Novi Sad J. Math.
Vol. 43, No. 2, 2013, 21-37

# INTEGRABLE DISTRIBUTIONS AND $\phi$-FOURIER TRANSFORM

**Ganesan Chinnaraman**[1]

**Abstract.** A new concept of integration of distributions is introduced and studied. It is proved that the space of all integrable distributions is properly larger than the space of Lebesgue integrable functions and compactly supported distributions. As an application of this concept, an alternate definition of Fourier transform namely $\phi$-Fourier transform is defined, on the space of integrable distributions into the space of continuous functions and its properties are established.

*AMS Mathematics Subject Classification* (2010): 42A38, 44A35, 46F12

*Key words and phrases:* Distributions, Integration, Integrable functions, Fourier transform

## 1. Introduction

In the earlier 1920s, the concept of singular functions (subclass of generalized functions) was first introduced and tacitly used by the British theoretical physicist Paul Dirac in his quantum mechanics studies. The mathematical foundation for the theory of generalized functions was independently formulated by S.L. Sobolev [21] and by L. Schwartz [15]-[19]. Later, different approaches for the development of generalized functions were made by many mathematicians (see [2, 3, 6, 22, 28]). The theory of distributions and their integral transforms has a wide range of applications in mathematical physics, applied mathematics and engineering sciences, etc. L. Schwartz extends the classical Fourier transform theory to the context of distributions and this extended Fourier transform was used by L. Ehrenpresis, B. Malgrange and by L. Hormander to successfully solve certain types of partial differential equations. For the study of integral transforms on distributions one can refer to [1, 5, 6, 28, 29].

The concepts of Lebesgue integration of $f \in L^1(\mathbb{R})$ and Fourier transform of functions in $L^1(\mathbb{R})$ are well known. The notion of integration of compactly supported distributions is introduced and studied by L. Schwartz [18]. In [27], R. Wawak has introduced the improper integrals of distributions, which is a slight modification of the definitions of [10, 20]. As a consequence of his new definition, a representation theorem, the convolution of two distributions were obtained. Further, the class of distributions having the improper integrals turns out to be the dual space of a space which can be called a space of functions with bounded variations in $\mathbb{R}^n$.

On the other hand, it was proved in [24] that the set of integrable distributions is a Banach space isometrically isomorphic to the Banach space $B_R$

[1]Department of Mathematics, V. H. N. S. N. College, Virudhunagar-626001, Tamil Nadu, India, e-mail: c.ganesan28@yahoo.com

of the space of all functions $F:\mathbb{R}\to\mathbb{R}$ such that (i) both left and right limits exist at all points in $\mathbb{R}$, (ii) left continuous on $\mathbb{R}$ and (iii) $F(-\infty)=0$; $F(\infty)\in\mathbb{R}$. Apart from proving standard properties of integrals, it was shown that the space of all integrable distributions is a module over the functions of bounded variation. Most recently [8], Svetlana Mincheva-Kaminska proved the equivalence of various conditions for integrability of distributions by extending the equivalence conditions given in [9]. One can refer to [4, 7, 9, 10, 23, 24, 25] for extensive study of various ways of defining the integral of distributions.

In this paper, motivated by the fact that the distributional derivative of a Schwartz distributions is again a Schwartz distributions, the Lebesgue integral of a certain type of Schwartz distribution is obtained as a Schwartz distribution. We define the Lebesgue integral of a distribution in a different way on a subclass of $\mathscr{D}'(\mathbb{R})$ which properly contains $L^1(\mathbb{R})\cup\mathcal{E}'(\mathbb{R})$. By using this integration of distributions, we define the Fourier transform of a distribution as a continuous function on $\mathbb{R}$, which is consistent with the distributional Fourier transform $\hat{}:\mathscr{D}^{'}(\mathbb{R})\to\mathcal{Z}^{'}(\mathbb{R})$ [11]. In order the present work be self-contained, we shall first list basic notations, definitions and some of the results from the literature.

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, real numbers and complex numbers respectively. Let $L^1(\mathbb{R})$, $C(\mathbb{R})$, $C_0(\mathbb{R})$, $C^\infty(\mathbb{R})$, $\mathscr{D}(\mathbb{R})$, $\mathscr{S}(\mathbb{R})$, $\mathscr{O}_M(\mathbb{R})$, $\mathscr{S}^{'}(\mathbb{R})$, $\mathscr{D}^{'}(\mathbb{R})$ and $\mathcal{E}^{'}(\mathbb{R})$ denote respectively the usual spaces of Lebesgue integrable functions, continuous functions, continuous functions vanishing at infinity, infinitely differentiable functions, $C^\infty$-functions with compact support, rapidly decreasing functions, infinitely differentiable functions with polynomial growth, tempered distributions, Schwartz distributions and compactly supported distributions. For the topology and the notion of convergence of the above mentioned spaces, we refer to [12, 13, 14, 26, 28].

For $f,g\in L^1(\mathbb{R})$, we define respectively the Fourier transform $\hat{f}\in C_0(\mathbb{R})$ of $f$ and the convolution $f*g\in L^1(\mathbb{R})$ by

$$\hat{f}(t)=\int_{\mathbb{R}} f(x)e^{-itx}\,dx \text{ and } (f*g)(t)=\int_{\mathbb{R}} f(t-y)g(y)\,dy,\ \ \forall t\in\mathbb{R}.$$

For $u\in\mathscr{D}^{'}(\mathbb{R})$ and $\varphi\in\mathscr{D}(\mathbb{R})$, we define

$$(u*\varphi)(x)=\langle u(y),\varphi(x-y)\rangle=u(\tau_x\check{\varphi}),\ \forall\, x\in\mathbb{R},$$

where $(\tau_x\varphi)(y)=\varphi(y-x)$ and $\check{\varphi}(y)=\varphi(-y)$, $\forall y\in\mathbb{R}$. In the same way we can also define $u*\varphi$ when $u\in\mathscr{S}^{'}(\mathbb{R})$ and $\varphi\in\mathscr{S}(\mathbb{R})$.

- Let $u\in\mathscr{D}'(\mathbb{R})$ and $a\in\mathbb{R}$. The derivative $u'$, the translation $\tau_a u$ and the dilatation $e_a u$ of $u$ are defined as
  (i) $\langle u'(x),\phi(x)\rangle=\langle u(x),-\phi'(x)\rangle,\forall\phi\in\mathscr{D}(\mathbb{R})$.
  (ii) $\langle(\tau_a u)(x),\phi(x)\rangle=\langle u(x),\phi(x+a)\rangle,\forall\phi\in\mathscr{D}(\mathbb{R})$.
  (iii) $\langle(e_a u)(x),\phi(x)\rangle=\langle u(x),e^{iax}\phi(x)\rangle,\forall\phi\in\mathscr{D}(\mathbb{R})$.
  Moreover, these operations are continuous from $\mathscr{D}'(\mathbb{R})$ into $\mathscr{D}'(\mathbb{R})$.

- The spaces $\mathscr{D}(\mathbb{R})$ and $\mathscr{S}(\mathbb{R})$ become an algebra with respect to the usual convolution $*$ as well as with respect to the point-wise multiplication.

- The usual Fourier transforms $\hat{}: \mathscr{D}(\mathbb{R}) \to \hat{\mathscr{D}}(\mathbb{R}) = \mathcal{Z}(\mathbb{R})$ and $\hat{}: \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ are linear, continuous, one-to-one and onto. Further, the inverse Fourier transforms are also continuous.

- The Fourier transform $\hat{}: \mathscr{D}^{'}(\mathbb{R}) \to \mathcal{Z}^{'}(\mathbb{R})$ is defined by $\langle \hat{v}, \phi \rangle = \langle v, \hat{\phi} \rangle$, $\forall \phi \in \mathscr{D}(\mathbb{R})$ is linear, continuous, one-to-one and onto. Here $\mathcal{Z}^{'}(\mathbb{R})$ is the dual space of $\mathcal{Z}(\mathbb{R})$ and will also be called as the space of ultra distributions.

- The Fourier transform $\hat{}: \mathscr{S}^{'}(\mathbb{R}) \to \mathscr{S}^{'}(\mathbb{R})$ is defined by $\langle \hat{v}, \phi \rangle = \langle v, \hat{\phi} \rangle$, $\forall \phi \in \mathscr{S}(\mathbb{R})$ is linear, continuous, one-to-one and onto.

- If $u \in \mathscr{D}'(\mathbb{R})\,(\mathscr{S}'(\mathbb{R}))$ and $\phi \in \mathscr{D}(\mathbb{R})\,(\mathscr{S}(\mathbb{R}))$ then $u*\phi \in C^\infty(\mathbb{R})\,(\mathscr{O}_M(\mathbb{R}))$ and $(u * \phi)\hat{} = \hat{\phi}\,\hat{u}$.

This paper is organized as follows. In Section 2, the definition of integration of distributions is defined and it is shown that the collection of all integrable distributions $\mathbb{E}_I$ is a subspace of $\mathscr{D}'(\mathbb{R})$. An example to show that the space $\mathbb{E}_I$ is properly larger than $L^1(\mathbb{R}) \cup \mathcal{E}^{'}(\mathbb{R})$ is also given in this section. It is also shown that this space $\mathbb{E}_I$ is closed under differentiation, translation, dilatation and convolution by elements of $\mathscr{D}(\mathbb{R})$. Further, the concept of locally integrable distributions is introduced. Just as every continuous function is locally integrable, it is nice to see that every distribution is locally integrable. In Section 3, the $\phi$-Fourier transformation is defined from $\mathbb{E}_I$ into $C(\mathbb{R})$, and some of its properties are studied.

## 2. Integration of Distributions

**Definition 2.1.** Let $\mathbb{E} = \{u \in \mathscr{D}^{'}(\mathbb{R}) /\, u * \varphi \in L^1(\mathbb{R}),\ \forall\, \varphi \in \mathscr{D}(\mathbb{R})\}$. For $u \in \mathbb{E}$, we define $\int u : \mathscr{D}(\mathbb{R}) \to \mathbb{C}$ as follows: $\langle \left(\int u\right)(x), \varphi(x) \rangle = \int_{\mathbb{R}} (u * \varphi)(x)\, dx$, $\forall\, \varphi \in \mathscr{D}(\mathbb{R})$, where $\int_{\mathbb{R}} (u * \varphi)(x)\, dx$ is the Lebesgue integral of $u * \varphi$ on $\mathbb{R}$.

It is easy to see that $\int u$ is linear on $\mathscr{D}(\mathbb{R})$.

**Definition 2.2.** A distribution $u \in \mathbb{E}$ is said to be an integrable distribution if $u * \varphi_n \to u * \varphi$ as $n \to \infty$ in $L^1(\mathbb{R})$ whenever $\varphi_n \to \varphi$ as $n \to \infty$ in $\mathscr{D}(\mathbb{R})$. We denote the set of all integrable distributions by $\mathbb{E}_I$.

*Remark* 2.3. *If $u$ is an integrable distribution, then it easily follows from Definition* 2.2 *that, $\int u \in \mathscr{D}^{'}(\mathbb{R})$.*

**Lemma 2.4.** *For $u_1, u_2, u \in \mathbb{E}_I$ and $\alpha \in \mathbb{C}$, $\int (u_1 + u_2) = \int u_1 + \int u_2$ and $\int (\alpha u) = \alpha \int u$ and the set of all integrable distributions $\mathbb{E}_I$ is a subspace of $\mathscr{D}'(\mathbb{R})$.*

*Proof.* Since $u_1, u_2 \in \mathbb{E}_I$, we have $u_1 * \phi_n \to u_1 * \phi$ and $u_2 * \phi_n \to u_2 * \phi$ as $n \to \infty$ in $L^1(\mathbb{R})$, for every sequence $(\phi_n)$ in $\mathscr{D}(\mathbb{R})$ converging to $\phi$. Then $(u_1 + u_2) * \phi_n = u_1 * \phi_n + u_2 * \phi_n \to u_1 * \phi + u_2 * \phi = (u_1 + u_2) * \phi$ as $n \to \infty$ in $L^1(\mathbb{R})$. Thus $u_1 + u_2 \in \mathbb{E}_I$. Similarly, by using $(\alpha u) * \phi = \alpha(u * \phi)$ and the linearity of the Lebesgue integral on $L^1(\mathbb{R})$, we get $\int (\alpha u) = \alpha \int u$ and

$\alpha u \in \mathbb{E}_I$, whenever $u \in \mathbb{E}_I$. Thus, $\mathbb{E}_I$ is a subspace of $\mathscr{D}'(\mathbb{R})$. Let $u_1, u_2 \in \mathbb{E}_I$ and $\phi \in \mathscr{D}(\mathbb{R})$. Then by Definition 2.1, we have

$$\begin{aligned}
\left\langle \left(\int (u_1+u_2)\right)(x), \phi(x)\right\rangle &= \int_{\mathbb{R}} ((u_1+u_2)*\phi)(x)\,dx \\
&= \int_{\mathbb{R}} (u_1*\phi + u_2*\phi)(x)\,dx \\
&= \int_{\mathbb{R}} (u_1*\phi)(x)\,dx + \int (u_2*\phi)(x)\,dx. \\
&= \left\langle \left(\int u_1\right)(x), \phi(x)\right\rangle + \left\langle \left(\int u_2\right)(x), \phi(x)\right\rangle \\
&= \left\langle \left(\int u_1 + \int u_2\right)(x), \phi(x)\right\rangle.
\end{aligned}$$

□

The following example shows that $\int u$ need not belong to $\mathbb{E}_I$ even though $u \in \mathbb{E}_I$.

**Example 2.5.** If $\delta$ is the Dirac's delta distribution, then for any $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle (\textstyle\int \delta)(x), \varphi(x)\rangle = \int_{\mathbb{R}} (\delta * \varphi)(x)\,dx = \int_{\mathbb{R}} \varphi(x)\,dx = \langle 1(x), \varphi(x)\rangle.$$

**Lemma 2.6.** *If $u \in \mathbb{E}_I$ and $a \in \mathbb{R}$, then $\tau_a u \in \mathbb{E}_I$ and $\int_{\mathbb{R}} (\tau_a u) = \int_{\mathbb{R}} u$.*

*Proof.* For each $\phi \in \mathscr{D}(\mathbb{R})$, from [13, Theorem 6.35] we get that $\tau_a u * \phi = \tau_a(u*\phi) = u * \tau_a \phi$, which belongs to $L^1(\mathbb{R})$. This shows that $\tau_a u \in \mathbb{E}$. Let $\phi_n \to \phi$ as $n \to \infty$ in $\mathscr{D}(\mathbb{R})$. Since the translation operator $\tau_a : \mathscr{D}(\mathbb{R}) \to \mathscr{D}(\mathbb{R})$ is continuous and $u \in \mathbb{E}_I$, we get that $\tau_a u * \phi_n = u * \tau_a \phi_n \to u * \tau_a \phi = \tau_a u * \phi$ as $n \to \infty$ in $L^1(\mathbb{R})$. This shows that $\tau_a u \in \mathbb{E}_I$.

Further, $\langle (\int_{\mathbb{R}} \tau_a u)(x), \phi(x)\rangle = \int_{\mathbb{R}} ((\tau_a u) * \phi)(x)\,dx = \int_{\mathbb{R}} \tau_a (u*\phi)(x)\,dx = \int_{\mathbb{R}} (u*\phi)(x)\,dx = \langle (\int_{\mathbb{R}} u)(x), \phi(x)\rangle$, $\forall \phi \in \mathscr{D}(\mathbb{R})$. □

**Proposition 2.7.** $L^1(\mathbb{R}) \cup \mathcal{E}'(\mathbb{R}) \subset \mathbb{E}_I$.

*Proof.* Assume that $\varphi_n \to \varphi$ as $n \to \infty$ in $\mathscr{D}(\mathbb{R})$.

- Let $f \in L^1(\mathbb{R})$. We shall show that $f * \varphi_n \to f * \varphi$ as $n \to \infty$ in $L^1(\mathbb{R})$. Let $[a,b]$ be a compact interval of $\mathbb{R}$ which contains the support of $(\varphi_n - \varphi)$, $\forall n \in \mathbb{N}$. Then

  $$\begin{aligned}
  \|f * (\varphi_n - \varphi)\|_1 &\leq \|f\|_1 \, \|\varphi_n - \varphi\|_1 \\
  &\leq \|f\|_1 \, (b-a) \sup_{x \in [a,b]} |\varphi_n(x) - \varphi(x)| \to 0 \text{ as } n \to \infty.
  \end{aligned}$$

  This proves that $f \in \mathbb{E}_I$.

- Let $f \in \mathcal{E}'(\mathbb{R})$ and let $K_f$ be the support of $f$. Choose a compact set $K \in \mathbb{R}$ such that $K_f +$ supp $(\varphi_n - \varphi) \subset K$. Since the mapping $\psi \mapsto f * \psi$ is continuous linear from $\mathscr{D}(\mathbb{R})$ into $C^\infty(\mathbb{R})$ [13, Theorem 6.33], we have $f * (\varphi_n - \varphi) \to 0$ as $n \to \infty$ uniformly on each compact subset of $\mathbb{R}$. Hence for a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

  $$\sup_{x \in K} |f * (\varphi_n - \varphi)(x)| < \epsilon, \ \forall n \geq N.$$

  Hence

  $$\int_{\mathbb{R}} |(f * (\varphi_n - \varphi))(x)|\,dx = \int_K |(f * (\varphi_n - \varphi))(x)|\,dx \leq \epsilon\, m(K),$$

  where $m(K)$ is the Lebesgue measure of $K$. Therefore $f \in \mathbb{E}_I$.

$\square$

The following example shows that the inclusion in the above proposition is proper.

**Example 2.8.** *Consider the bounded $C^\infty$-function defined by*

$$f(x) = e^{e^{-x^2}}, \forall x \in \mathbb{R}.$$

*Since $f \in \mathscr{O}_M(\mathbb{R})$ and the Fourier transform on $\mathscr{S}'(\mathbb{R})$ is bijective, we can choose $u \in \mathscr{S}'(\mathbb{R})$ such that $\hat{u} = f$. We first claim that*

$$u \notin L^1(\mathbb{R}) \cup \mathcal{E}'(\mathbb{R}).$$

*Since $f(x) \to 1$ as $x \to \pm\infty$, we have $f \notin C_0(\mathbb{R})$ and hence $u \notin L^1(\mathbb{R})$. Also, it is true that $f$ can be extended as an entire function $f(z) = e^{e^{-z^2}}$, $\forall z \in \mathbb{C}$. But when $z$ is restricted to the imaginary axis, we have $|f(z)| = |f(iy)| = |e^{e^{y^2}}|$ and hence there are no real $\gamma$, $N \in \mathbb{N}$ and $r \geq 0$ such that the inequality*

$$|f(iy)| \leq \gamma(1 + |iy|)^N e^{r|Im(iy)|}$$

*holds for all $y \in \mathbb{R}$. Hence, by the Paley-Wiener theorem [13], $u \notin \mathcal{E}'(\mathbb{R})$.*

*We shall show that this $u \in \mathbb{E}_I$. Since $f$ and all its derivatives belong to $\mathscr{O}_M(\mathbb{R})$, we have*

$$f\psi \in \mathscr{S}(\mathbb{R}) \text{ for all } \psi \in \mathscr{S}(\mathbb{R})$$

*and also*

$$f\psi_n \to f\psi \text{ as } n \to \infty \text{ in } \mathscr{S}(\mathbb{R}), \text{ whenever } \psi_n \to \psi \text{ as } n \to \infty \text{ in } \mathscr{S}(\mathbb{R}).$$

*This shows that $u * \varphi = (f\check{\hat{\varphi}})\check{} \in \mathscr{S}(\mathbb{R})$ for every $\varphi \in \mathscr{D}(\mathbb{R})$ and*

$$u * \varphi_n \to u * \varphi \text{ as } n \to \infty \text{ in } \mathscr{S}(\mathbb{R}), \text{ whenever } \varphi_n \to \varphi \text{ as } n \to \infty \text{ in } \mathscr{D}(\mathbb{R}).$$

*Hence $u * \varphi_n \to u * \varphi$ as $n \to \infty$ in $L^1(\mathbb{R})$. Thus $u \in \mathbb{E}_I$.*

**Proposition 2.9** (Consistency)**.**

*(i) If $f \in L^1(\mathbb{R})$ then $\langle (\int f)(x), \varphi(x)\rangle = \langle \int_{\mathbb{R}} f(x)dx, \varphi(x)\rangle$, $\forall \varphi \in \mathscr{D}(\mathbb{R})$.*

*(ii) If $f \in \mathcal{E}'(\mathbb{R})$ then $\langle (\int f)(x), \varphi(x)\rangle = \langle (\langle f(x), 1(x)\rangle)(s), \varphi(s)\rangle$, $\forall \varphi \in \mathscr{D}(\mathbb{R})$.*

*Proof.* $(i)$ Let $f \in L^1(\mathbb{R})$. For an arbitrary $\varphi \in \mathscr{D}(\mathbb{R}) \subset L^1(\mathbb{R})$, we have

$$\begin{aligned} \langle (\textstyle\int f)(x), \varphi(x)\rangle &= \int_{\mathbb{R}} (f * \varphi)(x)\, dx &&= \left(\int_{\mathbb{R}} f(x)\, dx\right)\left(\int_{\mathbb{R}} \varphi(x)\, dx\right) \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x)\, dx\right) \varphi(x)\, dx &&= \langle \int_{\mathbb{R}} f(x)dx, \varphi(x)\rangle. \end{aligned}$$

This completes the proof of $(i)$.

$(ii)$ Let $f \in \mathcal{E}'(\mathbb{R})$. Let $\varphi \in \mathscr{D}(\mathbb{R})$. Choose a compact set $K$ such that supp $f$, supp $\varphi$ and supp $(f*\varphi) \subset K$ (note that this is possible since $f*\varphi \in \mathscr{D}(\mathbb{R})$). Now

$$\begin{aligned} \langle (\textstyle\int f)(x), \varphi(x)\rangle &= \int_{\mathbb{R}} (f * \varphi)(x)\, dx &&= \int_{\mathbb{R}} (f * \varphi)(x) 1(y - x)\, dx \\ &= ((f * \varphi) * 1)(y) &&= (f * (\varphi * 1))(y) \\ &= \langle f(x), (\varphi * 1)(y - x)\rangle &&= \left\langle f(x), \int_{\mathbb{R}} \varphi(y - x - t) 1(t)\, dt\right\rangle \\ &= \left\langle f(x), \left(\int_{\mathbb{R}} \varphi(s)\, ds\right)(x)\right\rangle &&= \langle f(x), 1(x)\rangle \left(\int_{\mathbb{R}} \varphi(s)\, ds\right) \\ &= \int_{\mathbb{R}} \langle f(x), 1(x)\rangle \varphi(s)\, ds &&= \langle (\langle f(x), 1(x)\rangle)(s), \varphi(s)\rangle. \end{aligned}$$

$\square$

**Lemma 2.10.** *If $u \in \mathbb{E}_I$, then its distributional derivative $u' \in \mathbb{E}_I$ and* $\langle \left(\int u'\right)(x), \varphi(x)\rangle = \langle -\left(\int u\right)'(x), \varphi(x)\rangle,\ \forall \varphi \in \mathscr{D}(\mathbb{R})$.

*Proof.* Let $u \in \mathbb{E}_I$. Since the differential operator $\varphi \mapsto \varphi'$ is continuous from $\mathscr{D}(\mathbb{R})$ into $\mathscr{D}(\mathbb{R})$, we get $u' \in \mathbb{E}_I$. By using the fact that $\int u \in \mathscr{D}'(\mathbb{R})$ and the definition of $\int u$, for each $\varphi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned} \left\langle \left(\int u'\right)(x), \varphi(x)\right\rangle &= \int_{\mathbb{R}} (u' * \varphi)(x)\,dx &&= \int_{\mathbb{R}} (u * \varphi')(x)\,dx \\ &= \left\langle \left(\int u\right)(x), \varphi'(x)\right\rangle &&= \left\langle -\left(\int u\right)'(x), \varphi(x)\right\rangle. \end{aligned}$$
□

**Corollary 2.11.** *For every $f \in L^1(\mathbb{R}) \cup \mathcal{E}'(\mathbb{R})$, $\int f' = 0$.*

*Proof.* From Proposition 2.9, we have

$$\langle \int f, \phi\rangle = \begin{cases} \langle \int_{\mathbb{R}} f(x)dx, \phi(x)\rangle, & \text{if } f \in L^1(\mathbb{R}) \\ \langle\langle f(x), 1(x)\rangle, \phi(x)\rangle, & \text{if } f \in \mathcal{E}'(\mathbb{R}). \end{cases}$$

In both cases, $\int f$ represents a regular distribution given by a constant and hence $(\int f)' = 0$. Now, the proof follows from Lemma 2.10. □

**Theorem 2.12.** *If $u \in \mathscr{S}'(\mathbb{R})$ such that $\hat{u}^{(k)}(x) \in \mathscr{O}_M(\mathbb{R})$ for all $k = 0, 1, 2, \ldots$, then $u \in \mathbb{E}_I$.*

*Proof.* Let $u \in \mathscr{S}'(\mathbb{R})$ such that $\hat{u}^{(k)}(x) \in \mathscr{O}_M(\mathbb{R})$, $\forall k = 0, 1, 2, \ldots$. Let $\varphi_n \to \varphi$ as $n \to \infty$ in $\mathscr{D}(\mathbb{R})$. Then $\hat{\varphi}_n \to \hat{\varphi}$ as $n \to \infty$ in $\mathscr{S}(\mathbb{R})$. Now $\hat{\varphi}_n\hat{u}$, $\hat{\varphi}\hat{u} \in \mathscr{S}(\mathbb{R})$ for $n = 1, 2, \ldots$ and also $\hat{\varphi}_n\hat{u} \to \hat{\varphi}\hat{u}$ as $n \to \infty$ in $\mathscr{S}(\mathbb{R})$ (here we have used the Leibniz formula and the fact that $\hat{u}^{(k)}(x) \in \mathscr{O}_M(\mathbb{R})$ for $k = 0, 1, 2, \ldots$). This will imply that $(\hat{\varphi}_n\hat{u})\check{} \to (\hat{\varphi}\hat{u})\check{}$ as $n \to \infty$ in $\mathscr{S}(\mathbb{R})$. That is, $u * \varphi_n \to u * \varphi$ as $n \to \infty$ in $\mathscr{S}(\mathbb{R})$ and hence in $L^1(\mathbb{R})$. This shows that $u \in \mathbb{E}_I$. □

We shall denote the space of all $u$ satisfying the hypothesis of the above theorem by $\mathcal{C}$.

**Definition 2.13.** Let $H \subset \mathbb{R}$ be Lebesgue measurable and $u \in \mathscr{D}'(\mathbb{R})$. We define $\int_H u$ as follows: $\langle \left(\int_H u\right)(x), \varphi(x)\rangle = \int_H (u*\varphi)(x)\,dx$, whenever the right hand side integral exists in the Lebesgue sense for every $\varphi \in \mathscr{D}(\mathbb{R})$.

Analogous to Definition 2.2, we shall define the following.

**Definition 2.14.** A distribution $u$ is said to be Lebesgue integrable over $H$ if $\langle \int_H u, \varphi\rangle$ exists for every $\varphi \in \mathscr{D}(\mathbb{R})$ and $u * \varphi_n \to u * \varphi$ as $n \to \infty$ in $L^1(H)$, whenever $\varphi_n \to \varphi$ as $n \to \infty$ in $\mathscr{D}(\mathbb{R})$.

*Remark* 2.15. *From Theorem 6.33 [13, p.173], we have if $u \in \mathscr{D}'(\mathbb{R})$, then $u * \varphi_n \to u * \varphi$ as $n \to \infty$ in $C^\infty(\mathbb{R})$, whenever $\varphi_n \to \varphi$ as $n \to \infty$ in $\mathscr{D}(\mathbb{R})$. Hence $u * \varphi_n \to u * \varphi$ uniformly on each compact subset of $\mathbb{R}$. This in turn implies that for each $u \in \mathscr{D}'(\mathbb{R})$, and for each compact set $K \subset \mathbb{R}$, we have $\int_K u \in \mathscr{D}'(\mathbb{R})$. Hence we prefer to say that* every distribution is locally integrable.

**Lemma 2.16.** *If $H$ is any bounded measurable subset of $\mathbb{R}$, then $\int_H u \in \mathscr{D}'(\mathbb{R})$, for every $u \in \mathscr{D}'(\mathbb{R})$.*

*Proof.* Since $H$ is bounded, there exists $M > 0$ such that $H \subset [-M, M]$. Now the proof follows easily from Theorem 6.33 [13, p. 173] and the fact that

$$\int_H |(u * \varphi)(x)|\, dx \leq \int_{-M}^{M} |(u * \varphi)(x)|\, dx, \ \ \forall \varphi \in \mathscr{D}(\mathbb{R}).$$

□

**Theorem 2.17.** *A necessary condition for $0 \neq u \in \mathbb{E}_I$ is that $u' \neq 0$.*

*Proof.* For $u \in \mathscr{D}'(\mathbb{R})$, we shall first prove that if $u' = 0$ then $u$ must be a regular distribution representing some constant function. Since non-zero constant functions are not in $\mathbb{E}_I$, the proof follows immediately. Assume that $u \in \mathscr{D}'(\mathbb{R})$ such that $u' = 0$. Fix $a \in \mathbb{R}$. Then for $y > a$ and $\varphi \in \mathscr{D}(\mathbb{R})$, by using $(u * \varphi)' = (u' * \varphi)$, we get

$$(u * \varphi)(y) - (u * \varphi)(a) = \int_a^y (u * \varphi)'(x)\, dx = \int_a^y \langle u'(t), \varphi(x - t)\rangle\, dx = 0.$$

This shows that $(u * \varphi)(y) = (u * \varphi)(a) = \alpha_\varphi$ (a constant depending on $\varphi$). Since $u * \varphi$ is a $C^\infty$-function we have

$$(u * \varphi)(y) = (u * \varphi)(a) = \alpha_\varphi, \ \text{for all } y \in \mathbb{R}.$$

Now choose a sequence $(\varphi_n)$ in $\mathscr{D}(\mathbb{R})$ such that $u * \varphi_n \to u$ as $n \to \infty$ in $\mathscr{D}'(\mathbb{R})$. Then for every $\eta \in \mathscr{D}(\mathbb{R})$, for a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|\langle \alpha_{\varphi_n}(t), \eta(t)\rangle - \langle u(t), \eta(t)\rangle| = |\langle (u * \varphi_n)(t), \eta(t)\rangle - \langle u(t), \eta(t)\rangle| < \epsilon, \ \forall n \geq N.$$

This proves that $\alpha_{\varphi_n} \to u$ as $n \to \infty$ in $\mathscr{D}'(\mathbb{R})$. Now, for an $\eta$ with $\int_{\mathbb{R}} \eta(x)\, dx = 1$, we have

$$\begin{aligned} |\alpha_{\varphi_n} - \alpha_{\varphi_m}| &= |\int_{\mathbb{R}} \alpha_{\varphi_n} \eta(x)\, dx - \int_{\mathbb{R}} \alpha_{\varphi_m} \eta(x)\, dx| \\ &= |\langle \alpha_{\varphi_n}(t), \eta(t)\rangle - \langle \alpha_{\varphi_m}(t), \eta(t)\rangle| \\ &\leq |\langle \alpha_{\varphi_n}(t), \eta(t)\rangle - \langle u(t), \eta(t)\rangle| + |\langle \alpha_{\varphi_m}(t), \eta(t)\rangle - \langle u(t), \eta(t)\rangle| \\ &\quad \to 0 \ \text{as both } n \text{ and } m \text{ tend to } \infty. \end{aligned}$$

Hence, there exists some $\alpha \in \mathbb{C}$ such that $\alpha_{\varphi_n} \to \alpha$ as $n \to \infty$ in $\mathbb{C}$ and hence in $\mathscr{D}'(\mathbb{R})$ which in turn implies that $u = \alpha$. □

*Remark* 2.18. *By using the above result, we see that the converse of Corollary 2.11 does not hold. For example, one may take $u$ as a regular distribution representing a constant function.*

## 3. $\phi$-Fourier transform and its properties

Throughout this section, we fix $\phi \in \mathscr{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \phi(x)\, dx = 1$.

**Theorem 3.1.** *If $u \in \mathbb{E}_I$, then $e_t u \in \mathbb{E}_I$ for all $t \in \mathbb{R}$.*

*Proof.* Let $u \in \mathbb{E}_I$ and let $t \in \mathbb{R}$. For $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\begin{aligned}(e_t u * \varphi)(x) &= \langle e^{ity}u(y), \varphi(x-y)\rangle &&= \langle u(y), e^{ity}\varphi(x-y)\rangle \\ &= e^{itx}\langle u(y), e^{-it(x-y)}\varphi(x-y)\rangle &&= e^{itx}\langle u(y), (e_{-t}\varphi)(x-y)\rangle \\ &= e^{itx}(u * e_{-t}\varphi)(x).\end{aligned}$$

Let $\varphi_n \to \varphi$ as $n \to \infty$ in $\mathscr{D}(\mathbb{R})$. Using the above equality and the fact that for each $s \in \mathbb{R}$, the mapping $\psi \mapsto e_s\psi$ is continuous from $\mathscr{D}(\mathbb{R})$ into $\mathscr{D}(\mathbb{R})$, we get

$$e_t u * \varphi_n = u * e_t\varphi_n \to u * e_t\varphi = e_t u * \varphi \text{ as } n \to \infty.$$

This shows that $e_t u \in \mathbb{E}_I$, for all $t \in \mathbb{R}$. □

**Definition 3.2.** For $u \in \mathbb{E}_I$, we define the $\phi$-Fourier transform $F_\phi(u)$ of $u$ as $[F_\phi(u)](t) = \langle(\int e_{-t}u)(x), \phi(x)\rangle = \int_{\mathbb{R}} e^{-itx}(u * e_t\phi)(x)\,dx$ for all $t \in \mathbb{R}$.

**Theorem 3.3.** *If $u \in \mathbb{E}_I$, then $F_\phi(u) \in C(\mathbb{R})$.*

*Proof.* Let $t_n \to t$ as $n \to \infty$ in $\mathbb{R}$. We shall show that $[F_\phi(u)](t_n) \to [F_\phi(u)](t)$ as $n \to \infty$. Consider

$$|[F_\phi(u)](t_n) - [F_\phi(u)](t)| = |(u * e_{t_n}\phi)\hat{}(t_n) - (u * e_t\phi)\hat{}(t)|\,. \tag{3.1}$$

Now we claim that

$$e_{t_n}\phi \to e_t\phi \text{ as } n \to \infty \text{ in } \mathscr{D}(\mathbb{R}).$$

Once this is done, then by using the fact that $u \in \mathbb{E}_I$, we have $u * e_{t_n}\phi \to u * e_t\phi$ as $n \to \infty$ in $L^1(\mathbb{R})$. This will imply that $(u * e_{t_n}\phi)\hat{} \to (u * e_t\phi)\hat{}$ as $n \to \infty$ in $C_0(\mathbb{R})$ (since the Fourier transform $\hat{}: L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is continuous) and hence

$$(u * e_{t_n}\phi)\hat{}(y) \to (u * e_t\phi)\hat{}(y) \text{ uniformly on } \mathbb{R} \text{ as } n \to \infty.$$

Finally, from a result in [12, p.166], we have

$$(u * e_{t_n}\phi)\hat{}(y_n) \to (u * e_t\phi)\hat{}(y) \text{ as } n \to \infty,$$

whenever $y_n \to y$ as $n \to \infty$ in $\mathbb{R}$. If we take $y_n = t_n$, then the right-hand side of (3.1) tends to 0 as $n \to \infty$, proving the result.

Let $k$ be any non-negative integer and let $\psi_t(y) = e^{ity}\phi(y)$. Using mean value theorem and Leibniz formula, we have

$|D^k[e^{it_n y}\phi(y) - e^{ity}\phi(y)]|$

$$\begin{aligned}
&|D^k[e^{it_n y}\phi(y) - e^{ity}\phi(y)]| \\
=&|D^k[e^{ity}\phi(y)\left(e^{i(t_n-t)y}-1\right)]| \\
=&|D^k[\psi_t(y)\left(e^{i(t_n-t)y}-1\right)]| \\
=&\left|\sum_{m=0}^{k} C_{m,k} D^{(k-m)}\psi_t(y)\, D^{(m)}\left(e^{i(t_n-t)y}-1\right)\right| \\
\leq&\sum_{m=0}^{k} C_{m,k}\left|D^{(k-m)}\psi_t(y)\right|\left|D^{(m)}\left(e^{i(t_n-t)y}-1\right)\right| \\
=&\left|C_{0,k}D^{(k)}\psi_t(y)\right|\left|\left(e^{i(t_n-t)y}-1\right)\right| \\
&+\sum_{m=1}^{k} C_{m,k}\left|D^{(k-m)}\psi_t(y)\right|\left|D^{(m)}\left(e^{i(t_n-t)y}-1\right)\right| \\
\leq&\left|C_{0,k}D^{(k)}\psi_t(y)\right| |t_n-t|\,|y|\,|e^{iys_n}| \\
&+\sum_{m=1}^{k} C_{m,k}\left|D^{(k-m)}\psi_t(y)\right| |(t_n-t)|^m\,|e^{iy(t_n-t)}| \\
\leq& M_1|t_n-t| + M_2 k\,|t_n-t|^m,
\end{aligned}$$

where $s_n = \lambda t_n + (1-\lambda)t$ for some $\lambda \in [0,1]$ depending on $n$, $M_1 = \sup_{y\in\mathbb{R}}\left|D^{(k)}\psi_t(y)\right| |y|$ and $M_2 = \max_{1\leq m\leq k}\left(\sup_{y\in\mathbb{R}} C_{m,k}\left|D^{(k-m)}\psi_t(y)\right|\right)$. (Note that here we used the fact that $\phi \in \mathscr{D}(\mathbb{R})$). Thew above arguments imply that

$$\|e_{t_n}\phi - e_t\phi\|_k \leq M_1|t_n-t| + M_2 k\,|t_n-t|^m \to 0 \text{ as } n\to\infty.$$

Thus $e_{t_n}\phi \to e_t\phi$ as $n\to\infty$ in $\mathscr{D}(\mathbb{R})$ and hence our claim. □

**Proposition 3.4.** *If $f \in L^1(\mathbb{R})$, then $\hat{f}(t) = [F_\phi(f)](t)$, for all $t\in\mathbb{R}$.*

*Proof.* Let $f \in L^1(\mathbb{R})$. Then for $t\in\mathbb{R}$, we have

$$[F_\phi(f)](t) = \langle(\textstyle\int e_{-t}f)(x), \phi(x)\rangle = \displaystyle\int_{\mathbb{R}} (e_{-t}f * \phi)(x)\,dx.$$

Since $e_{-t}f \in L^1(\mathbb{R})$, we have

$$[F_\phi(f)](t) = \left(\int_{\mathbb{R}} (e_{-t}f)(s)\,ds\right)\left(\int_{\mathbb{R}} \phi(s)\,ds\right) = \int_{\mathbb{R}} (e_{-t}f)(s)\,ds = \hat{f}(t).$$

□

**Proposition 3.5.** *If $f \in \mathcal{E}'(\mathbb{R})$, then $\hat{f}(t) = [F_\phi(f)](t)$, for all $t\in\mathbb{R}$.*

*Proof.* Let $f \in \mathcal{E}'(\mathbb{R})$. Then for any $t \in \mathbb{R}$, we have $e_{-t}f \in \mathcal{E}'(\mathbb{R})$ and supp $e_{-t}f =$ supp $f$. Also we have

$$\begin{aligned}
[F_\phi(f)](t) &= \langle (\textstyle\int e_{-t}f)(x), \phi(x)\rangle = \displaystyle\int_{\mathbb{R}} (e_{-t}f * \phi)(x)\, dx \\
&= \int_{\mathbb{R}} (e_{-t}f * \phi)(x) 1(y-x)\, dx = ((e_{-t}f * \phi) * 1)(y) \ (\forall y \in \mathbb{R}) \\
&= (e_{-t}f * (\phi * 1))(y) = \langle (e_{-t}f)(s), (\phi * 1)(y-s)\rangle \\
&= \langle (e_{-t}f)(s), \textstyle\int_{\mathbb{R}} \phi(y-s-v) 1(v)\, dv\rangle = \langle (e_{-t}f)(s), \int_{\mathbb{R}} \phi(v)\, dv\rangle \\
&= \langle (e_{-t}f)(s), 1(s)\rangle = \langle f(s), e^{-its}\rangle = \hat{f}(t).
\end{aligned}$$

This completes the proof. □

*Remark* 3.6. *Note that the above result is also true for every* $t \in \mathbb{C}$.

**Theorem 3.7.** *Let* $u \in \mathbb{E}_I$. *Then for any* $a \in \mathbb{R}$ *and* $\psi \in \mathscr{D}(\mathbb{R})$, *we have*
*(i)* $[F_\phi(\tau_a u)](t) = e^{-iat}[F_\phi(u)](t)$.
*(ii)* $[F_\phi(e_a u)](t) = [F_\phi(u)](t-a)$.
*(iii)* $[F_\phi(u * \psi)](t) = [F_\phi(u)](t)\,[F_\phi(\psi)](t)$.
*(iv)* $[F_\phi(u')] = (it)[F_\phi(u)](t)$.

*Proof.* For any $t \in \mathbb{R}$, we have

*(i)*

$$\begin{aligned}
[F_\phi(\tau_a u)](t) &= \textstyle\int_{\mathbb{R}} (e_{-t}\tau_a u * \phi)(s)\, ds = \int_{\mathbb{R}} \langle (e_{-t}\tau_a u)(y), \phi(s-y)\rangle\, ds \\
&= \textstyle\int_{\mathbb{R}} \langle (\tau_a u)(y), e^{-ity}\phi(s-y)\rangle\, ds \\
&= \textstyle\int_{\mathbb{R}} \langle u(y), e^{-it(y+a)}\phi(s+a-y)\rangle\, ds \\
&= e^{-ita} \textstyle\int_{\mathbb{R}} (e_{-t}u * \phi)(s+a)\, ds = e^{-ita} \int_{\mathbb{R}} (e_{-t}u * \phi)(s)\, ds \\
&= e^{-ita}[F_\phi(u)](t).
\end{aligned}$$

*(ii)*

$$\begin{aligned}
[F_\phi(e_a u)](t) &= \textstyle\int_{\mathbb{R}} (e_{-t}e_a u * \phi)(s)\, ds = \int_{\mathbb{R}} \langle u(y), e^{-i(t-a)y}\phi(s-y)\rangle\, ds \\
&= \textstyle\int_{\mathbb{R}} (e_{a-t}u * \phi)(s)\, ds = [F_\phi(u)](t-a).
\end{aligned}$$

*(iii)*

$$\begin{aligned}
[F_\phi(u * \psi)](t) &= \textstyle\int_{\mathbb{R}} (e_{-t}(u * \psi) * \phi)(x)\, dx = \int_{\mathbb{R}} e^{-itx}(u * \psi * e_t\phi)(x)\, dx \\
&= \textstyle\int_{\mathbb{R}} e^{-itx}(u * e_t\phi * \psi)(x)\, dx = (u * e_t\phi)\hat{}(t)\, \hat{\psi}(t) \\
&= \left(\textstyle\int_{\mathbb{R}} e^{-itx}(u * e_t\phi)(x)\, dx\right)\ \hat{\psi}(t) \\
&= \textstyle\int_{\mathbb{R}} (e_t u * \phi)(x)\, dx\ \hat{\psi}(t) = [F_\phi(u)](t)\,[F_\phi(\psi)](t).
\end{aligned}$$

*(iv)*

$$\begin{aligned}
[F_\phi(u')](t) &= \textstyle\int_{\mathbb{R}} (e_{-t}u' * \phi)(x)\, dx = \int_{\mathbb{R}} e^{-itx}(u' * e_t\phi)(x)\, dx \\
&= \textstyle\int_{\mathbb{R}} e^{-itx}(u * e_t\phi)'(x)\, dx = [(u * e_t\phi)'(x)]\hat{}\ (t) \\
&= (it)[(u * e_t\phi)(x)]\hat{}\ (t). = (it)[F_\phi(u)](t).
\end{aligned}$$

□

**Proposition 3.8.** *Let* $\mathcal{A} = \{u \in \mathbb{E}_I / \hat{u} \in C(\mathbb{R})\}$, *where* $\hat{u}$ *denotes the original distributional Fourier transform of* $u$. *Then,* $[F_\phi(u)](t) = \hat{u}(t)$ *for all* $t \in \mathbb{R}$.

*Proof.* Let $u \in \mathcal{A}$, and let $t \in \mathbb{R}$. Now for $\psi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned}\langle (u * e_t\phi)\hat{}, \psi\rangle &= \langle u * e_t\phi, \hat{\psi}\rangle = \langle u, \hat{\psi} * (e_t\phi)\check{}\rangle \\ &= \langle \hat{u}, [\hat{\psi} * (e_t\phi)\check{}]\check{\hat{}}\rangle = \langle \hat{u}, \check{\hat{\check{\hat{\psi}}}}(e_t\phi)\check{\hat{}}\rangle \\ &= \langle \hat{u}, \psi(e_t\phi)\hat{}\rangle = \langle (e_t\phi)\hat{}\hat{u}, \psi\rangle.\end{aligned}$$

Since both $(u * e_t\phi)\hat{}$ and $(e_t\phi)\hat{}\hat{u}$ are continuous functions we must have

$$(u * e_t\phi)\hat{}(s) = (e_t\phi)\hat{}(s)\,\hat{u}(s) \text{ for any } s \in \mathbb{R}.$$

Thus we have

$$[F_\phi(u)](t) = (u * e_t\phi)\hat{}(t) = \hat{u}(t)(e_t\phi)\hat{}(t) = \hat{u}(t)\hat{\phi}(0) = \hat{u}(t). \tag{3.2}$$

This completes the proof. □

*Remark* 3.9. *In the previous proposition, the continuity of $\hat{u}$ is used to obtain (3.2) for all $t \in \mathbb{R}$. But (3.2) holds true for almost all $t \in \mathbb{R}$ if we assume $\hat{u}$ as a regular distribution instead $\hat{u} \in C(\mathbb{R})$.*

**Theorem 3.10.** *Let $u \in \mathbb{E}_I \cap \mathscr{S}'(\mathbb{R})$. Then $([F_\phi(u)](t), \psi(t)) = (\hat{u}(t), \psi(t))$ for all $\psi \in \mathscr{D}(\mathbb{R})$.*

**Proof:** Let $u \in \mathbb{E}_I \cap \mathscr{S}'(\mathbb{R})$ and let $\psi \in \mathscr{D}(\mathbb{R})$ with $supp\,\psi = K$. Consider

$$\langle [F_\phi(u)](t), \psi(t)\rangle = \int_K \psi(t)[F_\phi(u)](t)\,dt = \int_K \psi(t)\left(\int_{\mathbb{R}} (e_{-t}u * \phi)(x)\,dx\right) dt. \tag{3.3}$$

Put $H(x,t) = \psi(t)(e_{-t}u * \phi)(x) = \psi(t)e^{-itx}(u * e_t\phi)(x)$. Then $H : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is continuous and hence it is Borel measurable (since (i) $(x,t) \mapsto t \mapsto \psi(t)$ is continuous, (ii) $(x,t) \mapsto -itx \mapsto e^{-itx}$ is continuous and (iii) $(x,t) \mapsto e^{ity}\phi(x-y) \mapsto \langle u(y), e^{ity}\phi(x-y)\rangle$ is continuous).

Next we claim that one of the iterated integrals, namely

$$\int_{\mathbb{R}}\left(\int_K |\psi(t)\,(e_{-t}u * \phi)(x)|\,dt\right) dx \tag{3.4}$$

is finite. Define $G(x,t) = |\psi(t)|\,|(e_{-t}u * \phi)(x)| = |\psi(t)|\,|(u * e_t\phi)(x)|$. Then for each real $x$, $G(x,t)$ as a function of $t$ is continuous on $\mathbb{R}$ (Note that $t \mapsto |\psi(t)|$ is continuous and the mapping $\eta \mapsto u * \eta$ is continuous from $\mathscr{D}$ into $C^\infty(\mathbb{R})$). Thus there exists $t_0 \in K$ such that $G(x,t) \le G(x,t_0)$ for all $t \in K$. i.e.,

$$|\psi(t)|\,|(u * e_t\phi)(x)| \le |\psi(t_0)|\,|(u * e_{t_0}\phi)(x)|, \quad \forall\, t \in K.$$

Using this fact in (3.4), we get that

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\int_K |\psi(t)\,(e_{-t}u*\phi)(x)|\,dt\right)dx\\
&\quad\le \int_{\mathbb{R}}\left(\int_K |\psi(t_0)\,(u*e_{t_0}\phi)(x)|\,dt\right)dx\\
&\quad\le M\int_{\mathbb{R}} |(u*e_{t_0}\phi)(x)|\,dx \text{ with } M=|\psi(t_0)|m(K)\\
&\quad\le M\int_{\mathbb{R}} |(e_{-t_0}u*\phi)(x)|\,dx\\
&\quad< +\infty,\ \text{since } u\in\mathbb{E}_I.
\end{aligned}$$

Thus, we can change the order of the integration in (3.3) to get

$$\begin{aligned}
&\langle [F_\phi(u)](t),\psi(t)\rangle\\
&\quad= \int_{\mathbb{R}}\left(\int_K \psi(t)\,(e_{-t}u*\phi)(x)\,dt\right)dx\\
&\quad= \int_{\mathbb{R}}\left(\int_K \langle u(y),\psi(t)e^{-ity}\phi(x-y)\rangle\,dt\right)dx \text{ (using Pettis integral)}\\
&\quad= \int_{\mathbb{R}} \langle u(y),\int_K \psi(t)e^{-ity}\phi(x-y)\,dt\rangle\,dx
\end{aligned}$$

$$= \int_{\mathbb{R}} \langle u(y),\hat{\psi}(y)\phi(x-y)\rangle\,dx. \tag{3.5}$$

Now we claim that $\int_{\mathbb{R}}\langle u(y),\hat{\psi}(y)\phi(x-y)\rangle\,dx = \langle u(y),\int_{\mathbb{R}}\hat{\psi}(y)\phi(x-y)\,dx\rangle = \langle u(y),\hat{\psi}(y)\rangle$, since $\int_{\mathbb{R}}\phi(x)dx = 1$.

Choose a sequence $\psi_n\in\mathscr{S}$ such that $\hat{\psi}_n\in\mathscr{D}$ and $\hat{\psi}_n\to\hat{\psi}$ as $n\to\infty$ in $\mathscr{S}$. Using Pettis integral for each $n$ we get,

$$\int_{\mathbb{R}}\langle u(y),\hat{\psi}_n(y)\phi(x-y)\rangle\,dx = \langle u(y),\int_{\mathbb{R}}\hat{\psi}_n(y)\phi(x-y)\,dx\rangle = \langle u(y),\hat{\psi}_n(y)\rangle.$$

(Note that for each fixed $y\in\mathbb{R}$, $supp\,\hat{\psi}_n(y)\phi(x-y)$ is compact, and hence the above Pettis integral exists). Taking limit as $n\to\infty$ and using the fact that $u\in\mathscr{S}'$ we get that

$$\lim_{n\to\infty}\int_{\mathbb{R}}\langle u(y),\hat{\psi}_n(y)\phi(x-y)\rangle\,dx = \lim_{n\to\infty}\langle u(y),\hat{\psi}_n(y)\rangle = \langle u(y),\psi(y)\rangle. \tag{3.6}$$

Next we claim that
$\lim_{n\to\infty}\int_{\mathbb{R}}\langle u(y),\hat{\psi}_n(y)\phi(x-y)\rangle\,dx = \int_{\mathbb{R}}\lim_{n\to\infty}\langle u(y),\hat{\psi}_n(y)\phi(x-y)\rangle\,dx.$

To prove this claim we have to verify the hypothesis of Lebesgue Dominated Convergence Theorem for the sequence $\{\langle u(y),\hat{\psi}_n(y)\phi(x-y)\rangle\}$. Now

$$\langle u(y),\hat{\psi}_n(y)\phi(x-y)\rangle = \langle(\hat{\psi}_n u)(y),\phi(x-y)\rangle = (\hat{\psi}_n u*\phi)(x)$$

implies that as a function of $x$, $\langle u(y),\hat{\psi}_n(y)\phi(x-y)\rangle$ is continuous and hence it is measurable for each $n$. Also, for each fixed $x\in\mathbb{R}$, the sequence $\{\hat{\psi}_n(y)\phi(x-y)\}$

converges to $\hat{\psi}(y)\phi(x-y)$ in $\mathscr{S}$ (Note that $\hat{\psi}_n \to \hat{\psi}$ as $n \to \infty$ in $\mathscr{S}$ and $\phi(x-y)$ as a function of $y$ is in $\mathscr{D}$ and hence $\phi(x-y)\hat{\psi}_n(y) \to \phi(x-y)\hat{\psi}(y)$ as $n \to \infty$ in $\mathscr{S}$). Hence $\langle u(y), \phi(x-y)\hat{\psi}_n(y)\rangle \to \langle u(y), \phi(x-y)\hat{\psi}(y)\rangle$ as $n \to \infty$. Thus we have established the fact that a point-wise limit exists. Next we shall show that this sequence of $L^1$-functions is dominated by some $L^1$-function. For this we shall first show that

$$(3.7)\quad (1+x^2)|\langle u(y), \phi(x-y)\hat{\psi}_n(y)\rangle - \langle u(y), \phi(x-y)\hat{\psi}(y)\rangle| \to 0 \text{ as } n \to \infty.$$

If this is done, then we have a stage $N$ such that for all $n \geq N$, we have

$$\begin{aligned} &|\langle u(y), \phi(x-y)\hat{\psi}_n(y)\rangle| - |\langle u(y), \phi(x-y)\hat{\psi}(y)\rangle| \\ &\qquad \leq |\langle u(y), \phi(x-y)\hat{\psi}_n(y)\rangle - \langle u(y), \phi(x-y)\hat{\psi}(y)\rangle| \\ &\qquad < \tfrac{1}{1+x^2} \in L^1(\mathbb{R}). \end{aligned}$$

The above argument also shows that as a function of $x$, $\langle u(y), \phi(x-y)\hat{\psi}(y)\rangle$ is in $L^1(\mathbb{R})$ since $\hat{\psi}_n u$ is in $\mathbb{E}_I$, as it is a compactly supported distribution for each $n$.

To prove (3.7), this it is enough to show that $(1+x^2)\phi(x-y)\hat{\psi}_n(y) \to (1+x^2)\phi(x-y)\psi(y)$ as $n \to \infty$ in $\mathscr{S}$ (consider them as functions of $y$). Now

$$\begin{aligned}
&(1+x^2)(1+|y|^2)^N |D^\alpha[\hat{\psi}_n(y) - \hat{\psi}(y)]\phi(x-y)| \\
&\leq (1+|x-y+y|^2)(1+|y|^2)^N \sum_{\beta\leq\alpha} c_{\alpha,\beta}|D^{\alpha-\beta}[\hat{\psi}_n(y) - \hat{\psi}(y)]|\,|D^\beta\phi(x-y)| \\
&\leq M_1(1+|x-y|^2)(1+|y|^2)^{N+1} \sum_{\beta\leq\alpha} c_{\alpha,\beta}|D^{\alpha-\beta}[\hat{\psi}_n(y) - \hat{\psi}(y)]|\,|D^\beta\phi(x-y)| \\
&= M_1(1+|y|^2)^{N+1} \sum_{\beta\leq\alpha} c_{\alpha,\beta}|D^{\alpha-\beta}[\hat{\psi}_n(y) - \hat{\psi}(y)]|\,(1+|x-y|^2)|D^\beta\phi(x-y)| \\
&\leq M_1(1+|y|^2)^{N+1} \sum_{\beta\leq\alpha} c_\beta\, c_{\alpha,\beta}|D^{\alpha-\beta}[\hat{\psi}_n(y) - \hat{\psi}(y)]| \\
&\qquad \text{where } c_\beta = \sup_{z\in\mathbb{R}} |\phi^{(\beta)}(z)|(1+|z|^2) \\
&= M_1 \sum_{\beta\leq\alpha} c_\beta\, c_{\alpha,\beta}(1+|y|^2)^{N+1}|D^{\alpha-\beta}[\hat{\psi}_n(y) - \hat{\psi}(y)]| \\
&\leq M_1 M_{\alpha,\beta}\|\hat{\psi}_n(y) - \hat{\psi}(y)\|_{N+1} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $(1+x^2)\phi(x-y)\hat{\psi}_n(y) \to (1+x^2)\phi(x-y)\psi(y)$ as $n \to \infty$ in $\mathscr{S}$. Hence (3.7). Thus all requirement for changing the limit under the integral sign has been established and hence we have proved that

$$\begin{aligned}
(3.8)\lim_{n\to\infty} \int_{\mathbb{R}} \langle u(y), \hat{\psi}_n(y)\phi(x-y)\rangle\, dx &= \int_{\mathbb{R}} \lim_{n\to\infty} \langle u(y), \hat{\psi}_n(y)\phi(x-y)\rangle\, dx \\
&= \int_{\mathbb{R}} \langle u(y), \hat{\psi}(y)\phi(x-y)\rangle\, dx.
\end{aligned}$$

Using (3.9) and (3.6) in (3.5), we get that

$$\begin{aligned}\langle [F_\phi(u)](t), \psi(t)\rangle &= \textstyle\int_{\mathbb{R}} \langle u(y), \hat{\psi}(y)\phi(x-y)\rangle\, dx \\ &= \lim_{n\to\infty} \textstyle\int_{\mathbb{R}} \langle u(y), \hat{\psi}_n(y)\phi(x-y)\rangle\, dx \\ &= \langle u(y), \hat{\psi}(y)\rangle = \langle \hat{u}(y), \psi(y)\rangle.\end{aligned}$$

Hence our result is proved.

**Theorem 3.11.** *If $F$ lies in the range of the $\phi$-Fourier transform then there exists a sequence $\{f_n\}$ in $\mathcal{C}$ such that $f_n \to F$ as $n \to \infty$ in $\mathscr{D}'(\mathbb{R})$.*

*Proof.* Let $F(t) = [F_\phi(u)](t)$ for some $u \in \mathbb{E}_I$. Define

$$g_n(x) = \int_{-n}^{n} \left(1 - \frac{|t|}{n}\right) F(t)e^{itx}\, dt, \quad (x \in \mathbb{R}).$$

Then it is clear that $g_n^{(k)}(x) \in \mathscr{O}_M(\mathbb{R})$ for all $k = 0, 1, 2, \ldots$ (since for each $k$, $g_n^{(k)}$ is the Fourier transform of some compactly supported distribution). For each $n$ choose $f_n \in \mathscr{S}'(\mathbb{R})$ such that $f_n = \check{\hat{g}}_n$. Hence $f_n \in \mathcal{C}$ for $n \in \mathbb{N}$. Now, for any $\psi \in \mathscr{D}(\mathbb{R})$, we have the following equality

$$\begin{aligned}\langle f_n(x), \psi(x)\rangle &= \langle \check{\hat{g}}_n(x), \psi(x)\rangle = \langle g_n(x), \check{\hat{\psi}}(x)\rangle = \int_{\mathbb{R}} g_n(x)\, \check{\hat{\psi}}(x)\, dx \\ &= \int_{\mathbb{R}} \left( \int_{-n}^{n} \left(1 - \frac{|t|}{n}\right) F(t)e^{itx}\, dt \right) \check{\hat{\psi}}(x)\, dx.\end{aligned}$$

Changing the order of integration we get

$$\langle f_n(x), \psi(x)\rangle = \int_{-n}^{n} \left(1 - \frac{|t|}{n}\right) F(t) \left( \int_{\mathbb{R}} \check{\hat{\psi}}(x)e^{itx}\, dx \right) dt = \int_{-n}^{n} \left(1 - \frac{|t|}{n}\right) F(t)\psi(t)\, dt.$$

Since $supp\, \psi$ is compact we have

$$\lim_{n\to\infty} \langle f_n(t), \psi(t)\rangle = \lim_{n\to\infty} \int_{-n}^{n} \left(1 - \frac{|t|}{n}\right) F(t)\psi(t)\, dt = \int_{\mathbb{R}} F(t)\psi(t)\, dt = \langle F(t), \psi(t)\rangle.$$

This proves that $f_n \to F$ as $n \to \infty$ in $\mathscr{D}'(\mathbb{R})$. □

**Theorem 3.12.** *Suppose $u \in \mathbb{E}_I$ such that*

$$(1 + x^2)\, (g_n u * \phi)(x) \to (1 + x^2)\, (gu * \phi)(x) \ \text{ as } n \to \infty,$$

*uniformly on $\mathbb{R}$, whenever $g_n^{(k)}(x) \to g^{(k)}(x)$ as $n \to \infty$ uniformly on $\mathbb{R}$ for every $k = 0, 1, 2, \ldots$. Then $f_n \to u$ as $n \to \infty$ in $\mathscr{D}'(\mathbb{R})$, where $f_n(x) = \int_{-n}^{n} [F_\phi u](t)e^{itx}\, dt$.*

*Proof.* Let $\psi \in \mathscr{D}(\mathbb{R})$ with $supp\,\psi = K$. Using Fubini's theorem, wherever required we have

$$\begin{aligned}
\langle f_n(x), \psi(x)\rangle &= \int_{\mathbb{R}} f_n(x)\psi(x)\,dx = \int_K \psi(x)\left(\int_{-n}^{n} [F_\phi u](t)e^{itx}\,dt\right) dx \\
&= \int_{-n}^{n} [F_\phi u](t)\left(\int_K \psi(x)e^{itx}\,dx\right) dt = \int_{-n}^{n} [F_\phi u](t)\check{\hat{\psi}}(t)\,dt \\
&= \int_{-n}^{n} \check{\hat{\psi}}(t)\left(\int_{\mathbb{R}} (e_{-t}u * \phi)(x)\,dx\right) dt \\
&= \int_{\mathbb{R}} \left(\int_{-n}^{n} \check{\hat{\psi}}(t)\,(e_{-t}u * \phi)(x)\,dt\right) dx \\
&= \int_{\mathbb{R}} \left(\int_{-n}^{n} u(e^{-ity}\check{\hat{\psi}}(t)\phi(x-y))\,dt\right) dx.
\end{aligned}$$

Now using the Pettis integral, we get

$$\begin{aligned}
\langle f_n(x), \psi(x)\rangle &= \int_{\mathbb{R}} u\left(\phi(x-y)\int_{-n}^{n} \check{\hat{\psi}}(t)e^{-ity}\,dt\right) dx \\
&= \int_{\mathbb{R}} u(\phi(x-y)g_n(y))\,dx, \quad \text{where } g_n(y) = \int_{-n}^{n} \check{\hat{\psi}}(t)e^{-ity}\,dt \\
&= \int_{\mathbb{R}} g_n u(\phi(x-y))\,dx = \int_{\mathbb{R}} (g_n u * \phi)(x)\,dx.
\end{aligned}$$

It is clear that $g_n^{(k)}(x) \to g^{(k)}(x)$ as $n \to \infty$ uniformly on $\mathbb{R}$ for every $k = 0, 1, 2, \ldots$ where $g(x) = \int_{\mathbb{R}} \check{\hat{\psi}}(t)e^{-itx}\,dt = \psi(x)$. Thus $(1 + x^2)\,(g_n u * \phi)(x) \to (1 + x^2)\,(gu * \phi)(x)$ as $n \to \infty$ uniformly on $\mathbb{R}$. This will imply that $g_n u * \phi \in L^1(\mathbb{R})$ for all sufficiently large $n$ and the hypothesis of Lebesgue dominated convergence theorem is also satisfied. Hence we have

$$\begin{aligned}
\lim_{n\to\infty} \langle f_n(x), \psi(x)\rangle &= \lim_{n\to\infty} \int_{\mathbb{R}} g_n u(\phi(x-y))\,dx &&= \int_{\mathbb{R}} (g_n u * \phi)(x)\,dx. \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} (g_n u * \phi)(x)\,dx &&= \int_{\mathbb{R}} (gu * \phi)(x)\,dx \\
&= \int_{\mathbb{R}} (\psi u * \phi)(x)\,dx &&= \int_{\mathbb{R}} u(\psi(y)\phi(x-y))\,dx \\
&= \langle u(x), \psi(x)\rangle, \ \text{(using Pettis integral).}
\end{aligned}$$

This completes the proof. □

## Acknowledgement

The author is very much thankful to the referees for their valuable comments towards the improvement of this paper.

## References

[1] Bremermann, H.J., Distributions, Complex variables and Fourier transforms. Addition Wesley Publishing Company, Inc. 1965.

[2] Colombeau, J.F., New generalized functions and multiplication of distributions. North Holland Mathematics Studies, 84 (1984).

[3] Colombeau, J.F., An introduction to new generalized functions. North Holland Mathematics Studies, 113 (1985).

[4] Dierolf, P., Voigt, J., Calculation of the bidual for some function spaces. Integrable distributions, Math. Ann. 253 (1980), 63–87.

[5] Gelfand, I.M., Shilov, G. E., Generalized functions, Vols 1-3. New York: Academic Press 1964.

[6] Howell, K. B., A new theory for Fourier analysis Part I, The space of test functions. J. Math. Anal. Appl., 168 (1992), 342–350.

[7] Schmeelk, J., Stieltjes transforms on new generalized functions, Internat. J. Math. and Math. Sci., Vol. 25 No. 6 (2001), 421–427.

[8] Mincheva-Kaminska, S., Sequential approach to integrable distributions. Novi Sad J. Math. Vol. 41 No. 1 (2011), 123–131.

[9] Mincheva-Kaminska, S., Equivalence of sequential definitions of the convolutions of distributions. Rend. Sem. Mat. Univ. Politec. Torino.

[10] Musielak, J., A note on integrals of distributions. Prace Mat. 8 (1963/64), 1-7.

[11] Pathak, R. S., Integral transforms of generalized functions and their applications. Amsterdam: Gordon and Breach Science Publishers 1997.

[12] Rudin, W., Principles of Mathematical Analysis. Singapore: McGraw-Hill International Editions 1976.

[13] Rudin, W., Functional Analysis. New Delhi: Tata McGraw-Hill Publishing Company Ltd 2006.

[14] Rudin, W., Real and Complex Analysis. New York: McGraw-Hill Book Company 1987.

[15] Schwartz, L., Causalité et analyticté, Anais da Academia Brasileria de Ciencias, Vol. 34 (1962), 13–21.

[16] Schwartz, L., Généralisation de la notion de fonction, de dérivation, de transformation de Fourier, et applications mathematiques dt physiques. Annales Univ. Grenoble, 21 (1945), 57–74.

[17] Schwartz, L., Mathematics for the physical sciences. Addison-Wesley Publishing Company 1966.

[18] Schwartz, L., Théorie des distributiones. Vols.1.2, Paris: Hermann (1957, 1959).

[19] Schwartz, L., Transformation de Laplace des distributions. Seminaire Mathematique de l'Université de Lund, Tome Supplémentaire, dedié á M. Reiez, (1952), 196–206.

[20] Sikorski, R., Integrals of distributions. Studia Math. 20 (1960), 119–139.

[21] Sobolev, S., Methode nouvelle a resoudre la probleme de Cauchy pour les equations hyperboliques normales. Recueil Math. (Math.Sbornik), 1 (1936), 39–71.

[22] Szmydt, Z., Fourier transformation and linear partial differential equations. Dordrecht: PWN and Reidel Warszawa, 1977.

[23] Talvila, E., The distributional Denjoy integral. Preprint (to appear in Real analysis Exchange.)

[24] Talvila, E., The regulated primitive integral. Preprint (to appear in Illinois Journal of Mathematics.)

[25] Talvila, E., Integrals and Banach spaces for finite order distributions. Preprint (to appear in Czechoslovak Mathematical Journal.)

[26] Vladimirov, V. S., Methods of the Theory of Generalized Functions. London-NewYork: Taylor & Francis, 2002.

[27] Wawak, R., Improper integrals of distributions. Studia Math. 86 (1987), 205–220.

[28] Zemanian, A. H., Distribution theory and transform analysis, New York: Mc-Graw Hill 1965.

[29] Zemanian, A. H., Generalized integral transformation, New York: John Wiley and Sons, Inc. 1968.

*Received by the editors July 21, 2011*