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RIGHT IDEALS OF Γ-SEMIRINGS

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Abstract. In this paper we introduce the concepts of a right weakly regular Γ-semiring and a fully prime right Γ-semiring. Several characterizations of them are furnished. Also, discuss the topological space of prime ideals of a Γ-semiring.

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1. Introduction

The notion of a Γ-ring was introduced by Nobusawa [10]. As a generalization of rings, semirings were introduced by Vandiver in [12]. Also, as a generalization of a Γ-ring and a semiring, the notion of Γ-semiring was introduced by Rao [11]. Characterizations of ideals in a semigroup were given by Lajos in [9], while ideals in semirings were characterized by Ahsan in [1], Iseki in [7, 8] and Shabir and Iqubal in [12]. Properties of prime and semiprime ideals in Γ-semirings were discussed in detail by Dutta and Sardar in [3, 4]. The present authors discussed quasi-ideals in Γ-semiring [5, 6].

In this paper, efforts are made to introduce and characterize a right weakly regular Γ-semiring and a fully prime right Γ-semiring. Furthermore, we give topological characterizations of the space of prime right ideals in a Γ-semiring.

2. Preliminaries

First we recall some definitions of the basic concepts of Γ-semirings that we need in the sequel. For this we follow Dutta and Sardar [3].

Definition 2.1. Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ denoted by $a\alpha b$; for all $a, b \in S$ *and* $\alpha \in \Gamma$ satisfying the following conditions:

(i) $a\alpha\,(b + c) = (a\ \alpha b) + (a\ \alpha c)$

(ii) $(b + c)\,\alpha a = (b\ \alpha a) + (c\ \alpha a)$

(iii) $a(\alpha + \beta)c = (a\ \alpha c) + (a\ \beta c)$

(iv) $a\alpha\,(b\beta c) = (a\alpha b)\,\beta c$; for all $a, b, c \in S$ *and for all* $\alpha, \beta \in \Gamma$.

Obviously, every semiring S is a Γ-semiring. Let S be a semiring and Γ be a commutative semigroup. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by, $a\alpha b = ab$; for all $a, b \in S$ and $\alpha \in \Gamma$. Then S is a Γ-semiring.

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Definition 2.2. An element $0 \in S$ is said to be an *absorbing zero* if $0\alpha a = 0 = a\alpha 0, a + 0 = 0 + a = a$; for all $a \in S$ *and* $\alpha \in \Gamma$.

Definition 2.3. A non-empty subset T of S is said to be a *sub*-Γ-*semiring* of S if $(T,+)$ is a subsemigroup of $(S,+)$ and $a\alpha b \in T$; for all $a, b \in T$ and $\alpha \in \Gamma$.

Definition 2.4. A non-empty subset T of S is called a *left* (respectively *right*) *ideal* of S if S is a subsemigroup of $(S,+)$ and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T$, $x \in S$ and $\alpha \in \Gamma$.

Definition 2.5. If T is both left and right ideal of S, then T is known as an *ideal* of S.

Definition 2.6. A right ideal P of S is called a *prime right ideal* if $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, for any right ideals A and B of S.

Definition 2.7. A right ideal P of S is called a *semiprime right ideal* if $A^2 = A\Gamma A \subseteq P$ implies $A \subseteq P$, for any right ideal A of S.

Obviously, every prime right ideal in S is a semiprime right ideal.

Definition 2.8. A right ideal P of S is called an *irreducible right ideal* if $A \cap B = P$ implies $A = P$ or $B = P$, for any right ideals A and B of S.

Definition 2.9. A right ideal P of S is called a *strongly irreducible right ideal* if $A \cap B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, for any right ideals A and B of S.

Definition 2.10. A proper ideal M of S is said to be a *maximal ideal* if there does not exist any other proper ideal of S containing M properly.

Definition 2.11. An element $1 \in S$ is said to be an *unit element* if $1\alpha a = a$ and $a\alpha 1 = a$ for all $a \in S$ and $\alpha \in \Gamma$.

Definition 2.12. A Γ-semiring S is said to be a *commutative* Γ-*semiring* if $a\alpha b = b\alpha a$; for all $a, b \in S$ and $\alpha \in \Gamma$.

From now onwards S will denote a Γ-semiring with an absorbing zero and a unit element 1 unless otherwise stated.

Remark 2.13. Let $\mathcal{L}_R$ denote the family of all right ideals of S. Then $\langle \mathcal{L}_R, \subseteq \rangle$ is a partially ordered set. As $\{0\}$, $S \in \mathcal{L}_R$ and $\bigcap_{\alpha \in \Delta} I_\alpha \in \mathcal{L}_R$, $\mathcal{L}_R$ is a complete lattice under $I \vee J = I + J$ and $I \wedge J = I \cap J$ (see [2]).

3. Prime right ideal

Proposition 3.1. *A right ideal P of S is a prime right ideal of S if and only if $a\Gamma S\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$, for any $a, b \in S$.*

Proof. Suppose that P is a prime right ideal of S. Let $a\Gamma S\Gamma b \subseteq P$,for $a, b \in S$. Then $a\Gamma S\Gamma b\Gamma S \subseteq P \Rightarrow (a\Gamma S)\Gamma(b\Gamma S) \subseteq P$. By $a\Gamma S$ and $b\Gamma S$ are right ideals of S and P is a prime right ideal, $a\Gamma S \subseteq P$ or $b\Gamma S \subseteq P$. Therefore, $a \in P$ or $b \in P$. Conversely, assume the given statement holds. Let A and B be any two right ideals of S such that $A\Gamma B \subseteq P$. If $A \subseteq P$, then the result holds. Suppose that $A \nsubseteq P$. Hence, there exists an element $a \in A$ such that $a \notin P$. For any $b \in B$, $a\Gamma S\Gamma b = (a\Gamma S)\Gamma b \subseteq A\Gamma B \subseteq P$. Therefore by the assumption $b \in P$ implies $B \subseteq P$. Therefore, P is a prime right ideal of S. □

Proposition 3.2. *A right ideal P of S is a semiprime right ideal of S if and only if $a\Gamma S\Gamma a \subseteq P$ implies $a \in P$, for any $a \in S$.*

Proof. Suppose that P is a semiprime right ideal of S. Let $a\Gamma S\Gamma a \subseteq P$, for $a \in S$. Then, $a\Gamma S\Gamma a\Gamma S \subseteq P \Rightarrow (a\Gamma S)\Gamma(a\Gamma S) \subseteq P$. By $a\Gamma S$ is a right ideal of S and P is a semiprime right ideal, $a\Gamma S \subseteq P$. Then $a \in P$. Conversely, assume given statement holds. Let A be any right ideal of S such that $A\Gamma A \subseteq P$. For any $a \in A$, $a\Gamma S\Gamma a = (a\Gamma S)\Gamma a \subseteq A\Gamma A \subseteq P$. Therefore, by assumption $a \in P$ implies $A \subseteq P$. Hence P is a semiprime right ideal of S. □

Proposition 3.3. *If P is a prime right ideal of S, then $(P : a) = \{x \in S | a\Gamma x \subseteq P\}$ is also a prime right ideal of S for any, $a \in S \backslash P$.*

Proof. Let P be a prime right ideal of S and $(P : a) = \{x \in S | a\Gamma x \subseteq P\}$. Let $x, y \in (P : a)$. Therefore, $a\Gamma x \subseteq P$, $a\Gamma y \subseteq P$. $a\Gamma(x + y) = a\Gamma x + a\Gamma y \subseteq P$ implies $x + y \in (P : a)$. Let $x \in (P : a)$, $t \in S$ and $\alpha \in \Gamma$. Then, $a\Gamma(x\alpha t) \subseteq a\Gamma(x\Gamma t) = (a\Gamma x)\Gamma t \subseteq P$ gives $x\alpha t \in (P : a)$. This shows $(P : a)$ is a right ideal. To show $(P : a)$ is a prime right ideal let A and B be any two right ideals of S such that $A\Gamma B \subseteq (P : a)$. Then, $a\Gamma(A\Gamma B) \subseteq P$. $a\Gamma A$ and $a\Gamma B$ are right ideals of S. $(a\Gamma A)\Gamma(a\Gamma B) = (a\Gamma A\Gamma a)\Gamma B \subseteq a\Gamma A\Gamma B = a\Gamma(A\Gamma B) \subseteq P$. As P is a prime right ideal of S, $a\Gamma A \subseteq P$ or $a\Gamma B \subseteq P$. Therefore, $A \subseteq (P : a)$ or $B \subseteq (P : a)$, which shows that $(P : a)$ is a prime right ideal of S. □

Similarly to as in the proof of Proposition 3.3, we immeditely get the following proposition.

Proposition 3.4. *If P is a prime right ideal of S, then $I = \{x \in S | S\Gamma x \subseteq P\}$ is the largest two-sided ideal of S contained in P.*

The necessary condition for a right ideal to be prime is given in the following proposition.

Proposition 3.5. *Every semiprime and strongly irreducible right ideal is a prime right ideal of S.*

Proof. Let P be a strongly irreducible and a semiprime right ideal of S. For any right ideals A and B of S, $(A\Gamma B) \subseteq P$. $A \cap B$ is a right ideal of S. Hence $(A \cap B)^2 = (A \cap B)\Gamma(A \cap B) \subseteq A\Gamma B \subseteq P$. By P is a semiprime right ideal, $A \cap B \subseteq P$. Therefore, $A \subseteq P$ or $B \subseteq P$, since P is a strongly irreducible right ideal. Thus P is a prime right ideal of S. □

Proposition 3.6. *Any maximal right ideal of S is a prime right ideal.*

Proof. Let M be any maximal ideal of S. To show that M is a prime let $a\Gamma S\Gamma b \subseteq M$. Suppose that $a \notin M$. $a\Gamma S$ is a right ideal of S which contains an element a. By M is a maximal right ideal, $M + a\Gamma S = S$. As $1 \in S$, $1 = m + \sum_i a\alpha_i x_i$. Then, $1\alpha b = m\alpha b + (\sum_i a\alpha_i x_i)\alpha b \subseteq M + a\Gamma S\Gamma b \subseteq M$. Therefore, $b \in M$. This shows that M is a prime right ideal. □

Proposition 3.7. *If R is a right ideal of S and a is a nonzero element of S such that $a \notin R$, then there exists an irreducible right ideal P of S such that $R \subseteq P$ and $a \notin P$.*

Proof. Let $\mathcal{B}$ be the family of all right ideals of S containing I and not containing an element a. Then $\mathcal{B}$ is nonempty as $R \in \mathcal{B}$. This family of all right ideals of S forms a partially ordered set under the inclusion of sets. Hence, by Zorn's lemma there exists a maximal right ideal P in $\mathcal{B}$. Therefore, $R \subseteq P$ and $a \notin P$. Now, to show that P is an irreducible right ideal of S let A and B be any two right ideals of S such that $A \cap B = P$. Suppose that A and B both are contained in P properly. Since P is a maximal right ideal in $\mathcal{B}$, we get $a \in A$ and $a \in B$. Therefore, $a \in A \cap B = P$ which is an absurd. Thus, either $A = P$ or $B = P$. Therefore, P is an irreducible right ideal of S. □

Proposition 3.8. *Any proper right ideal of S is the intersection of irreducible right ideals of S which contain it.*

Proof. Let R be any proper ideal of S and $\{A_k/k \in \Delta\}$ be a family of irreducible right ideals of S which contain R, where Δ denotes the indexed set. Then clearly $R \subseteq \bigcap_k A_k$. To show that $\bigcap_k A_k \subseteq R$. Suppose that $\bigcap_k A_k \subset R$. Therefore, there is an element $a \in \bigcap_k A_k$ such that $a \notin R$. Then by Proposition 3.7, there exists an irreducible ideal P such that $R \subseteq P$ and $a \notin P$. This establishes the existence of irreducible right ideal P such that $a \notin P$ and $R \subseteq P$. Therefore, $a \notin \bigcap_k A_k$ for every $a \notin R$. Hence, by the contrapositive method $\bigcap_k A_k \subseteq R$. Therefore $\bigcap_k A_k = R$. □

4. Right weakly regular Γ-semiring

Definition 4.1. A Γ-semiring S is said to be *right weakly regular* if $a \in (a\Gamma S)^2$, for any $a \in S$.

In the following theorems we furnish the characterizations for a right weakly regular Γ-semiring.

Theorem 4.2. *The following statements are equivalent in S*

(1) S is right weakly regular.

(2) $R^2 = R$, for each right ideal R of S.

(3) $R \cap I = R\Gamma I$, for any right ideal R and two-sided ideal I of S.

Proof. (1) $\Longrightarrow$ (2) Suppose that S is right weakly regular. For any right ideal R of S, $R^2 = R\Gamma R \subseteq R\Gamma S \subseteq R$. Conversely, let $a \in R$. As S is right weakly regular, $a \in (a\Gamma S)^2$, then $a \in (a\Gamma S)^2 = (a\Gamma S)\,\Gamma\,(a\Gamma S) \subseteq (R\Gamma S)\,\Gamma\,(R\Gamma S) \subseteq R\Gamma R = R^2$. Thus, $R^2 = R$, for each right ideal R of S.

(2) $\Longrightarrow$ (1) Suppose that $R^2 = R$, for each right ideal R of S. For any $a \in S$, $a \in a\Gamma S$ and $a\Gamma S$ is a right ideal of S. By assumption $(a\Gamma S)^2 = (a\Gamma S)$. Therefore, $a \in (a\Gamma S)^2$, which shows that S is right weakly regular.

(2) $\Longrightarrow$ (3) Let R be a right ideal and I be a two-sided ideal of S. Then $R \cap I$ is a right ideal of S. By assumption $(R \cap I)^2 = R \cap I$. $R \cap I = (R \cap I)^2 = (R \cap I)\,\Gamma(R \cap I) \subseteq R\Gamma I$. Clearly, $R\Gamma I \subseteq R$ and $R\Gamma I \subseteq I$. Therefore, $R\Gamma I \subseteq R \cap I$. Thus we get $R \cap I = R\Gamma I$.

(3) $\Longrightarrow$ (2) Let R be a right ideal of S and (R) be a two-sided ideal generated by R. By Result 3.2 in [5], $(R) = S\Gamma R\Gamma S$. By assumption $R \cap (R) = R\Gamma(R)$. Then, $R = R\Gamma\,(S\Gamma R\Gamma S) = (R\Gamma S)\,\Gamma\,(R\Gamma S) \subseteq R\Gamma R = R^2$. Therefore, $R^2 = R$. □

Proposition 4.3. *S is right weakly regular if and only if every right ideal of S is semiprime.*

Proof. Suppose that S is right weakly regular. Let R be a right ideal of S such that $A\Gamma A \subseteq R$, for any right ideal A of S. $A = A\Gamma A$ as S is right weakly regular. Therefore $A \subseteq R$. Hence R is a semiprime right ideal of S. Conversely, suppose that every right ideal of S is semiprime. Let R be right ideal of S. $R\Gamma R$ is also a right ideal of S. By assumption $R\Gamma R$ is a semiprime right ideal of S. $R\Gamma R \subseteq R\Gamma R$ implies $R \subseteq R\Gamma R$. Therefore, $R^2 = R$. Hence, S is right weakly regular. □

Proposition 4.4. *If S is right weakly regular, then an ideal P of S is prime if and only if P is irreducible.*

Proof. Let S be a right weakly regular Γ-semiring and P be an ideal of S. If P is a prime ideal of S, then clearly P is an irreducible ideal. Suppose that P is an irreducible ideal of S. To show P is a prime ideal, let A and B be any two ideals of S such that $A\Gamma B \subseteq P$. Then, by Theorem 4.2, we have $A \cap B \subseteq P$. Therefore, $(A \cap B) + P = P$. But $\mathcal{L}_S$ lattice of all ideals of S being distributive $(A + P) \cap (B + P) = P$. As P is an irreducible ideal, $A+P = P$ or $B+P = P$. Then $A \subseteq P$ or $B \subseteq P$. Therefore, P is a prime ideal of S. □

Now we define a fully prime right Γ-semiring and a fully semiprime right Γ-semiring.

Definition 4.5. A Γ-semiring S is said to be a *fully prime (semiprime) right* Γ*-semiring* if all right ideals of S are prime (semiprime) right ideals.

The relation between a fully prime right Γ-semiring and a right weakly regular Γ-semiring is furnished in the following propositions.

Proposition 4.6. *If S is a fully prime right Γ-semiring, then S is right weakly regular and the set of ideals of S is totally ordered.*

Proof. Let S be a fully prime right Γ-semiring. Therefore, every right ideal of S is a prime right ideal. But every prime right ideal is a semiprime right ideal. Hence, by Proposition 4.3, S is right weakly regular. Let A and B be any two ideals of S. Then $A \cap B$ is a right ideal of S. By hypothesis $A \cap B$ is a prime right ideal of S. $A\Gamma B \subseteq A \cap B$ implies $A \subseteq A \cap B$ or $B \subseteq A \cap B$. Therefore, $A \cap B = A$ or $A \cap B = B$. Thus we get either $A \subseteq B$ or $B \subseteq A$. Hence, the set of ideals of S is totally ordered. □

Proposition 4.7. *If S is right weakly regular and the set of ideals of S is totally ordered, then S is a fully prime right* Γ*-semiring.*

Proof. Let S be a right weakly regular Γ-semiring and the set of ideals of S is totally ordered. To show that S is a fully prime right Γ-semiring, let P be any right ideal of S. To prove P is a prime right ideal of S, let A and B be any two right ideals of S such that $A\Gamma B \subseteq P$. By assumption, either $A \subseteq B$ or $B \subseteq A$ and $A^2 = A$, $B^2 = B$. We consider $A \subseteq B$. Then, $A = A^2 = A\Gamma A \subseteq A\Gamma B \subseteq P$. Therefore, P is a prime right ideal of S. Hence, S is a fully prime right Γ-semiring. □

The definition of a regular Γ-semiring given by Dutta and Sardar in [3] as follows:

A Γ-semiring S is said to be *regular* if $a \in a\Gamma S\Gamma a$, for any $a \in S$.

In general, the family of regular Γ-semirings forms a proper subclass of the family of right weakly regular Γ-semirings. But if S is a commutative Γ-semiring, then S is regular Γ-semiring if and only if S is right weakly regular Γ-semiring.

Proposition 4.8. *If S is a commutative Γ-semiring, then S is regular if and only if S is right weakly regular.*

Proof. Let S be a commutative Γ-semiring. Suppose that S is a right weakly regular Γ-semiring. Hence, for any $a \in S$, $a \in (a\Gamma S)^2$. $a \in (a\Gamma S)^2 = (a\Gamma S)\,\Gamma\,(a\Gamma S) = (a\Gamma S)\,\Gamma\,(S\Gamma a) \subseteq a\Gamma S\Gamma a$. Therefore, S is a regular Γ-semiring. Conversely, suppose S is a regular Γ-semiring. Let $a \in S$. Hence, $a \in a\Gamma S\Gamma a$. Then, $a \in a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a)\,\Gamma\,(S\Gamma a) \subseteq (a\Gamma S)\,\Gamma\,(a\Gamma S) = (a\Gamma S)^2$. This shows that S is a right weakly regular Γ-semiring. □

Proposition 4.9. *Each ideal of a right weakly regular Γ-semiring S is a right weakly regular (as a Γ-semiring).*

Proof. Let R be any ideal of a right weakly regular Γ-semiring S. Hence R itself is a sub-Γ-semiring of S. For any element $a \in R$, $a\Gamma R$ is a right ideal of S. S is a right weakly regular Γ-semiring implies $a \in (a\Gamma S)^2$ and $(a\Gamma R)^2 = a\Gamma R$. Hence we have, $a \in (a\Gamma S)^2 = (a\Gamma S)\,\Gamma\,(a\Gamma S) = a\Gamma\,(S\Gamma a\Gamma S) \subseteq a\Gamma\,(S\Gamma R\Gamma S) \subseteq a\Gamma R = (a\Gamma R)^2$. Therefore, $a \in (a\Gamma R)^2$ implies R is itself a right weakly regular Γ-semiring. □

Bi-ideals of a Γ-semiring are defined by the authors in [6] as follows: A nonempty subset B of a Γ-semiring S is said to be a *bi-ideal* of S if B is a sub-Γ-semiring of S and $B\Gamma S\Gamma B \subseteq B$.

Proposition 4.10. *S is right weakly regular if and only if $B \cap I \subseteq B\Gamma I$, for any bi-ideal B and an ideal I of S.*

Proof. Suppose that S is a right weakly regular Γ-semiring. Let B be a bi-ideal and I be an ideal of S. Let $a \in B \cap I$. Therefore, $a \in (a\Gamma S)^2$, since S is a right weakly regular. Then $a \in (a\Gamma S)^2 = (a\Gamma S)\,\Gamma\,(a\Gamma S) \subseteq (a\Gamma S)\,\Gamma\,(a\Gamma S)\,\Gamma\,(a\Gamma S)\,\Gamma S \subseteq (B\Gamma S\Gamma B)\,\Gamma(S\Gamma I\Gamma S) \subseteq B\Gamma I$. Therefore, $B \cap I \subseteq B\Gamma I$. Conversely, suppose that $B \cap I \subseteq B\Gamma I$, for any bi-ideal B and an ideal I of S. Let R be a right ideal of S. Then R itself a bi-ideal of S. By assumption $R = R \cap (R) \subseteq R\Gamma\,(R) = R\Gamma\,(S\Gamma R\Gamma S) = (R\Gamma S)\,\Gamma(R\Gamma S) \subseteq R\Gamma R$. Therefore $R = R\Gamma R = R^2$. Then by Theorem 4.2, S is a right weakly regular Γ-semiring. □

Proposition 4.11. *S is right weakly regular if and only if $B \cap I \cap R \subseteq B\Gamma I\Gamma R$, for any bi-ideal B, an ideal I and a right ideal R of S.*

Proof. Suppose that S is a right weakly regular Γ-semiring. Let B be a bi-ideal, I be an ideal and R be a right ideal of S. Let $a \in B \cap I \cap R$. Therefore $a \in$

$(a\Gamma S)^2$, since S is a right weakly regular. Then $a \in (a\Gamma S)^2 = (a\Gamma S)\,\Gamma\,(a\Gamma S) \subseteq (a\Gamma S)\,\Gamma\,(a\Gamma S)\,\Gamma\,(a\Gamma S)\,\Gamma S \subseteq B\Gamma\,(S\Gamma I\Gamma S)\,\Gamma(R\Gamma S) \subseteq B\Gamma I\Gamma R$. Therefore $B \cap I \cap R \subseteq B\Gamma I\Gamma R$. Conversely, suppose $B \cap I \cap R \subseteq B\Gamma I\Gamma R$, for any bi-ideal B and an ideal I and a right ideal R of S. For a right ideal R of S, R itself being a bi-ideal and S itself is being an ideal of S. By assumption $R \cap S \cap R \subseteq R\Gamma S\Gamma R = (R\Gamma \mathrm{S})\Gamma \mathrm{R} \subseteq R\Gamma R$. Therefore, $R \subseteq R\Gamma R$. Therefore, $R = R\Gamma R = R^2$. Then, by Theorem 4.2, S is a right weakly regular Γ-semiring. □

5. Right pure ideals

In this section we define a right pure ideal of a Γ-semiring S and furnish some of its characterizations.

Definition 5.1. An ideal I of Γ-semiring is said to be a *right pure ideal* if for any $x \in I$, $x \in x\Gamma I$.

Proposition 5.2. *An ideal I of S is right pure if and only if $R \cap I = R\Gamma I$, for any right ideal R of S.*

Proof. Let I be a right pure ideal and R be a right ideal of S. Then clearly $R\Gamma I \subseteq R \cap I$. Now let $a \in R \cap I$, gives $a \in R$ and $a \in I$. As I is a right pure ideal, $a \in a\Gamma I \subseteq R\Gamma I$. This gives $R \cap I \subseteq R\Gamma I$. By combining both inclusions we get $R \cap I = R\Gamma I$. Conversely, suppose $R \cap I = R\Gamma I$, for a right ideal R and an ideal I of S. Let I be an ideal of S and $a \in I$. $(a)_r$ denotes the right ideal generated by a and given by $(a)_r = N_0 a + a\Gamma I$, where N_0 is a set of non-negative integers. Then, $a \in (a)_r\Gamma I = (N_0 a + a\Gamma I)\Gamma I \subseteq a\Gamma I$. Therefore, I is a right pure ideal of S. □

Proposition 5.3. *The intersection of right pure ideals of S is a right pure ideal of S.*

Proof. Let A and B be right pure ideals of S. Then for any right ideal R of S we have, $R \cap A = R\Gamma A$ and $R \cap B = R\Gamma B$ by Proposition 5.2. We consider $R \cap (A \cap B) = (R \cap A) \cap B = (R\Gamma A) \cap B = (R\Gamma A)\,\Gamma B = R\Gamma\,(A\Gamma B) = R\Gamma(A \cap B)$. Therefore, $A \cap B$ is a right pure ideal of S. □

We characterize right weakly regular Γ-semirings in terms of right pure ideals in the following proposition.

Proposition 5.4. *S is right weakly regular if and only if any ideal of S is right pure.*

Proof. Suppose that S is a right weakly regular Γ-semiring. Let I be an ideal and R be a right ideal of S. Then by Theorem 4.2, $R \cap I = R\Gamma I$. Therefore, an ideal I of S is right pure by Proposition 5.2. Conversely, suppose any ideal of S is right pure. Then, from Proposition 5.2 and Theorem 4.2 we get S is a right weakly regular Γ-semiring. □

6. Space of prime ideals

Let S be a Γ-semiring and $\wp_S$ be the set of all prime ideals of S. For each ideal I of S define $\Theta_I = \{J \in \wp_S | I \nsubseteq J\}$ and $\zeta(\wp_S) = \{\Theta_I \ | I \ is \ an \ ideal \ of \ S\}$.

Theorem 6.1. *If S is a right weakly regular Γ-semiring, then $\zeta(\wp_S)$ forms a topology on the set $\wp_S$. There is an isomorphism between the lattice of ideals $\mathcal{L}_S$ and $\zeta(\wp_S)$ (lattice of open subsets of $\wp_S$).*

Proof. Since $\{0\}$ is an ideal of S and each ideal of S contains $\{0\}$, $\Theta_{\{0\}} = \{ J \in \wp_s | \{0\} \nsubseteq J\} = \Phi$. Therefore, $\Phi \in \zeta(\wp_S)$. Also, S itself is an ideal $\Theta_S = \{J \in \wp_S | S \nsubseteq J\} = \wp_S$ imply $\wp_S \in \zeta(\wp_S)$. Now let $\Theta_{I_k} \in \zeta(\wp_S)$ for $k \in \Lambda$, Λ is an indexing set and I_k is an ideal of S. Therefore, $\Theta_{I_k} = \{J \in \wp_S | I_k \nsubseteq J\}$. As $\bigcup_k \Theta_{I_k} = \{J \in \wp_S | \sum_k I_k \nsubseteq J\}$, $\sum_k I_k$ is an ideal of S. Therefore $\bigcup_k \Theta_{I_k} = \Theta_{\sum_k I_k} \in \zeta(\wp_S)$. Further, let Θ_A, $\Theta_B \in \zeta(\wp_S)$. Let $J \in \Theta_A \bigcap \Theta_B$, J is a prime ideal of S. Hence $A \nsubseteq J$ and $B \nsubseteq J$. Suppose that $A \cap B \subseteq J$. In a right weakly regular Γ-semiring prime ideals and strongly irreducible ideals coincide. Therefore, J is a strongly irreducible ideal of S. As J is a strongly irreducible ideal of S, $A \subseteq J$ or $B \subseteq J$, which is a contradiction to $A \nsubseteq J$ and $B \nsubseteq J$. Hence, $A \cap B \nsubseteq J$ implies $J \in \Theta_{A \cap B}$. Therefore, $\Theta_A \bigcap \Theta_B \subseteq \Theta_{A \cap B}$. Now let $J \in \Theta_{A \cap B}$. Then $A \cap B \nsubseteq J$ implies $A \nsubseteq J$ and $B \nsubseteq J$. Therefore, $J \in \Theta_A$ and $J \in \Theta_B$ imply $J \in \Theta_A \bigcap \Theta_B$. Thus, $\Theta_{A \cap B} \subseteq \Theta_A \bigcap \Theta_B$. $\Theta_A \bigcap \Theta_B = \Theta_{A \cap B} \in \zeta(\wp_S)$. Thus, $\zeta(\wp_S)$ forms a topology on the set $\wp_S$.

Now we define a function $\phi : \mathcal{L}_S \longrightarrow \zeta(\wp_S)$ by $\phi(I) = \Theta_I$, for all $I \in \mathcal{L}_S$. Let $I, K \in \mathcal{L}_S$. $\phi(I \cap K) = \Theta_{I \cap K} = \Theta_I \bigcap \Theta_K = \phi(I) \cap \phi(K)$. $\phi(I + K) = \phi(I + K) = \Theta_{I+K} = \Theta_I \cup \Theta_K = \phi(I) \cup \phi(K)$. Therefore, ϕ is a lattice homomorphism. Now consider $\phi(I) = \phi(K)$. Then $\Theta_I = \Theta_K$.

Suppose that $I \neq K$. Then there exists $a \in I$ such that $a \notin K$. As K is a proper ideal of S, there exists an irreducible ideal J of S such that $K \subseteq J$ and $a \notin J$ by Proposition 3.7. Hence, $I \nsubseteq J$. As S is a right weakly regular Γ-semiring, J is a prime ideal of S by Proposition 4.4. Therefore, $J \in \Theta_K = \Theta_I$ implies $I \subseteq J$; which is a contradiction. Therefore, $I = K$. Thus, $\phi(I) = \phi(K)$ implies $I = K$, and hence ϕ is one-one. As ϕ is onto, the result follows. □

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