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ORTHOGONALITY OF OPERATORS ON $(\mathbb{R}^n, \| \ \|_\infty)$[1]

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Abstract. We study the orthogonality of two linear operators S, T on $(\mathbb{R}^n, \| \ \|_\infty)$ in the sense of Birkhoff-James [4]. We find a necessary and sufficient condition for S to be orthogonal to T in the sense of Birkhoff-James with certain conditions on S. We give a sufficient condition for the existence of two operators S, T on $(\mathbb{R}^n, \| \ \|_\infty)$ with $S \perp_B T$ such that there does not exist any $x \in \mathbb{R}^n$ with $\|x\|_\infty = 1$, $Sx \perp_B Tx$ and$\|Sx\|_\infty = \|S\|_\infty$. We find a sufficient condition on S so that if $S \perp_B T$ then there exists $x \in \mathbb{R}^n$ with $\|x\|_\infty = 1$ such that $Sx \perp_B Tx$ and $\|Sx\|_\infty = \|S\|_\infty$. We also obtain the relations between the orthogonality of vectors in $(\mathbb{R}^n, \| \ \|_\infty)$ and the orthogonality of operators on $(\mathbb{R}^n, \| \ \|_\infty)$, both in the sense of Birkhoff-James.

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1. Introduction

Let $(X, \| \ \|)$ be a normed linear space. For any two elements x, y in X, x is said to be orthogonal to y in the sense of Birkhoff-James[4], written as $x \perp_B y$ iff $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in K (= \mathbb{R} \text{ or } \mathbb{C})$. In [5, 6, 7] James studied many important properties related to the notion of orthogonality. Let $B(X, X)$ denote the Banach algebra of all bounded linear operators from $(X, \| \ \|)$ to $(X, \| \ \|)$. For any two elements S, T in $B(X, X)$, S is orthogonal to T in the sense of Birkhoff-James, written as $S \perp_B T$, iff $\|S\| \leq \|S + \lambda T\|$ for all $\lambda \in K$.

In a finite dimensional Hilbert space X, Bhatia and Šemrl[3] and Paul et. al. [12] independently proved that $S \perp_B T$ iff there exists $x \in X$ with $\|x\| = 1$ such that $\|Sx\| = \|S\|$ and $Sx \perp_B Tx$. Bhatia and Šemrl in their paper conjectured that if X is a finite dimensional normed linear space and $S \perp_B T$ then there exists $x \in X$ with $\|x\| = 1$ such that $\|Sx\| = \|S\|$ and $Sx \perp_B Tx$. Li and Schneider [9] and Paul and Das [11] gave examples of normed spaces $(X, \| \ \|)$ in which there exist operators $S, T : X \longrightarrow X$ such that $S \perp_B T$ but there exists no $x \in X$, $\|x\| = 1$ such that $\|Sx\| = \|S\|$ and $Sx \perp_B Tx$, which shows that the conjecture of Bhatia and Šemrl is not true. The notion of orthogonality has been studied by many mathematicians over the time, a few of them are Alonso and Soriano [1], Benítez et. al. [2], Kapoor and Prasad [8] and Partington [10].

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In this paper we study the orthogonality of operators on $(\mathbb{R}^n, \| \|_\infty)$ in the sense of Birkhoff-James. We find a necessary and sufficient condition for an operator S on $(\mathbb{R}^n, \| \|_\infty)$ to be orthogonal to T in the sense of Birkhoff-James with certain conditions on S. If $S \perp_B T$ then it is not necessarily true that there exists x, $\|x\|_\infty = 1$ with $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \perp_B Tx$. We find a sufficient condition on S and T so that $S \perp_B T$, but there exists no such x, $\|x\|_\infty = 1$ for which $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \perp_B Tx$. We also find a sufficient condition on S and T so that $S \perp_B T$ implies that there exists x, $\|x\|_\infty = 1$ for which $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \perp_B Tx$.

2. Main Results

We first find a necessary and sufficient condition for two linear operators S and T on $(\mathbb{R}^n, \|.\|_\infty)$ to be orthogonal in the sense of Birkhoff-James.

Theorem 2.1. *Suppose $S = (a_{ij})_{n \times n}$ and $T = (b_{ij})_{n \times n}$ are two linear operators on $(\mathbb{R}^n, \|.\|_\infty)$ and there exists $i_0 \in \{1, 2, \cdots, n\}$ such that $a_{i_0 j} \neq 0$ for all $j \in \{1, 2, \cdots, n\}$ and*

$$|a_{i_0 1}| + |a_{i_0 2}| + \cdots + |a_{i_0 n}| > |a_{i1}| + |a_{i2}| + \cdots + |a_{in}| \quad \textit{for all } i \in \{1, 2, \ldots n\} - \{i_0\}.$$

Then

$$\|S\|_\infty \leq \|S + \lambda T\|_\infty, \ \textit{for all } \lambda \in \mathbb{R}$$

iff $(sgn\ a_{i_0 1}) b_{i_0 1} + \cdots + (sgn\ a_{i_0 n}) b_{i_0 n} = 0$ where

$$\begin{aligned} sgn\ (a_{ij}) &= +1, \ \textit{if } a_{ij} > 0 \\ &= -1, \ \textit{if } a_{ij} < 0 \\ &= \ \ 0, \ \textit{if } a_{ij} = 0. \end{aligned}$$

Proof. Let $(sgn\ a_{i_0 1}) b_{i_0 1} + \cdots + (sgn\ a_{i_0 n}) b_{i_0 n} = 0$.

Now, $S + \lambda T = (a_{ij} + \lambda\ b_{ij})_{n \times n}$ and

$$\begin{aligned} &\|S + \lambda T\|_\infty \\ &\geq |a_{i_0 1} + \lambda b_{i_0 1}| + \cdots + |a_{i_0 n} + \lambda b_{i_0 n}| \\ &= |(sgn\ a_{i_0 1})(a_{i_0 1} + \lambda b_{i_0 1})| + \cdots + |(sgn\ a_{i_0 n})(a_{i_0 n} + \lambda b_{i_0 n})| \\ &= ||a_{i_0 1}| + \lambda (sgn\ a_{i_0 1}) b_{i_0 1}| + \cdots + ||a_{i_0 n}| + \lambda (sgn\ a_{i_0 n}) b_{i_0 n}| \\ &\geq |(|a_{i_0 1}| + \cdots + |a_{i_0 n}|) + \lambda \left((sgn\ a_{i_0 1}) b_{i_0 1} + \cdots + (sgn\ a_{i_0 n}) b_{i_0 n} \right)| \\ &= |a_{i_0 1}| + \cdots + |a_{i_0 n}| \\ &= \|S\|_\infty \quad for\ all \quad \lambda \in \mathbb{R} \end{aligned}$$

Hence $\|S\|_\infty \leq \|S + \lambda T\|_\infty$ for all $\lambda \in \mathbb{R}$.

Conversely, let $\|S\|_\infty \leq \|S + \lambda T\|_\infty$ for all $\lambda \in \mathbb{R}$ and without loss of generality we assume that $i_0 = 1$ i.e.,

$$|a_{11}| + \cdots + |a_{1n}| > |a_{i1}| + \cdots + |a_{in}| \qquad \text{for all } \ i \in \{2, 3, \cdots, n\}.$$

Thus S attains its norm at the point $((sgn\ a_{11}), \cdots, (sgn\ a_{1n}))$ and

$$\|S\|_\infty = |a_{11}| + \cdots + |a_{1n}|.$$

We choose λ such that

$$(1) \qquad |\lambda| < \frac{(|a_{11}| + \cdots + |a_{1n}|) - \max_{2\leq i\leq n}(|a_{i1}| + \cdots + |a_{in}|)}{2\max_{2\leq i\leq n}(|b_{i1}| + \cdots + |b_{in}|)}$$

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $\|x\|_\infty = 1$. Then

$$(S + \lambda\ T)(x_1, x_2, \cdots, x_n) = (\sum_{j=1}^{n}(a_{1j} + \lambda b_{1j})x_j, \cdots, \sum_{j=1}^{n}(a_{nj} + \lambda b_{nj})x_j)$$

Therefore, for any λ satisfying (1), any $i \in \{2, 3, \cdots, n\}$ and $x = (x_1, x_2, \cdots, x_n)$ with $\|x\|_\infty = 1$, we have,

$$\begin{aligned} &|\sum_{j=1}^{n}(a_{ij} + \lambda b_{ij})x_j)| \\ &\leq |a_{i1} + \lambda b_{i1}||x_1| + \cdots + |a_{in} + \lambda b_{in}||x_n| \\ &\leq |a_{i1} + \lambda b_{i1}| + \cdots + |a_{in} + \lambda b_{in}| \\ &\leq (|a_{i1}| + \cdots + |a_{in}|) + |\lambda|(|b_{i1}| + |b_{i2}| + \cdots + |b_{in}|) \\ &< |a_{11}| + \cdots + |a_{1n}| \\ &= \|S\|_\infty \end{aligned}$$

Therefore, when λ satisfies (1), there exists $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ with $\|(x_1, x_2, \cdots, x_n)\|_\infty = 1$ such that

$$|(a_{11} + \lambda b_{11})x_1 + \cdots + (a_{1n} + \lambda b_{1n})x_n| \geq |a_{11}| + \cdots + |a_{1n}|.$$

We claim that, $(sgn\ a_{11})b_{11} + \cdots + (sgn\ a_{1n})b_{1n} = 0$. If not, then choose λ such that λ satisfies (1), λ is of opposite sign of

$$(sgn\ a_{11})b_{11} + (sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n} \text{ and } |\lambda| < \frac{1}{2}\min_{1\leq j\leq n, b_{1j}\neq 0}\frac{|a_{1j}|}{|b_{1j}|}$$

and $|\lambda\{(sgn\ a_{11})b_{11} + \cdots + (sgn\ a_{1n})b_{1n}\}| < \frac{1}{2}(|a_{11}| + \cdots + |a_{1n}|)$. Then from

$$|(a_{11} + \lambda b_{11})x_1 + \cdots + (a_{1n} + \lambda b_{1n})x_n| \geq |a_{11}| + \cdots + |a_{1n}|,$$

we have

$$|(a_{11} + \lambda b_{11})(sgn\ a_{11}) + \cdots + (a_{1n} + \lambda b_{1n})(sgn\ a_{1n})| \geq |a_{11}| + \cdots + |a_{1n}|,$$

since a_{1j} and $a_{1j} + \lambda b_{1j}$ has the same sign for all $j \in \{1, 2 \cdots, n\}$ due to this particular choice of λ.

$$\begin{aligned} \Rightarrow &|a_{11}(sgn\ a_{11}) + \cdots + a_{1n}(sgn\ a_{1n}) + \lambda\{b_{11}(sgn\ a_{11}) + \cdots + b_{1n}(sgn\ a_{1n})\}| \\ &\geq |a_{11}| + \cdots + |a_{1n}|, \end{aligned}$$

which is clearly a contradiction as λ is of opposite sign of $b_{11}(sgn\ a_{11}) + \cdots + b_{1n}(sgn\ a_{1n})$ and

$$|\lambda\{b_{11}(sgn\ a_{11}) + \cdots + b_{1n}(sgn\ a_{1n})\}| < \frac{1}{2}(|a_{11}| + \cdots + |a_{1n}|).$$

Therefore we must have

$$(sgn\ a_{11})b_{11} + \cdots + (sgn\ a_{1n})b_{1n} = 0.$$

This completes the proof of the theorem. □

Example 2.2. Let

$$S = \begin{pmatrix} 1 & -2 & -5 \\ 2 & 3 & 2 \\ 4 & 0 & 1 \end{pmatrix} \text{ and } T = (b_{ij})_{3\times 3}.$$

Then for S to be orthogonal to T in the sense of Birkhoff-James we must have $b_{11} - b_{12} - b_{13} = 0$. In particular, we may choose

$$T = \begin{pmatrix} -4 & -2 & -2 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

But if we choose

$$T = \begin{pmatrix} 5 & -2 & -2 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

then S is not orthogonal to T in the sense of Birkhoff-James.

Our main objective is to study if $S \perp_B T$ then whether there exists x, with $\|x\|_\infty = 1$, for which $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \perp_B Tx$. In general, this is not true. We give below an example in which $S \perp_B T$ but there exists no x, with $\|x\|_\infty = 1$, for which $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \perp_B Tx$.

Example 2.3. Let

$$S = \begin{pmatrix} 1 & -4 \\ 2 & 3 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then $\|S\|_\infty = 5$ and S attains its norm at the points $\pm(1,1)$ and $\pm(1,-1)$. It is easy to check that $\|S\|_\infty \leq \|S + \lambda T\|_\infty$ for all $\lambda \in \mathbb{R}$, but there exists no $x = (x_1, x_2) \in \mathbb{R}^2$ with $\|x\|_\infty = 1$ such that $\|S\|_\infty = \|Sx\|_\infty \leq \|(S + \lambda T)x\|_\infty$ for all $\lambda \in \mathbb{R}$.

We next give a sufficient condition for the existence of two operators

$$S, T : (\mathbb{R}^n, \|\ \|_\infty) \longrightarrow (\mathbb{R}^n, \|\ \|_\infty)$$

such that $S \perp_B T$ but there does not exist any $x \in \mathbb{R}^n$ with $\|x\|_\infty = 1$ such that $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \perp_B Tx$.

Theorem 2.4. *Let* $S = (a_{ij})_{n\times n} : (\mathbb{R}^n, \|\ \|_\infty) \longrightarrow (\mathbb{R}^n, \|\ \|_\infty)$ *with* $|a_{11}|+\cdots+|a_{1n}| = |a_{21}|+\cdots+|a_{2n}| > |a_{i1}|+\cdots+|a_{in}|$ *for all* $i \in \{3,4,\cdots,n\}$ *and with* $a_{11}, a_{12}, \cdots, a_{1,n-1} > 0, a_{1n} < 0$ *and* $a_{21}, a_{22}, \cdots, a_{2n} > 0$.

Let $T = (b_{ij})_{n\times n} : (\mathbb{R}^n, \|\ \|_\infty) \longrightarrow (\mathbb{R}^n, \|\ \|_\infty)$ *where* $b_{11} > 0$ *and* $b_{2n} < 0$ *and* $b_{ij} = 0$ *for all other i,j's. Then* $\|S\|_\infty \le \|S+\lambda T\|_\infty$ *for all* $\lambda \in \mathbb{R}$ *and there does not exist* $x \in \mathbb{R}^n$ *with* $\|x\|_\infty = 1$ *such that* $\|Sx\|_\infty = \|S\|_\infty$ *and* $Sx \perp_B Tx$.

Proof. Clearly

$$\|S\|_\infty = a_{11}+a_{12}+\cdots+a_{1,n-1}-a_{1n} = a_{21}+a_{22}+\cdots+a_{2,n-1}+a_{2n}$$

and S attains its norm only at $\pm(1,1,\cdots,-1), \pm(1,1,\cdots,1,1)$.

Now,

$$S+\lambda T = \begin{pmatrix} a_{11}+\lambda b_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n}+\lambda b_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix}$$

Therefore for any $\lambda \in \mathbb{R}$,

$$\begin{aligned} &\|S+\lambda T\|_\infty \\ &\ge \max\{|a_{11}+\lambda b_{11}|+|a_{12}|+\cdots+|a_{1n}|, |a_{21}|+|a_{22}|+\cdots+|a_{2n}+\lambda b_{2n}|\} \\ &= \max\{|a_{11}+\lambda b_{11}|+a_{12}+\cdots-a_{1n}, a_{21}+a_{22}+\cdots+|a_{2n}+\lambda b_{2n}|\} \\ &\ge a_{11}+a_{12}+\cdots+a_{1,n-1}-a_{1n} \quad (= a_{21}+a_{22}+\cdots+a_{2n}) \\ &= \|S\|_\infty \end{aligned}$$

Therefore $\|S\|_\infty \le \|S+\lambda T\|_\infty, \forall\ \lambda \in \mathbb{R}$

Now, $(S+\lambda T)(1,1,\cdots,1,-1) = (a_{11}+\lambda b_{11}+a_{12}+\cdots+a_{1,n-1}-a_{1n}, a_{21}+a_{22}+\cdots+a_{2,n-1}-a_{2n}-\lambda b_{2n}, \cdots, a_{n1}+a_{n2}+\cdots+a_{n,n-1}-a_{nn})$. Therefore $(\|S+\lambda T)(\pm(1,1,\cdots,1,-1))\|_\infty < a_{11}+a_{12}+\cdots+a_{1,n-1}-a_{1n} \quad (= a_{21}+a_{22}+\cdots+a_{2n} = \|S\|_\infty)$, when λ is -ve and sufficiently small. Also $(S+\lambda T)(1,1,\cdots,1,1) = (a_{11}+\lambda b_{11}+a_{12}+\cdots+a_{1n}, a_{21}+a_{22}+\cdots+a_{2n}+\lambda b_{2n}, \cdots, a_{n1}+a_{n2}+\cdots+a_{nn})$. Therefore, $(\|S+\lambda T)(\pm(1,1,\cdots,1,1))\|_\infty < a_{11}+a_{12}+\cdots+a_{1,n-1}-a_{1n} \quad (= a_{21}+a_{22}+\cdots+a_{2n} = \|S\|_\infty)$, when λ is +ve and sufficiently small. Hence $\|S\|_\infty \le \|S+\lambda T\|_\infty$ for all $\lambda \in \mathbb{R}$ and there does not exist $x \in \mathbb{R}^n$ with $\|x\|_\infty = 1$ such that $\|S\|_\infty = \|Sx\|_\infty$ and $Sx \perp_B Tx$. □

Note. This result negates the following conjecture of Bhatia and Šemrl [3] :

"Given any finite dimensional normed linear space $(X, \|\ \|)$ if

$$S, T : (X, \|\ \|) \to (X, \|\ \|)$$

are two linear operators with $S \perp_B T$ then there exists $x \in X$ with $\|x\| = 1$ such that $Sx \perp_B Tx$ and $\|S\| = \|Sx\|$."

In the next example we show that even if

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\mathbb{R}^2, \|\ \|_\infty) \to (\mathbb{R}^2, \|\ \|_\infty)$$

is such that $|a| + |b| = |c| + |d|$ then it may happen so that for any

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (\mathbb{R}^2, \|\ \|_\infty) \to (\mathbb{R}^2, \|\ \|_\infty)$$

which is orthogonal to S in the sense of Birkhoff-James, i.e. $\|S\|_\infty \leq \|S+\lambda T\|_\infty$ for all $\lambda \in \mathbb{R}$, there exists $x \in \mathbb{R}^2$ with $\|x\|_\infty = 1$ such that $\|S\|_\infty = \|Sx\|_\infty$ and $Sx \perp_B Tx$.

Example 2.5. Let $S = \begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix}$. Let

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (\mathbb{R}^2, \|\ \|_\infty) \to (\mathbb{R}^2, \|\ \|_\infty)$$

be such that $\|S\|_\infty \leq \|S + \lambda T\|_\infty$ for all $\lambda \in \mathbb{R}$. Clearly, $\|S\|_\infty = 5$ and S attains its norm at the points $\pm(1, -1)$.

We have

$$\begin{aligned} (S + \lambda T)(x_1, x_2) &= ((1 + \lambda\alpha)x_1 + (-4 + \lambda\beta)x_2, (-2 + \lambda\gamma)x_1 + (3 + \lambda\delta)x_2) \\ &= (x_1 - 4x_2 + \lambda(\alpha x_1 + \beta x_2), -2x_1 + 3x_2 + \lambda(\gamma x_1 + \delta x_2)). \end{aligned}$$

Therefore, $(S + \lambda T)(1, -1) = (5 + \lambda(\alpha - \beta), -5 + \lambda(\gamma - \delta))$. Clearly, if $\alpha - \beta$ and $\gamma - \delta$ are not of the opposite sign then $5 = \|S\|_\infty = \|S(1, -1)\|_\infty \leq \|(S + \lambda T)(1, -1)\|_\infty$ for all $\lambda \in \mathbb{R}$. If possible, suppose that $\alpha - \beta > 0$ and $\gamma - \delta < 0$.

Now, $\|S + \lambda T\|_\infty = \max\{|1 + \lambda\alpha| + | - 4 + \lambda\beta|, | - 2 + \lambda\gamma| + |3 + \lambda\delta|\}$. Then, choosing λ to be sufficiently small and -ve, $\|S + \lambda T\| < 5 = \|S\|_\infty$, a contradiction. Therefore it is not possible to have $\alpha - \beta > 0$ and $\gamma - \delta < 0$. Similarly, it is not possible to have $\alpha - \beta < 0$ and $\gamma - \delta > 0$. This proves our claim.

We next find a sufficient condition on S so that if $S \perp_B T$, then there exists $x \in \mathbb{R}^n$ with $\|x\|_\infty = 1$ such that $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \perp_B Tx$.

Theorem 2.6. *Let $S = (a_{ij})_{n\times n} : (\mathbb{R}^n, \|\ \|_\infty) \longrightarrow (\mathbb{R}^n, \|\ \|_\infty)$ be such that there exists $i_0 \in \{1, 2, \cdots, n\}$ such that*

$$|a_{i_0 1}| + \cdots + |a_{i_0 n}| > |a_{i1}| + \cdots + |a_{in}| \ \forall\ i_0 \in \{1, 2, \ldots, n\} - \{i_0\}.$$

Let $T = (b_{ij})_{n\times n} : (\mathbb{R}^n, \|\ \|_\infty) \longrightarrow (\mathbb{R}^n, \|\ \|_\infty)$ is such that $S \perp_B T$. Then there exists $x \in \mathbb{R}^n$ with $\|x\|_\infty = 1$ such that $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \perp_B Tx$.

Proof. Without loss of generality, we assume $i_0 = 1$.
If $a_{1j} \neq 0$ for all $j \in \{1, 2, ..., n\}$ then, from Theorem 2.1 we have,

$$(sgn\ a_{11})b_{11} + \cdots + (sgn\ a_{1n})b_{1n} = 0 \tag{2}$$

Then, S attains its norm at the point $x = (sgn\ a_{11}, \cdots, sgn\ a_{1n})$ i.e. $\|S\|_\infty = \|Sx\|_\infty = |a_{11}| + \cdots + |a_{1n}|$.

Now, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned} \|(S+\lambda\ T)x\|_\infty &\geq |(a_{11}+\lambda b_{11})(sgn\ a_{11}) + \cdots + (a_{1n}+\lambda b_{1n})(sgn\ a_{1n})| \\ &= |a_{11}| + \cdots + |a_{1n}|, \qquad \text{by} \quad (2) \\ &= \|Sx\|_\infty \end{aligned}$$

So $Sx \perp_B Tx$.

Now we consider the case $a_{11} = 0$ and $a_{1j} \neq 0 \ \forall \ j \in \{2, 3, \cdots, n\}$. In that case, $\|S\|_\infty = |a_{12}| + |a_{13}| + \cdots + |a_{1n}|$ and S attains its norm at $x = (x_1, sgn\ a_{12}, \cdots, sgn\ a_{1n})$ where $-1 \leq x_1 \leq 1$. If $(sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n} = 0$, then we are done as before. Let $(sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n} \neq 0$.

Choose, $x_1 = -\frac{b_{12}(sgn\ a_{12}) + \cdots + b_{1n}(sgn\ a_{1n})}{b_{11}}$. Now, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned} &\|(S+\lambda T)x\|_\infty \\ &\geq |\lambda b_{11}x_1 + (a_{12}+\lambda b_{12})(sgn\ a_{12}) + \cdots + (a_{1n}+\lambda b_{1n})(sgn\ a_{1n})| \\ &= |a_{12}(sgn\ a_{12}) + \cdots + a_{1n}(sgn\ a_{1n}) + \\ &\qquad \lambda\big(b_{11}x_1 + b_{12}(sgn\ a_{12}) + \cdots + b_{1n}(sgn\ a_{1n})\big)| \\ &= |a_{12}| + \cdots + |a_{1n}| \\ &= \|Sx\|_\infty \\ &= \|S\|_\infty. \end{aligned}$$

Therefore, if we can show that $-1 \leq x_1 \leq 1$ then we have

$$x = (x_1, (sgn\ a_{12}), \cdots, (sgn\ a_{1n})) \in \mathbb{R}^n$$

such that $\|x\|_\infty = 1$ and $\|S\|_\infty = \|Sx\|_\infty \leq \|(S+\lambda T)x\|_\infty \ \ \forall \ \lambda \in \mathbb{R}$. Therefore, all we have to show is that

$$|b_{11}| \geq |(sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n}|$$

If possible, suppose that

$$|b_{11}| < |(sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n}|$$

Now, two cases may arise:

Case I: $(sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n} > 0$.

Choose λ such that λ satisfies (1), λ is -ve and λ is sufficiently small so that $a_{1j} + \lambda b_{1j}$ and a_{1j} have the same sign for all $j \in \{2, 3, \cdots, n\}$.

Then

$$\begin{aligned} &\|S+\lambda T\|_\infty \\ &= |\lambda b_{11}| + |a_{12}+\lambda b_{12}| + \cdots + |a_{1n}+\lambda b_{1n}| \\ &= -\lambda|b_{11}| + (a_{12}+\lambda b_{12})(sgn\ a_{12}) + \cdots + (a_{1n}+\lambda b_{1n})(sgn\ a_{1n}) \\ &= (|a_{12}| + \cdots + |a_{1n}|) + \lambda(-|b_{11}| + (sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n}) \\ &< |a_{12}| + \cdots + |a_{1n}|, \qquad \text{due to this particular choosing of } \lambda \\ &= \|S\|_\infty, \qquad \text{a contradiction.} \end{aligned}$$

Case II: $(sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n} < 0$.

Choose λ such that λ satisfies (1), λ is +ve, λ is sufficiently small so that $a_{1j} + \lambda b_{1j}$ and a_{1j} have the same sign $\forall\ j \in \{2, 3, \cdots n\}$.

Then

$$\begin{aligned}
&\|S + \lambda T\|_\infty \\
&= |\lambda b_{11}| + |a_{12} + \lambda b_{12}| + \cdots + |a_{1n} + \lambda b_{1n}| \\
&= \lambda|b_{11}| + (a_{12} + \lambda b_{12})(sgn\ a_{12}) + \cdots + (a_{1n} + \lambda b_{1n})(sgn\ a_{1n}) \\
&= (|a_{12}| + \cdots + |a_{1n}|) + \lambda(|b_{11}| + (sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n}) \\
&< |a_{12}| + \cdots + |a_{1n}|, \qquad \text{due to this particular choosing of } \lambda \\
&= \|S\|_\infty, \qquad \text{a contradiction.}
\end{aligned}$$

Therefore, in any case, we must have $|b_{11}| \geq |(sgn\ a_{12})b_{12} + \cdots + (sgn\ a_{1n})b_{1n}|$ and so $x = (-\frac{(sgn\ a_{12})b_{12}+\cdots+(sgn\ a_{1n})b_{1n}}{b_{11}}, sgn\ a_{12}, \cdots, sgn\ a_{1n}) \in \mathbb{R}^n$ such that $\|x\|_\infty = 1$ and $\|Sx\|_\infty = \|S\|_\infty$, $Sx \bot_B Tx$. Similarly, if $a_{1j} = 0$ for $j \in \{1, 2, \cdots, n\}$ then we can find $x \in \mathbb{R}^n$ with $\|x\|_\infty = 1$ such that $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \bot_B Tx$.

This completes the proof of the theorem. □

Example 2.7. Let $S, T : (\mathbb{R}^3 \| \ \|_\infty) \to (\mathbb{R}^3 \| \ \|_\infty)$ be given by

$$S = \begin{pmatrix} 1 & 4 & -2 \\ 3 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix} \text{ and } T = \begin{pmatrix} 3 & 2 & 5 \\ 1 & 1 & 0 \\ 0 & 8 & 3 \end{pmatrix}.$$

Then, $S \bot_B T$, and there exists $x = (1, 1, -1) \in \mathbb{R}^3$ with $\|x\|_\infty = 1$ such that $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \bot_B Tx$.

Example 2.8. Let $S, T : (\mathbb{R}^3 \| \ \|_\infty) \to (\mathbb{R}^3 \| \ \|_\infty)$ be given by

$$S = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -2 & 0 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then $S \bot_B T$ and there exists $x = (0, 1, 1) \in \mathbb{R}^3$ with $\|x\|_\infty = 1$ such that $\|Sx\|_\infty = \|S\|_\infty$ and $Sx \bot_B Tx$.

Remark 2.9. We have obtained the relations between the orthogonality of vectors in $(\mathbb{R}^n, \| \ \|_\infty)$ and the orthogonality of operators on $(\mathbb{R}^n, \| \ \|_\infty)$, both in the sense of Birkhoff-James. In the same way we can explore the relations between the orthogonality of vectors in $(\mathbb{R}^n, \| \ \|_1)$ and the orthogonality of operators on $(\mathbb{R}^n, \| \ \|_1)$, both in the sense of Birkhoff-James.

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