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THE VARIATION PROBLEM IN GENERALIZED LAGRANGE-HAMILTON SPACES

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Abstract. Many significant geometers have contributed to the generalization of Riemann spaces in different directions. In this way arise Finsler spaces, Lagrange spaces, Hamilton spaces, k-Lagrange and k-Hamilton spaces, Lagrange spaces of order k and Hamilton spaces of order k. In references [1–19] an incomplete selection of papers and books connected with these spaces is given. In all these spaces the variation problem is solved. Here, this problem is examined in generalized Lagrange-Hamilton spaces, $(GLH)^{(nk)}$, introduced in [9]. All the spaces mentioned above appear as special cases of $(GLH)^{(nk)}$.

In the first section, the group of coordinates transformation is given and the natural bases $\bar{B}$ and $\bar{B}^*$ of tangent and cotangent spaces $T(GLH)^{(nk)}$ and $T^*(GLH)^{(nk)}$ are examined.

In the second section, the solution of the variation problem of the integral of action for the extreme value of the fundamental function $F(x, y^1, \dots, y^k, p_1, \dots, p_k)$ is obtained. Here, the modified Liouville vectors $I_A(v, h)$ are applied. The connection between notations used here and in [13–15] can be easily established. The generalized Euler-Lagrange (E-L) equations in $(GLH)^{(nk)}$ reduce to the known (E-L) equations in generalized Lagrange spaces.

In the third section, the generalizations of Craig-Synge covectors are given and some important theorems connected with this problem in $(GLH)^{(nk)}$ are proved. The method of proofs is the same as in [13].

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1. Group of transformations, tangent and cotangent spaces

Generalized Lagrange-Hamilton spaces are introduced in [9]. We shall recall only the basic notions which are necessary for understanding the variation problem in these spaces.

Let us denote by $(LH)^{(nk)}$ the $(2k+1)n$ dimensional C^∞ manifold in which a point $(y, p) = (x = y^{(0)}, y^{(1)}, y^{(2)}, \dots, y^{(k)}, p_{(1)}, p_{(2)}, \dots, p_{(k)})$ has the coordinates

$$(x^a = y^{0a}, y^{1a}, y^{2a}, \dots, y^{ka}, p_{1a}, p_{2a}, \dots, p_{ka}), \quad a = \overline{1, n}.$$

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Some curve $c \in (LH)^{(nk)}$ is given by $c : t \in [a,b] \to c(t) \in (LH)^{(nk)}$. A point $(y,p) \in c(t)$ has the coordinates

$$(x^a(t) = y^{0a}(t), y^{1a}(t), \dots, y^{ka}(t), p_{1a}(t), \dots, p_{ka}(t)),$$

where

$$y^{Aa}(t) = d_t^A y^{0a}(t) \quad A = \overline{1,k}, \quad d_t^A = \frac{d^A}{dt^A}, \tag{1.1}$$

$$p_{\alpha a}(t) = d_t^{\alpha-1} p_{1a}(t), \quad \alpha = \overline{1,k}, \quad d_t^{\alpha-1} = \frac{d^{\alpha-1}}{dt^{\alpha-1}}.$$

The allowable coordinate transformations are given by

$$x^{a'} = x^{a'}(x^a) \Leftrightarrow x^a = x^a(x^{a'}) \tag{1.2}$$

$$\begin{aligned}
y^{1a'} &= B_a^{a'} y^{1a}, \quad B_a^{a'} = \partial_{0a} x^{a'} = \partial_a x^{a'}, \\
\partial_{Aa} &= \frac{\partial}{\partial y^{Aa}} \quad A = \overline{0,k}, \quad rank(B_a^{a'}) = n, \dots, \\
y^{Aa'} &= \binom{A-1}{0} (d_t^{A-1} B_a^{a'}) y^{1a} + \binom{A-1}{1} (d_t^{A-2} B_a^{a'}) y^{2a} + \cdots \\
&\cdots + \binom{A-1}{A-1} B_a^{a'} y^{Aa} = d_t^{A-1}(B_a^{a'} y^{1a}), \dots, \\
y^{ka'} &= \binom{k-1}{0} (d_t^{k-1} B_a^{a'}) y^{1a} + \binom{k-1}{1} (d_t^{k-2} B_a^{a'}) y^{2a} + \cdots \\
&\cdots + \binom{k-1}{k-1} B_a^{a'} y^{ka} = d_t^{k-1}(B_a^{a'} y^{1a}), \\
p_{1a'} &= B_{a'}^a p_{1a} \quad B_{a'}^a = \partial_{0a'} x^a = \frac{\partial x^a}{\partial x^{a'}} = B_{a'}^a(t), \dots, \\
p_{\alpha a'} &= \binom{\alpha-1}{0} (d_t^{\alpha-1} B_{a'}^a) p_{1a} + \binom{\alpha-1}{1} (d_t^{\alpha-2} B_{a'}^a) p_{2a} + \cdots \\
&\cdots + \binom{\alpha-1}{\alpha-1} B_{a'}^a p_{\alpha a}, \dots, \\
p_{ka'} &= \binom{k-1}{0} (d_t^{k-1} B_{a'}^a) p_{1a} + \binom{k-1}{1} (d_t^{k-2} B_{a'}^a) p_{2a} + \cdots \\
&\cdots + \binom{k-1}{k-1} B_{a'}^a p_{ka}.
\end{aligned}$$

Theorem 1.1. *The transformations of type* (1.2) *on the common domain form a group.*

Definition 1.1. *The generalized Lagrange-Hamilton space* $(GLH)^{(nk)}$ *of order* k *is a* $(LH)^{(nk)}$ *space, where the group of allowable transformations is given by* (1.2), *and in which a fundamental function*

$$F(x, y^{(1)}, y^{(2)}, \dots, y^{(k)}, p_{(1)}, p_{(2)}, \dots, p_{(k)})$$

is given, where $F : U \to R$ *is differentiable on* $\tilde{U}$ *(rank* $[y^{1a}] = 1$, *rank* $[p_{1a}] = 1$*) and continuous at those points of* U, *where* y^{1a} *and* p_{1a} *are equal to zero,* U *is a domain in* $(GLH)^{(nk)}$.

The natural basis, $\bar{B}_{LH}$ of $T(GLH)^{(nk)}$, as usual, consists of partial derivatives of variables, i.e.

$$\bar{B}_{LH} = \{\partial_{0a}, \partial_{1a}, \dots, \partial_{ka}, \partial^{1a}, \partial^{2a}, \dots, \partial^{ka}\}, \tag{1.3}$$

$$\partial_{0a} = \partial_a = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}}, \quad \partial_{Aa} = \frac{\partial}{\partial y^{Aa}}\; A = \overline{1,k}, \quad \partial^{\alpha a} = \frac{\partial}{\partial p_{\alpha a}}, \; \alpha = \overline{1,k}.$$

Theorem 1.2. *The elements of* $\bar{B}_{LH}$ *transform in the following way:*

(1.4)

$$\begin{aligned}
\partial_{0a} &= (\partial_{0a} y^{0a'})\partial_{0a'} + (\partial_{0a} y^{1a'})\partial_{1a'} + (\partial_{0a} y^{2a'})\partial_{2a'} + (\partial_{0a} y^{3a'})\partial_{3a'} + \dots + (\partial_{0a} y^{ka'})\partial_{ka'} \\
&\quad + (\partial_{0a} p_{1a'})\partial^{1a'} + (\partial_{0a} p_{2a'})\partial^{2a'} + (\partial_{0a} p_{3a'})\partial^{3a'} + \dots + (\partial_{0a} p_{ka'})\partial^{ka'}, \\
\partial_{1a} &= (\partial_{1a} y^{1a'})\partial_{1a'} + (\partial_{1a} y^{2a'})\partial_{2a'} + (\partial_{1a} y^{3a'})\partial_{3a'} + \dots + (\partial_{1a} y^{ka'})\partial_{ka'} \\
&\quad + (\partial_{1a} p_{2a'})\partial^{2a'} + (\partial_{1a} p_{3a'})\partial^{3a'} + \dots + (\partial_{1a} p_{ka'})\partial^{ka'}, \dots \\
\partial_{ka} &= (\partial_{ka} y^{ka'})\partial_{ka'} \\
\partial^{1a} &= (\partial^{1a} p_{1a'})\partial^{1a'} + (\partial^{1a} p_{2a'})\partial^{2a'} + (\partial^{1a} p_{3a'})\partial^{3a'} + \dots + (\partial^{1a} p_{ka'})\partial^{ka'}, \\
\partial^{2a} &= (\partial^{2a} p_{2a'})\partial^{2a'} + (\partial^{2a} p_{3a'})\partial^{3a'} + \dots + (\partial^{2a} p_{ka'})\partial^{ka'}, \dots, \\
\partial^{ka} &= (\partial^{ka} p_{ka'})\partial^{ka'}.
\end{aligned}$$

The natural basis of $T^*(GLH)^{(nk)}$ is

$$\bar{B}^*_{LH} = \{dy^{0a}, dy^{1a}, \dots, dy^{ka}, dp_{1a}, dp_{2a}, \dots, dp_{ka}\}.$$

Theorem 1.3. *The elements of* $\bar{B}^*_{LH}$ *transform in the following way:*

(1.5)

$$\begin{aligned}
dy^{0a'} &= (\partial_{0a} y^{0a'})dy^{0a} \\
dy^{1a'} &= (\partial_{0a} y^{1a'})dy^{0a} + (\partial_{1a} y^{1a'})dy^{1a}, \dots, \\
dy^{ka'} &= (\partial_{0a} y^{ka'})dy^{0a} + (\partial_{1a} y^{ka'})dy^{1a} + \dots + (\partial_{ka} y^{ka'})dy^{ka}, \\
dp_{1a'} &= (\partial_{0a} p_{1a'})dy^{0a} + (\partial^{1a} p_{1a'})dp_{1a},
\end{aligned}$$

$$dp_{2a'} = (\partial_{0a}p_{2a'})dy^{0a} + (\partial_{1a}p_{2a'})dy^{1a} + (\partial^{1a}p_{2a'})dp_{1a} + (\partial^{2a}p_{2a'})dp_{2a}, \ldots,$$

$$dp_{ka'} = (\partial_{0a}p_{ka'})dy^{0a} + (\partial_{1a}p_{ka'})dy^{1a} + \cdots + (\partial_{(k-1)a}p_{ka'})dy^{(k-1)a} +$$

$$(\partial^{1a}p_{ka'})dp_{1a} + \cdots + (\partial^{ka}p_{ka'})dp_{ka}.$$

It is obvious that the elements of $\bar{B}_{LH}$ and $\bar{B}^*_{LH}$ are not transforming as tensors (except for ∂_{ka}, ∂^{ka} and dy^{0a}). Using the J structure in [9], special adapted bases B_{LH} and $\bar{B}^*_{LH}$ are constructed, such that their elements are tensors. Here, these bases will not be used, so their construction is omitted. For the further application we shall define the special Lagrange-Hamilton $(SLH)^{(nk)}$ spaces by

Definition 1.2. *The $(SLH)^{(nk)}$ are such $(LH)^{(nk)}$ spaces in which the group of transformation is reduced to a linear group, i.e. elements of the matrix $(B^{a'}_a)$ are real numbers.*

From Definition 1.2 and (1.2) it follows that in $(SLH)^{(nk)}$ the group of transformation is given by:

$$y^{0a'} = B^{a'}_a y^{0a}, y^{1a'} = B^{a'}_a y^{1a}, \ldots, y^{ka'} = B^{a'}_a y^{ka},$$
$$p_{1a'} = B^a_{a'} p_{1a}, \ldots, p_{ka'} = B^a_{a'} p_{ka}. \tag{1.6}$$

From (1.6) it follows that in $(SLH)^{(nk)}$ the elements of $\bar{B}_{SLH}$ and $\bar{B}^*_{SLH}$ are the same as the corresponding elements of $\bar{B}_{LH}$ and $\bar{B}^*_{LH}$. But, their elements are transforming as tensors, namely from (1.4) and (1.5) it follows

$$\partial_{0a} = B^{a'}_a \partial_{0a'}, \ldots, \partial_{ka} = B^{a'}_a \partial_{ka'}, B^{a'}_a = \partial_{0a} y^{0a'}$$
$$\partial^{1a} = B^a_{a'} \partial^{1a'}, \ldots, \partial^{ka} = B^a_{a'} \partial^{ka'}$$
$$dy^{0a'} = B^{a'}_a dy^{0a}, \ldots, dy^{ka'} = B^{a'}_a dy^{ka},$$
$$dp_{1a'} = B^a_{a'} dp_{1a}, \ldots, dp_{ka'} = B^a_{a'} dp_{ka}. \tag{1.7}$$

2. The variation problem in $(GLH)^{(nk)}$

Let us consider the differentiable curve

$$c^* : t \in [0,1] \to c^*(t) \subset U \subset (GLH)^{(nk)}$$

U is an open set and

$$c^*(t) = r(t) = y^{0a}(t)\partial_{0a} + y^{1a}(t)\partial_{1a} + \cdots$$
$$\cdots + y^{ka}(t)\partial_{ka} + p_{1a}(t)\partial^{1a} + \cdots + p_{ka}(t)\partial^{ka},$$

$$y^{Aa}(t) = d^A_t y^{0a}(t), \quad A = \overline{1,k}, \qquad p_{\alpha a}(t) = d^{\alpha-1}_t p_{1a}(t), \quad \alpha = \overline{2,k}.$$

The integral of action I_{c^*} for the fundamental function

$$F(y^0, y^1, \dots, y^k, p_1, \dots, p_k)$$

is given by

$$(2.1) \qquad I_{c^*} = \int_0^1 F(y^{0a}(t), y^{1a}(t), \dots, y^{ka}(t), p_{1a}(t), \dots, p_{ka}(t))dt.$$

The curve $c^*_\varepsilon(t) = r(t) + \varepsilon\delta r(t)$ is given by $c^*_\varepsilon : t \in [0,1] \to c^*_\varepsilon(t) \subset U \subset (GLH)^{(nk)}$, where for

$$(2.2) \quad \delta r(t) = v^{0a}(t)\partial_{0a} + v^{1a}(t)\partial_{1a} + \cdots + v^{ka}(t)\partial_{ka} + h_{1a}(t)\partial^{1a} + \cdots + h_{ka}(t)\partial^{ka}$$

the following relations are valid:

$$(2.3) \quad v^{Aa}(t) = d_t^A v^{0a}(t), \quad A = \overline{1,k}, \qquad h_{\alpha a}(t) = d_t^{\alpha-1} h_{1a}(t), \quad \alpha = \overline{2,k}.$$

We shall suppose that the curves $c^*_\varepsilon(t)$ for every small enough ε (positive or negative) such that $Imc^*_\varepsilon \subset U$, have the same endpoint and initial point as the curve $c^*(t)$, i.e.

$$c^*_\varepsilon(0) = c^*(0), \qquad c^*_\varepsilon(1) = c^*(1).$$

This will be satisfied if

$$(2.4) \quad v^{Aa}(0) = v^{Aa}(1) = 0, \quad A = \overline{1,k} \qquad h_{\alpha a}(0) = h_{\alpha a}(1) = 0, \quad \alpha = \overline{2,k}.$$

The integral of action $I_{c^*_\varepsilon}$ of F is

(2.5)

$$I_{c^*_\varepsilon} = \int_0^1 F(y^{0a}(t) + \varepsilon v^{0a}(t), \dots, y^{ka}(t) + \varepsilon v^{ka}(t), p_{1a}(t) + \varepsilon h_{1a}(t), \dots, p_{ka} + \varepsilon h_{ka}(t))dt.$$

Using Taylor's formula we get

$$(2.6) \qquad I_{c^*_\varepsilon} - I_{c^*} = \delta I + \delta^2 I + \varepsilon^3 R_3,$$

where

$$\delta I = \int_0^1 dF dt$$

$$(2.7) \quad = \varepsilon \int_0^1 (v^{0a}\partial_{0a} + v^{1a}\partial_{1a} + \cdots + v^{ka}\partial_{ka} + h_{1a}\partial^{1a} + \cdots + h_{ka}\partial^{ka})F dt,$$

$$\delta^2 I = \frac{1}{2}\int_0^1 d^2 F dt$$

$$= \frac{\varepsilon^2}{2} \int_0^1 [v^{0a}\partial_{0a} + v^{1a}\partial_{1a} + \cdots + v^{ka}\partial_{ka} + h_{1a}\partial^{1a} + \cdots + h_{ka}\partial^{ka}]^2 F dt.$$

As ε may be a positive or negative small number, so the necessary condition that $I_{c^*_\varepsilon} - I_{c^*}$ has the same signature for all ε is that δI be equal to zero. If $\delta I = 0$, $\delta^2 I > 0$, then I_{c^*} is minimum, if $\delta I = 0$, $\delta^2 I < 0$, then I_{c^*} is maximum.

The sufficient condition that $\delta I = 0$ is that the expression under integral (2.7) is equal to zero, but it is not a tensor equation. It will be a tensor for some special case of δr, namely if

$$dy^{Aa} = v^{Aa} dt, \quad A = \overline{0,k}, \qquad dp_{\alpha a} = h_{\alpha a} dt, \quad \alpha = \overline{1,k}.$$

In this case the sufficient condition for $\delta I = 0$ is

$$[dy^{0a}\partial_{0a} + dy^{1a}\partial_{1a} + \cdots + dy^{ka}\partial_{ka} + dp_{1a}\partial^{1a} + \cdots + dp_{ka}\partial^{ka}]F = 0,$$

which can be written in the form

$$\left[y^{1a}\partial_{0a} + y^{2a}\partial_{1a} + \cdots + \frac{dy^{ka}}{dt}\partial_{ka} + p_{2a}\partial^{1a} + \cdots + \frac{dp_{ka}}{dt}\partial^{ka}\right] F = 0$$

or

$$\frac{dF}{dt} = 0 \Leftrightarrow \Gamma_k F = 0,$$

where Γ_k is defined in [9].

In some books, the notation $v^{Aa} = \delta y^{Aa}$, $A = \overline{0,k}$ is used and it is called the variation of the variable y^{Aa}. Sometimes it is written as $\delta x, \delta\dot{x}, \delta\ddot{x}, \ldots$.

For the further examination we shall introduce the notations:

$$(2.8) \qquad I'_1(v) = \binom{k}{k} v^{0a}\partial_{ka}$$

$$I'_2(v) = \binom{k-1}{k-1} v^{0a}\partial_{(k-1)a} + \binom{k}{k-1} v^{1a}\partial_{ka}, \ldots,$$

$$I'_k(v) = \binom{1}{1} v^{0a}\partial_{1a} + \binom{2}{1} v^{1a}\partial_{2a} + \cdots + \binom{k}{1} v^{(k-1)a}\partial_{ka},$$

$$I''_2(h) = \binom{k-1}{k-1} h_{1a}\partial^{ka}$$

$$I''_3(h) = \binom{k-2}{k-2} h_{1a}\partial^{(k-1)a} + \binom{k-1}{k-2} h_{2a}\partial^{ka}, \ldots,$$

$$I''_k(h) = \binom{1}{1} h_{1a}\partial^{2a} + \binom{2}{1} h_{2a}\partial^{3a} + \cdots + \binom{k-1}{1} h_{(k-1)a}\partial^{ka}.$$

If the space $(GLH)^{(nk)}$ reduces to the generalized Lagrange space $(GL)^{(nk)}$ from (2.8) we can see that $I'_1(v), I'_2(v), \ldots, I'_k(v)$ are equal to $I^1_V, I^2_V, \ldots, I^k_V$ used by R. Miron in [13, 14] if we substitute v^{0i} by V^i and $\frac{y^{Ai}}{A!}$ by y^{Ai}.

Let us introduce the notations:

$$\bar{E}^0_a = \partial_{0a} - d^1_t\partial_{1a} + d^2_t\partial_{2a} - \cdots + (-1)^k d^k_t\partial_{ka}, \tag{2.9}$$

$$\overline{\overline{E}}^a_1 = \partial^{1a} - d^1_t\partial^{2a} + d^2_t\partial^{3a} - \cdots + (-1)^{k-1}d^{k-1}_t\partial^{ka}.$$

Using the above notations we can state the important identity given by

Theorem 2.1. *The following relation is valid:*

$$v^{0a}\partial_{0a} + v^{1a}\partial_{1a} + \cdots + v^{ka}\partial_{ka} + h_{1a}\partial^{1a} + \cdots + h_{ka}\partial^{ka} = \tag{2.10}$$

$$v^{0a}\bar{E}^0_a + h_{1a}\overline{\overline{E}}^a_1 + d^1_t(I'_k(v) + I''_k(h)) - d^2_t(I'_{k-1}(v) + I''_{k-1}(h)) +$$

$$\cdots + (-1)^{k-2}d^{k-1}_t(I'_2(v) + I''_2(h)) + (-1)^k d^k_t I'_1(v).$$

Remark. In $(GL)^{(nk)}$ (2.10) is shorter, because in this space $h_{1a}\partial^{1a} + \cdots + h_{ka}\partial^{ka} = 0, \overline{\overline{E}}^a_1 = 0, I''_k(h) = 0, I''_{k-1}(h) = 0, \ldots, I''_2(h) = 0$.

Proof. For the general case the proof is based on the following property of binomial coefficients:

$$\sum_{n=a}^{n=b}(-1)^n\binom{n}{a}\binom{b}{n} = 0 \qquad a < b,$$

$$a, b \in \{0, 1, 2, \ldots\}.$$

From (2.7) and (2.10) we get

$$\delta I = \int_0^1 (v^{0a}\bar{E}^{0a} + h_{1a}\overline{\overline{E}}^a_1)F dt. \tag{2.11}$$

□

Theorem 2.2. *The sufficient condition that I_{c^*} be the extremal value of $I_{c^*_\varepsilon}$ in $(GLH)^{(nk)}$ is the following equation:*

$$(v^{0a}\bar{E}^0_a + h_{1a}\overline{\overline{E}}^1_1)F = 0. \tag{2.12}$$

For the special case we have

Theorem 2.3. *For $v^{0a} = y^{1a}$ and $h_{1a} = p_{2a}$ in $(GLH)^{(nk)}$ we have*

$$y^{1a}\bar{E}^0_a + p_{2a}\overline{\overline{E}}^a_1 = y^{1a'}\bar{E}^0_{a'} + p_{2a'}\overline{\overline{E}}^{a'}_1,$$

i.e. the left-hand side of (2.12) *is a scalar field.*

Moreover, $\bar{E}^0_a$ and $\overline{\overline{E}}^a_1$ will be given in the next section.

3. Craig-Synge vectors and covectors

In 1935, Craig and Synge defined covector fields $\overset{(i)}{E}_a$, $i = \overline{0,k}$, in [4] and [19] which were connected with the higher order Finsler spaces. Similar covector fields are given in R. Miron's books [13], [14], ... and they are connected with Lagrange spaces of order k. Here, they will be examined in generalized Lagrange-Hamilton spaces $(GLH)^{(nk)}$. In these spaces we obtain two kinds of families: one of vector fields and the other "covector" fields.

Let us consider the curve $c^* : t \in [0,1] \to c^*(t) \in (GLH)^{(nk)}$ and the differentiable fundamental function $F = F(y^0, y^1, \dots, y^k, p_1, \dots, p_k)$. Now we have

Definition 3.1. *The Craig-Synge "covectors" in* $(GLH)^{(nk)}$ *along the curve* $c^*(t)$ *are defined by*

(3.1)
$$\begin{aligned}
\bar{E}^0_a(F) &= \left[\tbinom{0}{0}\partial_{0a} - \tbinom{1}{0}d^1_t\partial_{1a} + \tbinom{2}{0}d^2_t\partial_{2a} - \dots + (-1)^k\tbinom{k}{0}d^k_t\partial_{ka}\right](F),\\
\bar{E}^1_a(F) &= \left[-\tbinom{1}{1}\partial_{1a} + \tbinom{2}{1}d^1_t\partial_{2a} - \dots + (-1)^k\tbinom{k}{1}d^{k-1}_t\partial_{ka}\right](F),\\
\bar{E}^2_a(F) &= \left[\tbinom{2}{2}\partial_{2a} - \dots + (-1)^k\tbinom{k}{2}d^{k-2}_t\partial_{ka}\right](F), \dots,\\
\bar{E}^k_a(F) &= (-1)^k\tbinom{k}{k}\partial_{ka}(F).
\end{aligned}$$

Formally, $\bar{E}^A_a$, $A = \overline{0,k}$ are the same as the corresponding covectors in the Lagrange spaces of order k (see (8.4.1) in [13], only here $y^{Aa} = d^A_t y^{0a}$). The main difference is the fact, that in $(GLH)^{(nk)}$ ∂_{Aa}, $A = \overline{0,k}$ have different transformation law (see (1.4)). From this it follows

Theorem 3.1. *In* $(GLH)^{(nk)}$ $\bar{E}^0_a$ *defined by* (3.1) *is not covector.*

Proof. Let us restrict the proof for $k = 1$. Then, using (1.4) we get

$$\begin{aligned}
\bar{E}^0_a &= \partial_{0a} - d^1_t\partial_{1a}\\
&= (\partial_{0a}y^{0a'})\partial_{0a'} + (\partial_{0a}y^{1a'})\partial_{1a'} + (\partial_{0a}p_{1a'})\partial^{1a'} -\\
&\quad -d^1_t[\partial_{1a}y^{1a'})\partial_{1a'}].
\end{aligned}\tag{3.2}$$

We have

$$y^{1a'} = B^{a'}_a y^{1a}, \quad B^{a'}_a = \partial_{0a}y^{0a'}, \quad \partial_{1a}y^{1a'} = B^{a'}_a,$$

$$(\partial_{0a}y^{1a'})\partial_{1a'} = B^{a'}_{ab}y^{1b}\partial_{1a'}$$

$$d^1_t[(\partial_{1a}y^{1a'})\partial_{1a'}] = (B^{a'}_{ab}y^{1b})\partial_{1a'} + B^{a'}_a d^1_t\partial_{1a'}.$$

Substituting the last two equations into (3.2) we get

$$\bar{E}^0_a = B^{a'}_a(\partial_{0a'} - d^1_t\partial_{1a'}) + (\partial_{0a}p_{1a'})\partial^{1a'}$$

$$= \quad B_a^{a'} \bar{E}^0_{a'} + (\partial_{0a} p_{1a'}) \partial^{1a'}.$$

The above equation proves Theorem 3.1. □

If $(GLH)^{(nk)}$ reduces to $(GL)^{(nk)}$, then in (1.4) terms of the form $\partial_{Aa} p_{\alpha a'}$ $\alpha \geq A$ do not appear, and we obtain the known result: (see [13])

Theorem 3.2. *$\bar{E}^0_a$, defined by (3.2) in generalized Lagrange space $(GL)^{(nk)}$, is a covector.*

Proposition 3.1. *If $\phi = \phi(y^0, y^1, \dots, y^k, p_1, p_2, \dots, p_k)$ is a differentiable function in $(GLH)^{(nk)}$, such that $\partial_{ka}\phi = 0$, $\partial^{ka}\phi = 0$, then*

$$\begin{aligned} &\partial_{0a} d^1_t \phi = (d^1_t \partial_{0a})\phi, \\ &\partial_{1a} d^1_t \phi = (\partial_{0a} + d^1_t \partial_{1a})\phi, \\ &\partial_{2a} d^1_t \phi = (\partial_{1a} + d^1_t \partial_{2a})\phi, \dots, \\ &\partial_{(k-1)a} d^1_t \phi = (\partial_{(k-2)a} + d^1_t \partial_{(k-1)a})\phi, \\ &\partial_{ka} d^1_t \phi = \partial_{(k-1)a}\phi, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\partial^{1a}(d^1_t \phi) = (d^1_t \partial^{1a})\phi, \\ &\partial^{2a}(d^1_t \phi) = (\partial^{1a} + d^1_t \partial^{2a})\phi, \dots, \\ &\partial^{(k-1)a}(d^1_t \phi) = (\partial^{(k-2)a} + d^1_t \partial^{(k-1)a})\phi, \\ &\partial^{ka}(d^1_t \phi) = \partial^{(k-1)a}\phi. \end{aligned} \tag{3.4}$$

Proof. Using the assumptions $\partial_{ka}\phi = 0$, $\partial^{ka}\phi = 0$, we have

$$\begin{aligned} d^1_t \phi &= [(y^{1b}\partial_{0b} + y^{2b}\partial_{1b} + \cdots + y^{kb}\partial_{(k-1)b}) + \\ &\quad (p_{2b}\partial^{1b} + p_{3b}\partial^{2b} + \cdots + p_{kb}\partial^{(k-1)b})]\phi, \\ \partial_{0a} d^1_t &= [(y^{1b}\partial_{0a}\partial_{0b} + y^{2b}\partial_{0a}\partial_{1b} + \cdots + y^{kb}\partial_{0a}\partial_{(k-1)b}) + \\ &\quad (p_{2b}\partial_{0a}\partial^{1b} + p_{3b}\partial_{0a}\partial^{2b} + \cdots + p_{kb}\partial_{0a}\partial^{(k-1)b}]\phi. \end{aligned} \tag{3.5}$$

From the above two equations it follows $\partial_{0a} d^1_t \phi = d^1_t \partial_{0a}\phi$, which is the first equation of (3.3). From (3.5) it follows

$$\begin{aligned} \partial_{1a} d^1_t \phi &= [\partial_{0a} + (y^{1b}\partial_{1a}\partial_{0b} + y^{2b}\partial_{1a}\partial_{1b} + \cdots + y^{kb}\partial_{1a}\partial_{(k-1)b}) + \\ &\quad (p_{2b}\partial_{1a}\partial^{2b} + p_{3b}\partial_{1a}\partial^{2b} + \cdots + p_{kb}\partial_{1a}\partial^{(k-1)b})]\phi. \end{aligned}$$

From the above equation it follows

$$\partial_{1a} d_t^1 \phi = (\partial_{0a} + d_t^1 \partial_{1a})\phi,$$

which is the second equation of (3.3). As $\partial_{ka}\phi = 0$, from (3.5) it follows

$$\partial_{ka}(d_t^1\phi) = (\partial_{ka} y^{kb})\partial_{(k-1)b}\phi = \partial_{(k-1)a}\phi,$$

which is the last equation of (3.3). (3.4) can be proved using the same method. □

Proposition 3.2. *If $\phi = \phi(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k)$ is a differentiable function in $(GLH)^{(nk)}$, such that $\partial_{ka}\phi = 0$, $\partial^{ka}\phi = 0$, then*

$$\begin{aligned} \bar{E}_a^0(d_t^1\phi) &= 0 \\ \bar{E}_a^1(d_t^1\phi) &= -\bar{E}_a^0(\phi) \\ \bar{E}_a^2(d_t^1\phi) &= -\bar{E}_a^1(\phi), \ldots, \\ \bar{E}_a^k(d_t^1\phi) &= -\bar{E}_a^{(k-1)}\phi. \end{aligned} \tag{3.6}$$

The above equations are the extensions of the results of Caratheodory [3].

Proof. Using (3.3) and (3.1) we obtain:

$$\begin{aligned} \bar{E}_a^0(d_t^1\phi) &= (\partial_{0a} - d_t^1\partial_{1a} + d_t^2\partial_{2a} + \cdots + (-1)^k d_t^k \partial_{ka})(d_t^1\phi) \\ &= [d_t^1\partial_{0a} - d_t^1(\partial_{0a} + d_t^1\partial_{1a}) + d_t^2(\partial_{1a} + d_t^1\partial_{2a}) \\ &\quad -d_t^3(\partial_{2a} + d_t^1\partial_{3a}) + \cdots + (-1)^{k-1} d_t^{k-1}(\partial_{(k-2)a} + d_t^1\partial_{(k-1)a}) \\ &\quad +(-1)^k d_t^k \partial_{(k-1)a}]\phi. \end{aligned}$$

From the above it follows

$$\bar{E}_a^0(d_t^1\phi) = 0.$$

Using the well known relation: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ (3.1) and (3.3) we have:

$$\begin{aligned} &\bar{E}_a^1(d_t^1\phi) \\ &= [-\binom{1}{1}\partial_{1a} + \binom{2}{1} d_t^1\partial_{2a} - \binom{3}{1} d_t^2\partial_{3a} + \cdots + (-1)^k \binom{k}{1} d_t^{k-1}\partial_{ka}](d_t^1\phi) \\ &= [-\binom{1}{1}(\partial_{0a} + d_t^1\partial_{1a}) + \binom{2}{1} d_t^1(\partial_{1a} + d_t^1\partial_{2a}) - \binom{3}{1} d_t^2(\partial_{2a} + d_t^1\partial_{3a}) + \cdots \\ &\quad +(-1)^{k-1}\binom{k-1}{1} d_t^{k-2}(\partial_{(k-2)a} + d_t^1\partial_{(k-1)a}) + (-1)^k \binom{k}{1} d_t^{(k-1)}\partial_{(k-1)a}]\phi \end{aligned}$$

$$= [-\binom{0}{0}\partial_{0a} + [\binom{2}{1} - \binom{1}{1}]d_t^1\partial_{1a} - [\binom{3}{1} - \binom{2}{1}]d_t^2\partial_{2a} + [\binom{4}{1} - \binom{3}{1}]d_t^3\partial_{3a} - \cdots$$

$$+(-1)^k[\binom{k}{1} - \binom{k-1}{1}]d_t^{k-1}\partial_{(k-1)a} + (-1)^{k+1}\binom{k}{0}d_t^k\partial_{ka}]\phi.$$

The last term is equal to zero, because $\partial_{ka}\phi = 0$, so we obtain

$$\bar{E}_a^1(d_t^1\phi) = -[\binom{0}{0}\partial_{0a} - \binom{1}{0}d_t^1\partial_{1a} + \binom{2}{0}d_t^2\partial_{2a} + \binom{3}{0}d_t^3\partial_{3a} - \cdots + (-1)^{k-1}\binom{k-1}{0}d_t^{k-1}\partial_{(k-1)a} + (-1)^k\binom{k}{0}d_t^k\partial_{ka}]\phi,$$

i.e.

$$\bar{E}_a^1(d_t^1\phi) = -\bar{E}_a^0\phi.$$

The other relations from (3.6) can be proved in the same way. □

In $(GLH)^{(nk)}$ we can define vector fields by

Definition 3.2. *If $F(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k)$ is a differentiable function in $(GLH)^{(nk)}$, then along the curve $c^*(t)$ the Craig-Synge vector fields $\overline{\overline{E}}{}_\alpha^a$, $\alpha = \overline{1,k}$, are defined by*

(3.7)

$$\begin{aligned}
\overline{\overline{E}}{}_1^a(F) &= \left(\tbinom{0}{0}\partial^{1a} - \tbinom{1}{0}d_t^1\partial^{2a} + \tbinom{2}{0}d_t^2\partial^{3a} - \cdots + (-1)^{k-1}\tbinom{k-1}{0}d_t^{k-1}\partial^{ka}\right)F,\\
\overline{\overline{E}}{}_2^a(F) &= \left(-\tbinom{1}{1}\partial^{2a} + \tbinom{2}{1}d_t^1\partial^{3a} - \cdots + (-1)^{k-1}\tbinom{k-1}{1}d_t^{k-2}\partial^{ka}\right)F,\\
\overline{\overline{E}}{}_3^a(F) &= \left(\tbinom{2}{2}\partial^{3a} - \cdots + (-1)^{k-1}\tbinom{k-1}{2}d_t^{k-3}\partial^{ka}\right)F, \ldots,\\
\overline{\overline{E}}{}_k^a(F) &= (-1)^{k-1}\tbinom{k-1}{k-1}\partial^{ka}F.
\end{aligned}$$

Proposition 3.3. *If $\phi(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k)$ is a differentiable function in $(GLH)^{(nk)}$, such that $\partial_{ka}\phi = 0$, $\partial^{ka}\phi = 0$, then*

(3.8)
$$\begin{aligned}
\overline{\overline{E}}{}_1^a(d_t^1\phi) &= 0\\
\overline{\overline{E}}{}_2^a(d_t^1\phi) &= -\overline{\overline{E}}{}_1^a(\phi)\\
\overline{\overline{E}}{}_3^a(d_t^1\phi) &= -\overline{\overline{E}}{}_2^a(\phi)\\
\overline{\overline{E}}{}_k^a(d_t^1\phi) &= -\overline{\overline{E}}{}_{k-1}^a(\phi).
\end{aligned}$$

Proof. Using (3.4), (3.7) we have

$$\overline{\overline{E}}{}_1^a(d_t^1\phi) = (\partial^{1a} - d_t^1\partial^{2a} + d_t^2\partial^{3a} - \cdots + (-1)^{k-1}d_t^{k-1}\partial^{ka})(d_t^1\phi)$$

$$= (d_t^1\partial^{1a} - d_t^1(\partial^{1a} + d_t^1\partial^{2a}) + d_t^2(\partial^{2a} - d_t^1\partial^{3a}) + \cdots$$
$$\cdots + (-1)^{k-1}d_t^{k-1}\partial^{(k-1)a})\phi = 0.$$

$\overline{\overline{E}}{}_2^{\,a}(d_t^1\phi)$

$$= (-\binom{1}{1}\partial^{2a} + \binom{2}{1}d_t^1\partial^{3a} - \binom{3}{1}d_t^2\partial^{4a} + \cdots + (-1)^{k-1}\binom{k-3}{1}d_t^{k-2}\partial^{ka})(d_t^1\phi)$$

$$= [-\binom{1}{1}(\partial^{1a} + d_t^1(\partial^{2a}) + \binom{2}{1}d_t^1\partial^{2a} + d_t^1\partial^{3a}) - \binom{3}{1}d_t^2(\partial^{3a} - d_t^1\partial^{4a}) + \cdots]\phi$$

$$= [-\binom{0}{0}\partial^{1a} + [\binom{2}{1} - \binom{1}{1}]d_t^1\partial^{2a} - [\binom{3}{1} - \binom{2}{1}]d_t^2\partial^{3a} + \cdots]\phi = -\overline{\overline{E}}{}_1^{\,a}(\phi), \ldots,$$

$$\overline{\overline{E}}{}_k^{\,a}(d_t^1\phi) = [(-1)^{k-1}\binom{k-1}{k-1}\partial^{(k-1)a}$$
$$= -[(-1)^{k-2}\binom{k-2}{k-2}\partial^{(k-1)a} + (-1)^{(k-1)}\binom{k-1}{k-2}d_t^1\partial_a^k\phi.$$

The last term, which was added, is equal to zero because $\partial^{ka}\phi = 0$. So we obtain

$$\overline{\overline{E}}{}_k^{\,a}(d_t^1\phi) = -\overline{\overline{E}}{}_{k-1}^{\,a}(\phi).$$

□

Consequence:

Theorem 3.3. *In $(GLH)^{(nk)}$ the integrals of actions*

$$I_{c^*} = \int_0^1 F(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k)$$

$$I'_{c^*} = \int_0^1 [F(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k) + d_t^1\phi(y^0, y^1, \ldots, y^{k-1}, p_1, \ldots, p_{k-1})]dt$$

have the same extremal curves for any differentiable fundamental function F and any differentiable function ϕ, for which

$$\partial_{ka}\phi = 0, \quad \partial^{ka}\phi = 0.$$

Proof. The extremal curves of I_{c^*} are the solution of

$$(v^{0a}\bar{E}_a^0 + h_{1a}\overline{\overline{E}}{}_1^{\,a})F = 0$$

and those of I_{c^*} satisfy

$$(v^{0a}\bar{E}^0_a + h_{1a}\overline{\overline{E}}_1{}^a)(F + d^1_t\phi) = 0.$$

As $\bar{E}^0_a(d^1_t\phi) = 0$, $\overline{\overline{E}}_1{}^a(d^1_t\phi) = 0$, (see (3.6) and (3.8)), the extremal curves for both integrals are the solution of the same differential equation. □

Proposition 3.4. *If $\phi = \phi(t)$ and $F = F(y^0, y^1, \dots, y^k, p_1, \dots, p_k)$ are differentiable functions in $(GLH)^{(nk)}$, then the following relations are valid*

$$\begin{aligned}\bar{E}^0_a(\phi F) &= \phi\bar{E}^0_a(F) + (d^1_t\phi)\bar{E}^1_a(F) + \cdots + \\ &\quad (d^{k-1}_t\phi)E^{(k-1)}_a(F) + (d^k_t\phi)\bar{E}^k_a.\end{aligned} \tag{3.9}$$

Proof. From

$$\begin{aligned}\bar{E}^0_a(\phi F) &= [\binom{0}{0}\partial_{0a} - \binom{1}{0}d^1_t\partial_{1a} + \binom{2}{0}d^2_t\partial_{2a} - \binom{3}{0}d^3_t\partial_{3a} + \\ &\quad \binom{4}{0}d^4_t\partial_{4a} + \cdots + (-1)^k\binom{k}{0}d^k_t\partial_{ka}](\phi F).\end{aligned}$$

and $\phi = \phi(t)$ we have

$$\begin{aligned}
\partial_{0a}(\phi F) &= \binom{0}{0}\phi\partial_{0a}F, \\
-d^1_t(\partial_{1a}(\phi F)) &= -d^1_t(\phi\partial_{1a}F) = -[\binom{1}{1}(d^1_t\phi)\partial_{1a} + \binom{1}{0}\phi d^1_t\partial_{1a}]F, \\
d^2_t(\phi\partial_{2a}F) &= [\binom{2}{2}(d^2_t\phi)\partial_{2a} + \binom{2}{1}(d^1_t\phi)d^1_t\partial_{2a} + \binom{2}{0}\phi d^2_t\partial_{2a}]F, \\
-d^3_t(\phi\partial_{3a}F) &= -[\binom{3}{0}(d^3_t\phi)\partial_{3a} + \binom{3}{2}(d^2_t\phi)d^1_t\partial_{3a} \\
&\quad + \binom{3}{1}(d^1_t\phi)d^2_t\partial_{3a} + \binom{3}{0}\phi d^3_t\partial_{3a}]F, \dots, \\
(-1)^k d^k_t(\phi\partial_{ka}F) &= (-1)^k[\binom{k}{k}(d^k_t\phi)\partial_{ka} \\
&\quad + \binom{k}{k-1}d^{k-1}_t\phi d^1_t\partial_{ka} + \cdots + \binom{k}{0}\phi d^k_t\partial_{ka}]F.
\end{aligned}$$

The addition of former equations results (in the first line are the last terms, and so on) in:

$$\begin{aligned}
\bar{E}^0_a(\phi F) = \phi[\tbinom{0}{0}\partial_{0a} - \tbinom{1}{0}d^1_t\partial_{1a} + \tbinom{2}{0}d^2_t\partial_{2a} - \tbinom{3}{0}d^3_t\partial_{3a} + \cdots + (-1)^k\tbinom{k}{0}d^k_t\partial_{ka}]F+ \\
(d^1_t\phi)[-\tbinom{1}{1}\partial_{1a} + \tbinom{2}{1}d^1_t\partial_{2a} - \tbinom{3}{1}d^2_t\partial_{3a} + (-1)^k\tbinom{k}{1}d^{k-1}_t\partial_{ka}]F+ \\
(d^2_t\phi)[\tbinom{2}{2}\partial_{2a} - \tbinom{3}{2}d^1_t\partial_{3a} + \cdots + (-1)^k\tbinom{k}{2}d^{k-2}_t\partial_{ka}]F+ \\
(d^3_t\phi)[-\tbinom{3}{3}\partial_{3a} + \cdots + (-1)^k\tbinom{k}{3}d^{k-3}_t\partial_{ka}]F+ \\
\cdots + (d^k_t\phi)(-1)^k\tbinom{k}{k}\partial_{ka}F.
\end{aligned}$$

The comparison of the above equation with (3.1) gives (3.9). □

As a consequence of the previous proposition we have:

Proposition 3.5. *In $(GLH)^{(nk)}$, the following relations are valid*

$$\bar{E}^0_a(F) = \bar{E}^0_a(F), \tag{3.10}$$

$$\bar{E}^0_a(tF) = t\bar{E}^0_a(F) + \bar{E}^1_a(F),$$

$$\bar{E}^0_a(t^2F) = t^2E^0_a(F) + 2t\bar{E}^1_a(F) + 2!\bar{E}^2_a(F),$$

$$\bar{E}^0_a(t^kF) = t^kE^0_a(F) + kt^{k-1}\bar{E}^1_aF + \cdots + k!\bar{E}^k_aF.$$

Theorem 3.4. *If $(GLH)^{(nk)}$ is reduced to $(GL)^{(nk)}$, then $\bar{E}^0_a, \bar{E}^1_a, \dots, \bar{E}^k_a$ are covectors.*

Proof. It is known that $\bar{E}^0_a$ in $(GL)^{(nk)}$ is covector (Theorem 3.2). From the second equation of (3.10) we get

$$\bar{E}^1_a(F) = \bar{E}^0_a(tF) - t\bar{E}^0_a(F) = B^{a'}_a(\bar{E}^0_{a'}(tF) - t\bar{E}^0_a(F)) = B^{a'}_a\bar{E}^1_{a'}(F), \dots.$$

From the above it follows that $\bar{E}^1_a$ is a covector. For $\phi = t^2$ we get

$$\bar{E}^0_a(t^2F) = t^2\bar{E}^0_a(F) + 2t\bar{E}^1_a(F) + 2\bar{E}^2_a(F),$$

from which we conclude that $\bar{E}^2_a$ is a covector, and so on. □

Theorem 3.5. *If $(GLH)^{(nk)}$ is reduced to $(SLH)^{(nk)}$ then $\bar{E}^0_a, \bar{E}^1_a, \dots, \bar{E}^k_a$ are covectors.*

Proof. From (1.7) we have:

$$\partial_{Aa} = B^{a'}_a\partial_{Aa}, \qquad A = \overline{0,k}, d^A_tB^{a'}_a = 0, A = \overline{0,k}.$$

□

Proposition 3.6. *If $\phi = \phi(t)$ and $F = F(y^0, y^1, \dots, y^k, p_1, \dots, p_k)$ are differentiable functions in $(GLH)^{(nk)}$, then the following relations are valid*

$$\overline{\overline{E}}_1{}^a(\phi F) = \phi\overline{\overline{E}}_1{}^a(F) + (d^1_t\phi)\overline{\overline{E}}_2{}^a(F) + \cdots + (d^{(k-1)}_t\phi)\overline{\overline{E}}_a{}^k(F). \tag{3.11}$$

Proof. As $\phi = \phi(t)$ we have $\partial^{\alpha a}(\phi F) = \phi\partial^{\alpha a}F$, $\alpha = \overline{1,k}$. We get

$$\overline{\overline{E}}_1{}^a(\phi F) = [\binom{0}{0}\partial^{1a} - \binom{1}{0}d^1_t\partial^{2a} + \binom{2}{0}d^2_t\partial^{3a} - \cdots + (-1)^{k-1}\binom{k-1}{0}d^{k-1}_t\partial^{ka}](\phi F).$$

If we add all the following equations

$$\binom{0}{0}\partial^{1a}(\phi F) = \binom{0}{0}\phi\partial^{1a}F,$$

$$
\begin{aligned}
-\binom{1}{0} d_t^1 \partial^{2a}(\phi F) &= -\binom{1}{0}[\binom{1}{1}(d_t^1\phi)\partial^{2a} + \binom{1}{0}\phi d_t^1 \partial^{2a}]F,\\
+\binom{2}{0} d_t^2 \partial^{3a}(\phi F) &= \binom{2}{0}[\binom{2}{2}(d_t^2\phi)\partial^{3a} + \binom{2}{1} d_t^1\phi d_t^1 \partial^{3a}\\
&\quad + \binom{2}{0}\phi d_t^2 \partial^{3a}]F, \dots,\\
(-1)^{k-1}\binom{k-1}{0} d_t^{k-1}\partial^{ka}(\phi F) &= (-1)^{k-1}\binom{k-1}{0}[\binom{k-1}{k-1}(d_t^{k-1}\phi)\partial^{ka}\\
&\quad + \binom{k-1}{k-2}(d_t^{k-2}\phi)d_t^1\partial^{ka} + \dots + \binom{k-1}{0}\phi d_t^{k-1}\partial^{ka}]F,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\overline{\overline{E}}_1^{\,a}(\phi F) &= \phi[\binom{0}{0}\partial^{1a} - \binom{1}{0} d_t^1 \partial^{2a} + \binom{2}{0} d_t^2 \partial^{3a} - \cdots\\
&\quad \cdots + (-1)^{k-1}\binom{k-1}{0} d_t^{k-1}\partial^{ka}]F +\\
&\quad (d_t^1\phi)[-\binom{1}{1}\partial^{2a} + \binom{2}{1} d_t^1\partial^{3a} + \cdots\\
&\quad \cdots + (-1)^{k-1}\binom{k-1}{1} d_t^{k-1}\partial^{ka}]F + \cdots +\\
&\quad (d_t^{k-1}\phi)(-1)^{k-1}\binom{k-1}{k-1}\partial^{ka}F.
\end{aligned}
$$

□

The comparison of the above equation with (3.7) gives (3.11).

Theorem 3.6. *In* $(GLH)^{(nk)}\overline{\overline{E}}_1^{a}, \dots, \overline{\overline{E}}_k^{a}$ *are vector fields.*

Proof. $\overline{\overline{E}}_1^{a}$ is a vector field because $\partial^{1a}, \partial^{2a}, \dots, \partial^{ka}$ in $(GLH)^{(nk)}$ have the similar transformation law as $\partial_{0a}, \dots, \partial_{ka}$ in $(GL)^{(nk)}$, where $(\partial_{Aa} p_{\alpha a'}) = 0$, $A < \alpha$. □

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