in G \mid xN \in C_{\overline{G}}(\overline{H})\}$. Thus $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$ and $A_8 \leq G/K \leq \mathrm{Aut}(A_8)$. Then $G/K \cong A_8$ or $G/K \cong S_8$.

If $G/K \cong A_8$, then $|K| = 2$. We have $N \leq K$ and N is a maximal solvable normal subgroup of G, then $N = K$. Thus $H/N \cong A_8$ and $|N| = 2$. Then G has a normal subgroup N of order 2, generated by a central involution z. Let x be an element of order 7 in G. Since $xz = zx$ and $(o(x), o(z)) = 1$, $o(xz) = 14$. Hence $14 \in \pi_e(G)$. We know $14 \notin \pi_e(G)$, a contradiction.

If $G/K \cong S_8$, then $|K| = 1$ and $G \cong S_8$. Now the proof of the main theorem is complete.

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References

[1] Shao, C. G., Shi, W., Jiang, Q. H., Characterization of simple K_4-groups. Front Math China. 3 (2008), 355-370.

[2] Frobenius, G., Verallgemeinerung des sylowschen satze, Berliner sitz. (1895), 981-993.

[3] Miller, G., Addition to a theorem due to Frobenius. Bull. Am. Math. Soc. 11 (1904), 6-7.

[4] Hall, M., The Theory of Groups. New York: Macmillan 1959.

[5] Herzog, M., On finite simple groups of order divisible by three primes only. J. Algebra. 120 (10) (1968), 383-388.

[6] Khalili, A. R., Salehi, S. S., Iranmanesh, A., Tehranian, A., A New Characterization of Symmetric Groups for Some n. (unpublished manuscript)

[7] Khatami, M., Khosaravi, B., Akhlaghi, Z., A new characterization for some linear groups. Monatsh Math. 163 (2011), 39-50.

[8] Conway, J. H., Curtis, R. T., Norton, S. P., Wilson, R. A., Atlas of Finite Groups. Oxford: Clarendon 1985.

[9] Shen, R., Shao, C. G., Jiang, Q., Shi, W., Mazurov, V., A New Characterization of A_5. Monatsh Math. 160 (2010), 337-341.

[10] Shi, W. J., A new characterization of the sporadic simple groups in group theory. In Proceeding of the 1987 Singapore Group Theory Conference. Walter de Gruyter, Berlin. (1989), 531-540.

[11] Shi, W., The simple groups of order $2^a3^b5^c7^d$ and Janko's simple groups. J. Southwest China Normal University (Natural Science Edition). 12 (4) (1987), 1-8.

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