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A NEW VERSION OF THE ĆIRIĆ QUASI-CONTRACTION PRINCIPLE IN THE MODULAR SPACE

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Abstract. Let T be a mapping of a modular space (M_ρ, ρ) into itself which is (q, c)-quasi-contraction, i.e. if there exist numbers $0 \leq q < 1 < c$ such that T satisfies $\rho(c(Tx - Ty)) \leq q \max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), \rho(x - Ty), \rho(y - Tx)\}$ for all $x, y \in M_\rho$. In this article, the existence of a unique fixed point of T is proved. Moreover, the same result holds for the multi-valued (q, c)-quasi-contraction map.

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1. Introduction

A problem that mathematicians has dealt with for almost fifty years, is how to generalize the classical function spaces L^p. A first attempt was made by Birnhaum and Orlicz [2]. Their approach consisted of considering spaces of functions with some growth properties different from the power type growth control provided by the L^p-norm. This generalization found many applications in differential and integral equations with kernels of nonpower types. The main idea of another generalization is to consider, in a measure space, a functional that has the properties of a norm plus a monotonicity condition. A more abstract generalization was given by Nakano in 1950 [19] based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called a **modular**. This idea, which was the basis of the theory of modular spaces and initiated by Nakano in connection with the theory of order spaces, was redefined and generalized by Musielak and Orlicz in 1959 [17]. Modular spaces have been studied for almost forty years and there is a large set of known applications of them in various parts of analysis, probability and mathematical statistics. Moreover, it

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is possible to consider a nonlinear integral equation or to study the existence and behavior of an initial value problem such as

$$\begin{cases} u'(t) + (I - T)u(t) = 0, \\ u(0) = f, \end{cases}$$

where $A = I - T$ is a generator of a nonlinear semigroup and T is a nonexpansive mapping in the modular space (see [14] for more details or [13]). This problem has applications in engineering problems. In order to solve these kinds of problems, it is needed to apply fixed point theory on such kinds of spaces. Also, it is well-known that fixed point theory is one of the powerful tools in solving integral and differential equations. The existence of fixed point in various kinds of spaces such as metric spaces, fuzzy metric spaces, probabilistic metric spaces, etc., has been proved by mathematicians (see [1], [3], [4], [8], [9], [12], [14], [20], [21], [23], [24], [25] etc.).
As it is known, the Banach fixed point theorem is one of the basic theorems in the fixed point theory and it has a broad set of applications. Khamsi et al. [15] proved the Banach contraction principle for modular function spaces. Moreover, Ait Taleb et al. [1] presented a fixed point theorem of Banach type in modular space as well as its applications to a nonlinear integral equation in the Musielak-Orlicz space.

In this article, a generalization of the quasi-contraction theorem in the modular space is proved. This theorem can be regraded as a new generalization of the Banach fixed point theorem [15] in the modular space. In order to do this, we recall that a mapping T of a metric space X into itself is said to be a quasi-contraction if and only if there exists a number $0 \leq q < 1$, such that

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

holds for every $x, y \in X$. Ćirić [3] (see also [5], [6] and [16]) proved that if T is a quasi-contraction on a metric space X and if X is T-orbitally complete, then T has a unique fixed point.
In order to give a generalization of the quasi-contraction theorem in the modular space and for the sake of convenience, some definitions and notations are recalled from [7], [10], [11], [14], [17], [18] and [22].

Definition 1.1. Let M be a vector space over $K(= \mathbb{R}$ or $\mathbb{C})$. A function $\rho : M \to [0, +\infty)$ is called modular if:

(1) $\rho(x) = 0$ if and only if $x = 0$.
(2) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, and for all $x \in M$.
(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$, for all $x, y \in M$.

Definition 1.2. If (3) in Definition 1.1 is replaced by:

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$$

for $\alpha, \beta \geq 0, \alpha^s + \beta^s = 1$ with an $s \in (0, 1]$, then the modular ρ is called an s-convex modular; and if $s = 1$, ρ is called a convex modular.

Remark 1.3. *Every norm defined on M is a modular on M.*

Definition 1.4. A modular ρ defines a corresponding modular space M_ρ. The space M_ρ is given by

$$M_\rho = \{x \in M; \rho(\lambda x) \to 0 \text{ } as \text{ } \lambda \to 0\}.$$

Definition 1.5. Let M_ρ be a modular space.

(1) A sequence $\{x_n\}$ in M_ρ is said to be:
   (a) ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$.
   (b) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $m, n \to \infty$.

(2) M_ρ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

(3) M_ρ has Fatou property if and only if for any sequence $\{x_n\}$ which converges to x_0, $\rho(x_0) \leq \text{liminf}_n \rho(x_n)$.

(4) M_ρ satisfies Δ_2-condition if $\lim_{n\to\infty} \rho(2(x_n - x_0)) = 0$ whenever $\lim_{n\to\infty} \rho(x_n - x_0) = 0$.

The rest of the paper is organized as follows: In Section 2, the definition of quasi-contraction map in the modular space is given and two properties of it are proved. Section 3, is devoted to the existence of a fixed point for a quasi-contraction map in the modular space. Finally, the existence of a fixed point for a multi-valued quasi-contraction map in the space is proved.

2. (q, c)-quasi-contraction in the modular space

Let T be a mapping of a modular space M_ρ into itself. For $A \subset M_\rho$ let $\delta(A) = \sup\{\rho(a - b) : a, b \in A\}$ and for each $x \in M_\rho$,

$$\begin{aligned} O(x, n) &= \{x, Tx, \cdots, T^n x\}, \quad n = 1, 2, \cdots. \\ O(x, \infty) &= \{x, Tx, \cdots\}. \end{aligned}$$

Definition 2.1. A mapping $T : M_\rho \to M_\rho$ is called (q, c)-quasi-contraction, if there exist numbers $0 \leq q < 1 < c$, such that

$$\rho(c(Tx - Ty)) \leq q \max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), \rho(x - Ty), \rho(y - Tx)\},$$

for all $x, y \in M_\rho$.

Before stating the fixed point theorem, the following lemmas for a (q, c)-quasi-contraction map are presented.

Lemma 2.2. *Let T be a (q, c)-quasi-contraction and $x \in M_\rho$ be arbitrary fixed. Then for each $n \geq 1$, $\rho(c(T^i x - T^j x)) \leq q\delta(O(x, n))$ for all $i, j \in \{1, 2, \cdots, n\}$.*

Proof. Let $x \in M_\rho$ be arbitrary, n be any positive integer and $i, j \in \{1, 2, \cdots, n\}$. Then $T^{i-1}x, T^i x, T^{j-1}x, T^j x \in O(x, n)$. Since T is a (q, c)-quasi-contraction, we have

$$\begin{aligned} \rho(c(T^i x - T^j x)) = &\ \rho(c(TT^{i-1}x - TT^{j-1}x)) \\ \leq &\ q \max\{\rho(T^{i-1}x - T^{j-1}x), \rho(T^{i-1}x - T^i x) \\ &\quad \rho(T^{j-1}x - T^j x), \rho(T^{i-1}x - T^j x), \rho(T^i x - T^{j-1}x)\} \\ \leq &\ q\delta(O(x, n)), \end{aligned}$$

which proves the lemma. □

Lemma 2.3. *Let T be a (q,c)-quasi-contraction and $x \in M_\rho$ be arbitrary fixed. Then, for each $n \geq 1$, there exists a positive integer $k \leq n$, such that $\rho(x - T^k x) = \delta(O(x,n))$.*

Proof. By the property (3) in Definition 1.1 $\rho(x) \leq \rho(cx)$, for each $x \in M_\rho$. Therefore,

$$\rho(T^i x - T^j x) \leq \rho(c(T^i x - T^j x)),$$

for all i, j and all $x \in M_\rho$. The conclusion now follows from Lemma 2.2 and the definition of $\delta(O(x,n))$. □

Lemma 2.4. *Let T be a (q,c)-quasi-contraction and $x \in M_\rho$ be arbitrary fixed. Then*

$$\delta(O(x,\infty)) \leq (1/(1-q))\rho(\alpha(x - Tx))$$

where α is the conjugate of c, i.e. $1/\alpha + 1/c = 1$.

Proof. Let $x \in M_\rho$ be arbitrary. Note that the map $n \vdash \delta(O(x,n))$ is increasing. Moreover, $\delta(O(x,\infty)) = \sup\{\delta(O(x,n)) : n \in \mathbb{N}\}$. Now, it is enough to show that

$$\delta(O(x,n)) \leq 1/(1-q)\rho(\alpha(x - Tx))$$

for all $n \in \mathbb{N}$. Let n be any positive integer. By Lemma 2.3, there exists $T^k(x) \in O(x,n), (1 \leq k \leq n)$ such that $\rho(x - T^k x) = \delta(O(x,n))$. Then from Lemma 2.2, we get

$$\begin{aligned} \rho(x - T^k x) \leq &\quad \rho(\alpha(x - Tx)) + \rho(c(Tx - T^k x)) \\ \leq &\quad \rho(\alpha(x - Tx)) + q\delta(O(x,n)) \\ = &\quad \rho(\alpha(x - Tx)) + q\rho(x - T^k x). \end{aligned}$$

Therefore,

$$\delta(O(x,n)) = \rho(x - T^k x) \leq (1/(1-q))\rho(\alpha(x - Tx)).$$

Since n is arbitrary, the proof is completed. □

3. Fixed point theorem for (q,c)-quasi-contraction map

In this section we state a fixed point theorem for a (q,c)-quasi-contraction map as follows:

Theorem 3.1. *Let M_ρ be a ρ-complete modular space and ρ fulfills the Fatou property and the Δ_2-condition. Suppose that $T : M_\rho \to M_\rho$ is a (q,c)-quasi-contraction. Then*

(1) *T has a unique fixed point $u \in M_\rho$.*
(2) *$\rho(T^n x - u) \leq (q^n/(1-q))\rho(\alpha(x - Tx)))$ for each $x \in M\rho$ and for each $n \in \mathbb{N}$.*

Proof. Let x be an arbitrary point of M_ρ. It is enough to show that the sequence of iterates $\{T^n x\}$ is a ρ-Cauchy sequence.
Let n and m $(n < m)$ be any positive integers. Since T is a (q, c)-quasi-contraction map, it follows from Lemma 2.2 that

$$\begin{aligned} \rho(c(T^n x - T^m x)) = & \ \rho(c(TT^{n-1}x - T^{m-n+1}T^{n-1}x)) \\ \leq & \ q\delta(O(T^{n-1}x, m-n+1)). \end{aligned}$$

According to Lemma 2.3, there exists an integer $k_1, 1 \leq k_1 \leq m-n+1$, such that

$$\delta(O(T^{n-1}x, m-n+1)) = \rho(T^{n-1}x - T^{k_1}T^{n-1}x).$$

Applying Lemma 2.2 to obtain

$$\begin{aligned} \rho(c(T^{n-1}x - T^{k_1}T^{n-1}x)) = & \ \rho(c(TT^{n-2}x - T^{k_1+1}T^{n-2}x)) \\ \leq & \ q\delta(O(T^{n-2}x, k_1+1)) \\ \leq & \ q\delta(O(T^{n-2}x, m-n+2)). \end{aligned}$$

Therefore,

$$\begin{aligned} \rho(c(T^n x - T^m x)) \leq & \ q\delta(O(T^{n-1}x, m-n+1) \\ \leq & \ q^2\delta(O(T^{n-2}x, m-n+2)). \end{aligned}$$

After n iterations,

$$\begin{aligned} \rho(c(T^n x - T^m x)) \leq & \ q\delta(O(T^{n-1}x, m-n+1)) \\ \leq & \ \cdots \\ \leq & \ q^n\delta(O(x, m)). \end{aligned}$$

Then, it follows from Lemma 2.4 that

$$\rho(c(T^n x - T^m x)) \leq (q^n/(1-q))\rho(\alpha(x - Tx)).$$

Since $\lim_{n\to\infty} q^n = 0$, $\{T^n x\}$ is a ρ-Cauchy sequence. As M_ρ is ρ-complete, $\{T^n x\}$ has a limit point u in M. Now, Δ_2-condition, shows that

$$\lim_{n\to\infty} \rho(\alpha(u - T^{n+1}x)) = 0.$$

Consider the following inequalities,

$$\begin{aligned} \rho(c(u - Tu)) \leq & \ \operatorname{liminf}_n \rho(c(T^{n+1}x - Tu)) \\ \leq & \ \operatorname{liminf}_n q \max\Big\{\rho(T^n x - u), \rho(T^n x - T^{n+1}x) \\ & \qquad \rho(u - Tu), \rho(T^n x - Tu), \rho(T^{n+1}x - u)\Big\}. \end{aligned}$$

Since $\alpha > 1$, then

$$\begin{aligned} \rho(c(u - Tu)) \leq & \ \operatorname{liminf}_n q \max\Big\{\rho(\alpha(T^n x - u)), \rho(T^n x - T^{n+1}x) \\ & \qquad \rho(c(u - Tu)), \rho(T^n x - Tu), \rho(T^{n+1}x - u)\Big\}. \end{aligned}$$

Therefore

$$\begin{aligned}\rho(c(u-Tu)) \leq \quad & \liminf_n q\Big\{\rho(\alpha(T^n x-u))+\rho(T^n x-T^{n+1}x)\\ & \qquad +\rho(c(u-Tu))+\rho(T^{n+1}x-u)\Big\}.\end{aligned}$$

By Δ_2-condition,

$$\begin{aligned}\rho(c(u-Tu)) \leq \quad & 1/(1-q)\liminf_n q\{\rho(\alpha(T^n x-u))+\rho(T^n x-T^{n+1}x)\\ & +\rho(T^{n+1}x-u)\}\\ = \quad & 0\end{aligned}$$

Hence $\rho(u-Tu)=0$. Therefore, u is a fixed point for T. Let z be a fixed point different from u, thus

$$\begin{aligned}\rho(u-z) = \quad & \rho(Tu-Tz)\\ \leq \quad & \rho(c(Tu-Tz))\\ \leq \quad & q\max\{\rho(u-Tu),\rho(u-Tz),\rho(z-Tu),\rho(z-Tz),\rho(u-z)\}\\ = \quad & q\rho(u-z)\end{aligned}$$

and this is a contradiction. So $u=z$ and part (1) of the theorem is proved. Now, to prove part (2), note that

$$\liminf_m \rho(T^n x-T^m x)\leq (q^n/(1-q))\rho(\alpha(x-Tx)).$$

Using the Fatou property shows that

$$\rho(T^n x-u)\leq \liminf_m \rho(T^n x-T^m x).$$

Therefore

$$\rho(T^n x-u)\leq (q^n/(1-q))\rho(\alpha(x-Tx)),$$

and this completes the proof. □

Remark 3.2. *We are unable to prove whether the conclusion in Theorem 3.1 is true for $c=1$ and $0<q<1$. See in this direction Khamsi, Kozlowski and Riech [15, Theorem 2.4].*

The following corollary is immediate from the above theorem.

Corollary 3.3. *Let M_ρ be a ρ-complete modular space, where ρ fulfills the Fatou property and the Δ_2-condition. Suppose $T : M_\rho \to M_\rho$ is such that there exists a positive integer k in such a way that the iterate T^k is a (q,c)-quasi-contraction. Then*

(I) *T has a unique fixed point $u \in M_\rho$.*

(II) *for each $x \in M_\rho$ and each $n \geq k$ we have $\rho(T^n x-u)\leq \frac{q^s}{(1-q)}a(s)$, where $a(x)=\max\{\rho(\alpha(T^i x-T^{i+k}x)) : i=0,1,\cdots,k-1\}$ and s=E(n/k)(the greatest integer not exceeding n/k).*

4. Multi-valued (q,c)-quasi-contraction map

Let M_ρ be a modular space and A, B be two subsets of M_ρ. We denote

$$\begin{aligned}&\Gamma(A,B)=\sup\{\rho(a-b): a\in A, b\in B\},\\ &BN(M_\rho)=\{A: A\neq\emptyset, A\subset M_\rho \text{ and } \delta(A)<\infty\},\\ &D(A,B)=\inf\{\rho(a-b): a\in A, b\in B\}.\end{aligned}$$

Definition 4.1. Let $F: M_\rho\to BN(M_\rho)$ be a multi-valued function and $x_0\in M_\rho$. An orbit of F at x_0 is a sequence $\{x_n: x_n\in Fx_{n-1}, n=1,2,\cdots\}$. Moreover, a modular space M_ρ is said to be F-orbitally ρ-complete if and only if every ρ-Cauchy sequence which is a subsequence of an orbit of F at x for some $x\in M_\rho$, converges in M_ρ.

Definition 4.2. Let $F: M_\rho\to BN(M_\rho)$ be a multi-valued function on a modular space M_ρ. The element $u\in M_\rho$ is called a fixed point for F if and only if $u\in Fu$.

Now, we have the next theorem:

Theorem 4.3. *Let $F: M_\rho\to BN(M_\rho)$ be a multi-valued mapping on a modular space M_ρ and M_ρ is F-orbitally ρ-complete. If F satisfies*

$$\Gamma(cFx,cFy)\le q\max\{\rho(x-y),\Gamma(x,Fx),\Gamma(y,Fy),D(x,Fy),D(y,Fx)\}$$

for some $0\le q<1$ and $c>1$ and all $x,y\in M_\rho$, where $cFx=\{cy, y\in Fx\}$ then F has a unique fixed point u in M and $Fu=\{u\}$.

Proof. Let $a\in(0,1)$ be any number. Define a single-valued mapping $T: M_\rho\to M_\rho$ as follows:
for each $x\in M_\rho$, let Tx be a point of Fx, which satisfies $\rho(x-Tx)\ge q^a\Gamma(x,Fx)$. A mapping T is then a (q,c)-quasi-contraction with $q_1=q^{1-a}$. Indeed, for every $x,y\in M_\rho$ we have

$$\begin{aligned}\rho(c(Tx-Ty)) &\le \Gamma(cFx,cFy)\\ &\le qq^{-a}\max\{q^a\rho(x-y),q^a\Gamma(x,Fx),q^a\Gamma(y,Fy)\\ &\qquad\qquad q^aD(x,Fy),q^aD(y,Fx)\}\\ &\le q^{1-a}\max\{\rho(x-y),\rho(x-Tx),\rho(y-Ty)\\ &\qquad\qquad \rho(x-Ty),\rho(y-Tx)\},\end{aligned}$$

which means that T is a (q,c)-quasi-contraction, then there exists $u\in M_\rho$ such that $u=Tu$ which implies $u\in Fu$. From the contraction we have

$$\begin{aligned}\Gamma(Fu,Fu) &\le \Gamma(cFu,cFu)\\ &\le q\Gamma(u,Fu).\end{aligned}$$

This may happen only if $Fu=\{u\}$. □

References

[1] Ait Taleb, A., Hanebalyn, E. A fixed point theorem and its application to integral equations in modular function Spaces. Proc. Amer. Math. Soc. 128 (2000), 419-426.

[2] Birnhaum, Z., Orlicz, W., Uber die Verallgemeinerung des Begriffers der Zueinander Konjugierten Potenzen. Studia Math. 3 (1931), 1-67.

[3] Ćirić, L. B., A generalization of Banach's contraction principle. Proc. Amer. Math. Soc. 45 (1974), 267-273.

[4] Ćirić, L. B., Generalized contractions and fixed point theorems. Publ. Inst. Math. 12 (26) (1971), 19-26.

[5] Ćirić, L. B., Quasi-contractions in Banach spaces. Publ. Inst. Math. 21 (1977), 41-48.

[6] Ćirić, L. B., Quasi contraction non-self mappings on Banach spaces. Bull. Acad. Serb Sci. Arts Cl. Sci. Math. Natur. 23 (1998), 25-31.

[7] Dominguez Benavides, T., Khamsi, M. A., Samadi, S., Uniformly Lipschitzian mappings in modular function spaces. Nonlinear Anal. 46 (2001), 267–278.

[8] Edelstein, M., An extention of Banach's contraction principle. Proc. Amer. Math. Soc. 12 (1961), 7-10.

[9] Ghaemi, M. B., Razani, A., Fixed and periodic points in the probabilistic normed and metric spaces. Chaos, Solitons and Fractals 28 (2006), 1181-1187.

[10] Hajji, A., Hanebaly, E., Fixed point theorem and its application to perturbed integral equations in modular function spaces. Electron. J. Differential Equations 105 (2005), 11 pp.

[11] Hanebaly, E., Fixed point theorems in modular space. arXiv:math.FA/0511319v1, 12 Nov 2005.

[12] Kannan, R., Some results on fixed points II. American Mathematical Monthly 76 (1969), 405-408.

[13] Khamsi, M. A., Fixed point theory in modular function spaces. Proc. of the Workshop on Recent Advances on Metric, Fixed Point Theory (1995), 31-55.

[14] Khamsi, M. A., Nonlinear semigroups in modular function spaces. Math. Japon. 37 (1992), 291-299.

[15] Khamsi, M. A., Kozlowski, W. M., Reich, S., Fixed point theory in modular function spaces. Nonlinear Anal. 14 (1990), 935-953.

[16] Maiti, M., Pal, T. K., On Ćirić quasi-contractions. Bull. Calcutta Math. Soc. 72 (1980), 135-138.

[17] Musielak, J., Orlicz, W., On modular spaces. Studia Math. 18 (1959), 591-597.

[18] Nadler, S. B., Multi-valued contraction mappings. Pacific J. Math. 30 (1969), 475-488.

[19] Nakano, H., Modular semi-ordered spaces. Tokyo, (1950).

[20] Razani, A., Existence of fixed point for the nonexpansive mapping of intuitionistic fuzzy metric spaces. Chaos, Solitons and Fractals 30 (2006), 367-373.

[21] Razani, A., Fouladgar, K., Extension of contractive maps in the Menger probabilistic metric space. Chaos, Solitons and Fractals 34 (2007), 1724-1731.

[22] Razani, A., Nabizadeh, E., Beyg Mohamadi, M., Homaei Pour, S., Fixed points of nonlinear and asymptotic contractions in the modular space. Abstract and Applied Analysis (2007), Article ID 40575, 10 pages.

[23] Razani, A., Shirdaryazdi, M., A common fixed point theorem of compatible maps in Menger space. Chaos, Solitons and Fractals 37 (2007), 26-34.

[24] Smithson, R., Fixed point theorem for mappings with a contractive iterate. American Mathematical Society 23 (1969), 631-634.

[25] Smithson, R., Fixed points for contractive multifications. American Mathematical Society 27 (1971), 192-194.

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