

FIXED POINT FOR COMPATIBLE AND SUBSEQUENTIALLY CONTINUOUS MAPPINGS IN MENGER SPACES AND APPLICATIONS

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Abstract. We present a common fixed point theorem for two pairs of self-mappings by using the notions of compatibility and subsequential continuity (alternatively subcompatibility and reciprocal continuity) in Menger space and give some examples. As an application to our main result, we also obtain the corresponding common fixed point theorem in metric spaces. Our results improve several well-known results in the literature.

AMS Mathematics Subject Classification (2010): 54H25, 47H10

Key words and phrases: Menger space, compatible mappings, reciprocal continuity, subcompatible mappings, subsequential continuity, fixed point

1. Introduction

In 1991, Mishra [32] extended the notion of compatibility (introduced by Jungck [24] in metric spaces) to PM-space. Cho et al. [16] studied the notion of compatible mappings of type (A) (introduced by Jungck et al. [25] in metric spaces) in Menger spaces which is equivalent to the concept of compatible mappings under some conditions. Further, Pathak et al. [38] improved and generalized the results of Cho et al. [16] using the notion of weak compatibility of type (A) in Menger spaces. Most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. Pant [33] noticed these criteria for fixed points of contraction mappings and introduced a new continuity condition, known as reciprocal continuity and obtained a common fixed point theorem by using the compatibility in metric spaces. He also showed that in the setting of common fixed point theorems for compatible mappings satisfying contraction conditions, the notion of reciprocal continuity is weaker than the continuity of one of the mappings. Further, Jungck and Rhoades [26] termed a pair of self-mappings to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. In 2008, Al-Thagafi and Shahzad [4] gave a definition which is a proper generalization of nontrivial weakly compatible mappings which have coincidence points. Jungck

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and Rhoades [27] studied fixed point results for occasionally weakly compatible mappings.

Doric et al. [17] have shown that the condition of occasionally weak compatibility reduces to weak compatibility in the presence of a unique point of coincidence (or a unique common fixed point) of the given pair of mappings. Thus, no generalization can be obtained by replacing weak compatibility with occasionally weak compatibility. Most recently, Bouhadjera and Godet-Thobie [11] introduced two new notions, namely subsequential continuity and subcompatibility which are weaker than reciprocal continuity and compatibility respectively (see also [12]). Further, Imdad et al. [21] improved the results of Bouhadjera and Godet-Thobie [11] and showed that these results can be easily recovered by replacing subcompatibility with compatibility or subsequential continuity with reciprocally continuity. More recently, Gopal and Imdad [19] utilized these concepts and proved some results in (GV)-fuzzy metric spaces. Many authors have contributed to the development of fixed point theory in Menger spaces, for instance [2, 3, 5–10, 13–15, 18, 20, 22, 23, 29, 31, 34, 36, 37].

The purpose of this paper is to prove a common fixed point theorem for two pairs of self-mappings by using the notions of compatibility and subsequential continuity (alternatively subcompatibility and reciprocal continuity) in Menger spaces. We derive some examples in support of our main result. We also obtain the corresponding common fixed point theorems in metric spaces. Consequently, our results improve many known common fixed point theorems available in the existing literature.

2. Preliminaries

Definition 2.1. [41] A mapping $F : \mathbb{R} \rightarrow \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t \in \mathbb{R}} F(t) = 0$ and $\sup_{t \in \mathbb{R}} F(t) = 1$.

We shall denote by \mathfrak{S} the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.2. [41] A PM-space is an ordered pair (X, \mathcal{F}) , where X is a non-empty set of elements and \mathcal{F} is a mapping from $X \times X$ to \mathfrak{S} , the collection of all distribution functions. The value of \mathcal{F} at $(x, y) \in X \times X$ is represented by $F_{x,y}$. The functions $F_{x,y}$ are assumed to satisfy the following conditions:

- (i) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;
- (ii) $F_{x,y}(0) = 0$;
- (iii) $F_{x,y}(t) = F_{y,x}(t)$;
- (iv) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t+s) = 1$ for all $x, y, z \in X$ and $t, s > 0$.

Definition 2.3. [41] A mapping $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a triangular norm (briefly, t-norm) if the following conditions are satisfied: for all $a, b, c, d \in [0, 1]$

- (i) $\Delta(a, 1) = a$ for all $a \in [0, 1]$;
- (ii) $\Delta(a, b) = \Delta(b, a)$;
- (iii) $\Delta(a, b) \leq \Delta(c, d)$ for $a \leq c, b \leq d$;
- (iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$;

Examples of t-norms are $\Delta(a, b) = \min\{a, b\}$, $\Delta(a, b) = ab$ and $\Delta(a, b) = \max\{a + b - 1, 0\}$.

Definition 2.4. [30] A Menger space is a triplet (X, \mathcal{F}, Δ) , where (X, \mathcal{F}) is a PM-space and t-norm Δ is such that the inequality

$$F_{x,z}(t + s) \geq \Delta(F_{x,y}(t), F_{y,z}(s)),$$

holds for all $x, y, z \in X$ and all $t, s > 0$.

Every metric space (X, d) can be realized as a Menger space by taking $\mathcal{F} : X \times X \rightarrow \mathfrak{S}$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$.

Definition 2.5. [32] A pair (A, S) of self-mappings defined on a Menger space (X, \mathcal{F}, Δ) is said to be compatible if and only if $F_{ASx_n, SAx_n}(t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \rightarrow z$ for some $z \in X$ as $n \rightarrow \infty$.

Definition 2.6. [16] A pair (A, S) of self-mappings defined on a Menger space (X, \mathcal{F}, Δ) is said to be compatible of type (A) if $F_{SAx_n, AAx_n}(t) \rightarrow 1$ and $F_{ASx_n, SSx_n}(t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \rightarrow z$ for some $z \in X$ as $n \rightarrow \infty$.

Remark 2.7. [16] If the self-mappings A and S are both continuous then A and S are compatible if and only if they are compatible of type (A) .

It is noted that Remark 2.7 is not true if the self-mappings A and S are not continuous on X . For examples, we refer to Jungck and Rhoades [26].

Definition 2.8. [38] A pair (A, S) of self-mappings defined on a Menger space (X, \mathcal{F}, Δ) is said to be weak compatible of type (A) if

$$\lim_{n \rightarrow \infty} F_{ASx_n, SSx_n}(t) \geq \lim_{n \rightarrow \infty} F_{SAx_n, SSx_n}(t)$$

and

$$\lim_{n \rightarrow \infty} F_{SAx_n, AAx_n}(t) \geq \lim_{n \rightarrow \infty} F_{ASx_n, AAx_n}(t),$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \rightarrow z$ for some $z \in X$ as $n \rightarrow \infty$.

Remark 2.9. [38] If the self-mappings A and S are both continuous. Then

- (i) A and S are compatible of type (A) if and only if they are weak compatible of type (A) .
- (ii) A and S are compatible if and only if they are weak compatible of type (A) .

It is noted that Remark 2.9 is not true if the self-mappings A and S are not continuous on X . For examples, we refer to Pathak et al. [38].

Inspired by Aamri and Moutawakil [1], Kubiacyk and Sharma [28] defined the notion of property (E.A) in Menger spaces as follows:

Definition 2.10. A pair (A, S) of self-mappings defined on a Menger space (X, \mathcal{F}, Δ) is said to satisfy the property (E.A), if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z,$$

for some $z \in X$.

Note that weak compatibility and property (E.A) are independent of each other (see [39, Example 2.2]).

Remark 2.11. From Definition 2.5, it is inferred that two self-mappings A and S of a Menger space (X, \mathcal{F}, Δ) are non-compatible if and only if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$, but for some $t > 0$, $\lim_{n \rightarrow \infty} F_{ASx_n, SAx_n}(t)$ is either less than 1 or nonexistent.

Therefore, from Definition 2.10, it is easy to see that any non-compatible self-mappings of a Menger space (X, \mathcal{F}, Δ) satisfy the property (E.A), but two mappings satisfying the property (E.A) need not be non-compatible (see [18, Example 1]).

Definition 2.12. [27] Two self-mappings A and S of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az = Sz$ some $z \in X$, then $ASz = SAz$.

Remark 2.13. Two compatible self-mappings are weakly compatible, but the converse is not true (see [42, Example 1]). Therefore, the concept of weak compatibility is more general than that of compatibility.

Definition 2.14. [27] Two self-mappings A and S of a non-empty set X are occasionally weakly compatible if and only if there is a point $x \in X$ which is a coincidence point of A and S at which A and S commute.

The following definition is on the lines of Bouhadjera and Godet-Thobie [11].

Definition 2.15. A pair (A, S) of self-mappings defined on a Menger space (X, \mathcal{F}, Δ) is said to be subcompatible if and only if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z,$$

for some $z \in X$ and $\lim_{n \rightarrow \infty} F_{ASx_n, SAx_n}(t) = 1$, for all $t > 0$.

Remark 2.16. A pair of non-compatible or subcompatible mapping satisfies the property (E.A). Obviously, compatible mappings which satisfy the property (E.A) are subcompatible, but the converse statement does not hold in general (see [40, Example 2.3]).

Remark 2.17. Two occasionally weakly compatible mappings are subcompatible, however the converse is not true in general (see [12, Example 1.2]).

Definition 2.18. [29] A pair (A, S) of self-mappings defined on a Menger space (X, \mathcal{F}, Δ) is called reciprocally continuous if for a sequence $\{x_n\}$ in X , $\lim_{n \rightarrow \infty} ASx_n = Az$ and $\lim_{n \rightarrow \infty} SAx_n = Sz$, whenever

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z,$$

for some $z \in X$.

It is easy to see that if two self-mappings are continuous, then they are obviously reciprocally continuous but converse is not true. Moreover, in the setting of common fixed point theorems for compatible pair of self-mappings satisfying contractive conditions, the continuity of one of the mappings implies their reciprocal continuity, but not conversely (see [35]).

The notion of subsequentially continuous mappings (introduced by Bouhadjera and Godet-Thobie [11] in metric space) in Menger spaces is as follows:

Definition 2.19. A pair of self-mappings (A, S) defined on a Menger space (X, \mathcal{F}, Δ) is called subsequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that,

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z,$$

for some $z \in X$ and $\lim_{n \rightarrow \infty} ASx_n = Az$ and $\lim_{n \rightarrow \infty} SAx_n = Sz$.

one can easily check that if two self-mappings are continuous or reciprocally continuous, then they are naturally subsequentially continuous. However, there exist subsequentially continuous pair of mappings which are neither continuous nor reciprocally continuous (see [12, Example 1.4]).

Lemma 2.20. [32] Let (X, \mathcal{F}, Δ) be a Menger space, where Δ is a continuous t -norm. If there exists a constant $k \in (0, 1)$ such that

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $x, y \in X$ and $t > 0$ then $x = y$.

3. Results

Theorem 3.1. Let A, B, S and T be self-mappings of a Menger space (X, \mathcal{F}, Δ) , where Δ is a continuous t -norm. If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternatively subcompatible and reciprocally continuous), then

- (i) the pair (A, S) has a coincidence point,
- (ii) the pair (B, T) has a coincidence point,
- (iii) there exists a constant $k \in (0, 1)$ such that

$$(3.1) \quad (F_{Ax,By}(kt))^2 \geq \min \left\{ \begin{array}{l} (F_{Sx,Ty}(t))^2, F_{Sx,Ax}(t)F_{Ty,By}(t), \\ F_{Sx,By}(2t)F_{Ty,Ax}(t), F_{Ty,Ax}(t), F_{Sx,By}(2t)F_{Ty,By}(t) \end{array} \right\},$$

for all $x, y \in X$ and $t > 0$. Then A, B, S and T have a unique common fixed point in X .

Proof. **Case I:** Since the pair (A, S) (also (B, T)) is subsequentially continuous and compatible mappings, therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z,$$

for some $z \in X$, and

$$\lim_{n \rightarrow \infty} F_{ASx_n,SAx_n}(t) = F_{Az,Sz}(t) = 1,$$

for all $t > 0$ then $Az = Sz$, whereas in respect of the pair (B, T) , there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = w,$$

for some $w \in X$, and

$$\lim_{n \rightarrow \infty} F_{BTy_n,TDy_n}(t) = F_{Bw,Tw}(t) = 1,$$

for all $t > 0$ then $Bw = Tw$. Hence z is a coincidence point of the pair (A, S) , whereas w is a coincidence point of the pair (B, T) .

Now we prove that $z = w$. By putting $x = x_n$ and $y = y_n$ in inequality (3.1), we have

$$(F_{Ax_n,By_n}(kt))^2 \geq \min \left\{ \begin{array}{l} (F_{Sx_n,Ty_n}(t))^2, F_{Sx_n,Ax_n}(t)F_{Ty_n,By_n}(t), \\ F_{Sx_n,By_n}(2t)F_{Ty_n,Ax_n}(t), \\ F_{Ty_n,Ax_n}(t), F_{Sx_n,By_n}(2t)F_{Ty_n,By_n}(t) \end{array} \right\}.$$

Taking limit $n \rightarrow \infty$, we obtain

$$(F_{z,w}(kt))^2 \geq \min \left\{ \begin{array}{l} (F_{z,w}(t))^2, F_{z,z}(t)F_{w,w}(t), F_{z,w}(2t)F_{w,z}(t), \\ F_{w,z}(t), F_{z,w}(2t)F_{w,w}(t) \end{array} \right\} \\ = (F_{z,w}(t))^2.$$

On employing Lemma 2.20, we have $z = w$. Now we assert that $Az = z$. Putting $x = z$ and $y = y_n$ in inequality (3.1), we get

$$(F_{Az,By_n}(kt))^2 \geq \min \left\{ \begin{array}{l} (F_{Sz,Ty_n}(t))^2, F_{Sz,Az}(t)F_{Ty_n,By_n}(t), \\ F_{Sz,By_n}(2t)F_{Ty_n,Az}(t), \\ F_{Ty_n,Az}(t), F_{Sz,By_n}(2t)F_{Ty_n,By_n}(t) \end{array} \right\}.$$

Taking limit $n \rightarrow \infty$, we obtain

$$(F_{Az,w}(kt))^2 \geq \min \left\{ \begin{array}{l} (F_{Az,w}(t))^2, F_{Az,Az}(t)F_{w,w}(t), F_{Az,w}(2t)F_{w,Az}(t), \\ F_{w,Az}(t), F_{Az,w}(2t)F_{w,w}(t) \end{array} \right\},$$

and so

$$\begin{aligned} (F_{Az,z}(kt))^2 &\geq \min \left\{ \begin{array}{l} (F_{Az,z}(t))^2, F_{Az,Az}(t)F_{z,z}(t), F_{Az,z}(2t)F_{z,Az}(t), \\ F_{z,Az}(t), F_{Az,z}(2t)F_{z,z}(t) \end{array} \right\} \\ &= (F_{Az,z}(t))^2. \end{aligned}$$

From Lemma 2.20, we have $Az = z$. Therefore, $Az = Sz = z$. Now we show that $Bz = z$. Putting $x = x_n$ and $y = z$ in inequality (3.1), we have

$$(F_{Ax_n,Bz}(kt))^2 \geq \min \left\{ \begin{array}{l} (F_{Sx_n,Tz}(t))^2, F_{Sx_n,Ax_n}(t)F_{Tz,Bz}(t), \\ F_{Sx_n,Bz}(2t)F_{Tz,Ax_n}(t), \\ F_{Tz,Ax_n}(t), F_{Sx_n,Bz}(2t)F_{Tz,Bz}(t) \end{array} \right\}.$$

Taking limit $n \rightarrow \infty$, we obtain

$$\begin{aligned} (F_{z,Bz}(kt))^2 &\geq \min \left\{ \begin{array}{l} (F_{z,Bz}(t))^2, F_{z,z}(t)F_{Bz,Bz}(t), \\ F_{z,Bz}(2t)F_{Bz,z}(t), \\ F_{Bz,z}(t), F_{z,Bz}(2t)F_{Bz,Bz}(t) \end{array} \right\} \\ &= (F_{z,Bz}(t))^2. \end{aligned}$$

Appealing to Lemma 2.20, we have $Bz = z$. Thus $Bz = Sz = z$. Therefore, in all, $z = Az = Sz = Bz = Tz$, that is, z is the common fixed point of A, B, S and T . The uniqueness of common fixed point is an easy consequence of inequality (3.1). This completes the proof of the theorem.

Case II: Since the pair (A, S) (also (B, T)) is subcompatible and reciprocally continuous, therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z,$$

for some $z \in X$,

and

$$\lim_{n \rightarrow \infty} F_{ASx_n, SAx_n}(t) = \lim_{n \rightarrow \infty} F_{Az, Sz}(t) = 1,$$

for all $t > 0$, whereas in respect of the pair (B, T) , there exists a sequence $\{y_n\}$ in X with

$$\lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = w,$$

for some $w \in X$,

and

$$\lim_{n \rightarrow \infty} F_{BTy_n, TBy_n}(t) = \lim_{n \rightarrow \infty} F_{Bw, Tw}(t) = 1,$$

for all $t > 0$. Therefore, $Az = Sz$ and $Bw = Tw$ i.e. z is a coincidence point of the pair (A, S) , whereas w is a coincidence point of the pair (B, T) . The rest of the proof can be completed on the lines of Case I. \square

By setting $A = B$ in Theorem 3.1, we can derive a corollary for three mappings, which runs as follows.

Corollary 3.2. *Let A, S and T be self-mappings of a Menger space (X, \mathcal{F}, Δ) , where Δ is a continuous t -norm. If the pairs (A, S) and (A, T) are compatible and subsequentially continuous (alternatively subcompatible and reciprocally continuous), then*

- (i) *the pair (A, S) has a coincidence point,*
- (ii) *the pair (A, T) has a coincidence point,*
- (iii) *there exists a constant $k \in (0, 1)$ such that*

$$(3.2) \quad (F_{Ax,Ay}(kt))^2 \geq \min \left\{ \begin{matrix} (F_{Sx,Ty}(t))^2, F_{Sx,Ax}(t)F_{Ty,Ay}(t), F_{Sx,Ay}(2t)F_{Ty,Ax}(t), \\ F_{Ty,Ax}(t), F_{Sx,Ay}(2t)F_{Ty,Ay}(t) \end{matrix} \right\},$$

for all $x, y \in X$ and $t > 0$, then A, S and T have a unique common fixed point in X .

Alternatively, by setting $S = T$ in Theorem 3.1, we can also derive another corollary for three mappings, which runs as follows:

Corollary 3.3. *Let A, B and S be self-mappings of a Menger space (X, \mathcal{F}, Δ) , where Δ is a continuous t -norm. If the pairs (A, S) and (B, S) are compatible and subsequentially continuous (alternatively subcompatible and reciprocally continuous), then*

- (i) *the pair (A, S) has a coincidence point,*
- (ii) *the pair (B, S) has a coincidence point,*
- (iii) *there exists a constant $k \in (0, 1)$ such that*

$$(3.3) \quad F_{Ax,By}(kt))^2 \geq \min \left\{ \begin{matrix} (F_{Sx,Sy}(t))^2, F_{Sx,Ax}(t)F_{Sy,By}(t), F_{Sx,By}(2t)F_{Sy,Ax}(t), \\ F_{Sy,Ax}(t), F_{Sx,By}(2t)F_{Sy,By}(t) \end{matrix} \right\},$$

for all $x, y \in X$ and $t > 0$, then A, B and S have a unique common fixed point in X .

On taking $A = B$ and $S = T$ in Theorem 3.1, we get the following natural result.

Corollary 3.4. *Let A and S be self-mappings of a Menger space (X, \mathcal{F}, Δ) , where Δ is a continuous t -norm. If the pair (A, S) is compatible and subsequentially continuous (alternatively subcompatible and reciprocally continuous), then*

- (i) the pair (A, S) has a coincidence point,
- (ii) there exists a constant $k \in (0, 1)$ such that

$$(3.4) \quad F_{Ax,Ay}(kt)^2 \geq \min \left\{ \begin{array}{l} (F_{Sx,Sy}(t))^2, F_{Sx,Ax}(t)F_{Sy,Ay}(t), F_{Sx,Ay}(2t)F_{Sy,Ax}(t), \\ F_{Sy,Ax}(t), F_{Sx,Ay}(2t)F_{Sy,Ay}(t) \end{array} \right\},$$

for all $x, y \in X$ and $t > 0$, then A and S have a unique common fixed point in X .

Remark 3.5. Theorem 3.1 improves the results of Cho et al. [16, Theorem 4.2] and Pathak et al. [38, Theorem 4.2] in the sense that the conditions on completeness (or closedness) of the underlying space (or subspaces), containment of ranges amongst involved mappings together with conditions on continuity in respect of any one of the involved mappings are relaxed.

Example 3.6. Let $X = [0, \infty)$ and d be the usual metric on X and for each $t \in [0, 1]$, define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Clearly, (X, \mathcal{F}, Δ) be a Menger space. Set $A = B$ and $S = T$. Define the self-mappings A and S by

$$A(X) = \begin{cases} \frac{x}{6}, & \text{if } x \in [0, 1]; \\ 7x - 6, & \text{if } x \in (1, \infty). \end{cases} \quad S(X) = \begin{cases} \frac{x}{7}, & \text{if } x \in [0, 1]; \\ 3x - 2, & \text{if } x \in (1, \infty). \end{cases}$$

Consider a sequence $\{x_n\} = \{\frac{1}{n}\}_{n \in \mathbb{N}}$ in X . Then

$$\lim_{n \rightarrow \infty} A(x_n) = \lim_{n \rightarrow \infty} \left(\frac{1}{6n} \right) = 0 = \lim_{n \rightarrow \infty} \left(\frac{1}{7n} \right) = \lim_{n \rightarrow \infty} S(x_n).$$

Next,

$$\lim_{n \rightarrow \infty} AS(x_n) = \lim_{n \rightarrow \infty} A \left(\frac{1}{7n} \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{42n} \right) = 0 = A(0),$$

$$\lim_{n \rightarrow \infty} SA(x_n) = \lim_{n \rightarrow \infty} S \left(\frac{1}{6n} \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{42n} \right) = 0 = S(0),$$

and

$$\lim_{n \rightarrow \infty} F_{ASx_n, SAx_n}(t) = 1,$$

for all $t > 0$. Consider another sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n \in \mathbb{N}}$ in X . Then

$$\lim_{n \rightarrow \infty} A(x_n) = \lim_{n \rightarrow \infty} \left(7 + \frac{7}{n} - 6 \right) = 1 = \lim_{n \rightarrow \infty} \left(3 + \frac{3}{n} - 2 \right) = \lim_{n \rightarrow \infty} S(x_n).$$

Also,

$$\lim_{n \rightarrow \infty} AS(x_n) = \lim_{n \rightarrow \infty} A\left(1 + \frac{3}{n}\right) = \lim_{n \rightarrow \infty} \left(7 + \frac{21}{n} - 6\right) = 1 \neq A(1),$$

$$\lim_{n \rightarrow \infty} SA(x_n) = \lim_{n \rightarrow \infty} S\left(1 + \frac{7}{n}\right) = \lim_{n \rightarrow \infty} \left(3 + \frac{21}{n} - 2\right) = 1 \neq S(1),$$

but $\lim_{n \rightarrow \infty} F_{ASx_n, SAx_n}(t) = 1$. Thus, the pair (A, S) is compatible as well as subsequentially continuous but not reciprocally continuous. Therefore, all the conditions of Corollary 3.4 are satisfied for some fixed $k \in (0, 1)$. Here, 0 is a coincidence as well as a unique common fixed point of the pair (A, S) . It is noted that this example cannot be covered by those fixed point theorems which involve compatibility and reciprocal continuity both or by involving conditions on completeness (or closedness) of underlying space (or subspaces). Also, in this example neither X is complete nor any subspace are closed, that is, $A(X) = [0, \frac{1}{6}] \cup (1, \infty)$ and $S(X) = [0, \frac{1}{7}] \cup (1, \infty)$. It is noted that this example cannot be covered by those fixed point theorems which involve both compatibility and reciprocal continuity.

Example 3.7. Let $X = \mathbb{R}$ (set of real numbers) and d be the usual metric on X and for each $t \in [0, 1]$, define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Clearly (X, \mathcal{F}, Δ) be a Menger space. Set $A = B$ and $S = T$. Define the self-mappings A and S by

$$A(X) = \begin{cases} \frac{x}{4}, & \text{if } x \in (-\infty, 1); \\ 5x - 4, & \text{if } x \in [1, \infty). \end{cases} \quad S(X) = \begin{cases} x + 3, & \text{if } x \in (-\infty, 1); \\ 4x - 3, & \text{if } x \in [1, \infty). \end{cases}$$

Consider a sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n \in \mathbb{N}}$ in X . Then

$$\lim_{n \rightarrow \infty} A(x_n) = \lim_{n \rightarrow \infty} \left(5 + \frac{5}{n} - 4\right) = 1 = \lim_{n \rightarrow \infty} \left(4 + \frac{4}{n} - 3\right) = \lim_{n \rightarrow \infty} S(x_n).$$

Also,

$$\lim_{n \rightarrow \infty} AS(x_n) = \lim_{n \rightarrow \infty} A\left(1 + \frac{4}{n}\right) = \lim_{n \rightarrow \infty} \left(5 + \frac{20}{n} - 4\right) = 1 = A(1),$$

$$\lim_{n \rightarrow \infty} SA(x_n) = \lim_{n \rightarrow \infty} S\left(1 + \frac{5}{n}\right) = \lim_{n \rightarrow \infty} \left(4 + \frac{20}{n} - 3\right) = 1 = S(1),$$

and

$$\lim_{n \rightarrow \infty} F_{ASx_n, SAx_n}(t) = 1,$$

for all $t > 0$. Consider another sequence $\{x_n\} = \{\frac{1}{n} - 4\}_{n \in \mathbb{N}}$ in X . Then

$$\lim_{n \rightarrow \infty} A(x_n) = \lim_{n \rightarrow \infty} \left(\frac{1}{4n} - 1\right) = -1 = \lim_{n \rightarrow \infty} \left(\frac{1}{n} - 4 + 3\right) = \lim_{n \rightarrow \infty} S(x_n).$$

Next,

$$\lim_{n \rightarrow \infty} AS(x_n) = \lim_{n \rightarrow \infty} A\left(\frac{1}{n} - 1\right) = \lim_{n \rightarrow \infty} \left(\frac{1}{4n} - \frac{1}{4}\right) = -\frac{1}{4} = A(-1),$$

$$\lim_{n \rightarrow \infty} SA(x_n) = \lim_{n \rightarrow \infty} S\left(\frac{1}{4n} - 1\right) = \lim_{n \rightarrow \infty} \left(\frac{1}{4n} - 1 + 3\right) = 2 = S(-1),$$

and $\lim_{n \rightarrow \infty} F_{ASx_n, SAx_n}(t) \neq 1$. Thus, the pair (A, S) is reciprocally continuous as well as subcompatible but not compatible. Therefore, all the conditions of Corollary 3.4 are satisfied for some fixed $k \in (0, 1)$. Thus 1 is a coincidence as well as a unique common fixed point of the pair (A, S) . It is also noted that this example too cannot be covered by those fixed point theorems which involve both compatibility and reciprocal continuity.

4. Related results in metric spaces

In this section we utilize Theorem 3.1 to derive the corresponding common fixed point theorem in metric space.

Theorem 4.1. *Let A, B, S and T be self-mappings of a metric space (X, d) . If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternatively subcompatible and reciprocally continuous), then*

- (i) *the pair (A, S) has a coincidence point,*
- (ii) *the pair (B, T) has a coincidence point,*
- (iii) *there exists a constant $k \in (0, 1)$ such that*

$$(4.1) \quad (d(Ax, By))^2 \leq k \max \left\{ \begin{array}{l} (d(Sx, Ty))^2, d(Sx, Ax)d(Ty, By), \\ \frac{1}{2}d(Sx, By)d(Ty, Ax), d(Ty, Ax), \\ \frac{1}{2}d(Sx, By)d(Ty, By) \end{array} \right\}$$

for all $x, y \in X$ and $t > 0$. Then, A, B, S and T have a unique common fixed point in X

Proof. Define $F_{x,y}(t) = H(t - d(x, y))$ and $\Delta(a, b) = \min\{a, b\}$, for all $a, b \in [0, 1]$. Then the metric space (X, d) can be realized as a Menger space (X, \mathcal{F}, Δ) . It is straightforward to notice that Theorem 4.1 satisfies the conditions of Theorem 3.1. Also, inequality (4.1) of Theorem 4.1 implies inequality (3.1) of Theorem 3.1. For any $x, y \in X$ and $t > 0$, $F_{Ax, By}(kt) = 1$ if $kt > d(Ax, By)$ which confirms the verification of inequality (3.1) of Theorem 3.1. Otherwise, if $kt \leq d(Ax, By)$, then

$$t \leq \max \left\{ \begin{array}{l} (d(Sx, Ty))^2, d(Sx, Ax)d(Ty, By), \frac{1}{2}d(Sx, By)d(Ty, Ax), \\ d(Ty, Ax), \frac{1}{2}d(Sx, By)d(Ty, By) \end{array} \right\},$$

which shows that inequality (3.1) of Theorem 3.1 is completely satisfied. Thus, all the conditions of Theorem 3.1 are satisfied and, hence conclusions follow immediately from Theorem 3.1. \square

Remark 4.2. The results similar to Theorem 4.1 can also be outlined in view of Corollary 3.2, Corollary 3.3 and Corollary 3.4.

Remark 4.3. Theorem 4.1 improves the results of Cho et al. [16, Theorem 4.3] and Pathak et al. [38, Theorem 4.3].

Acknowledgement

Authors would like to thank Professor M. Imdad for providing preprint of the paper [40].

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Received by the editors June 16, 2012