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COMMON FIXED POINT THEOREM FOR FOUR MAPPINGS DEFINED ON MENGER PM-SPACES WITH NONLINEAR CONTRACTIVE TYPE CONDITION

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Abstract. This paper presents a common fixed point theorem for two pairs of self-mappings of which one is compatible and the other weakly compatible, defined on Menger PM-spaces, satisfying nonlinear generalized contractive type condition involving Φ-functions.

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1. Introduction

The notion of statistical metric spaces, as a generalization of metric spaces, was introduced by K. Menger [11] in 1942. Schweizer and Sklar [16] studied the properties of spaces introduced by K. Menger and gave some basic results on these spaces. They studied topology, convergence of sequences, continuity of mappings, defined the completeness of these spaces, etc. Following A. N. Šerstnev [20], H. Sherwood gave a notion of probabilistic metric spaces [18] and also proved a theorem of a characterization of nested, closed sequence of nonempty sets in complete probabilistic metric space.

Fixed point and common fixed point properties for mappings defined on probabilistic spaces were studied by many authors ([1], [17], [5], [19], [14], [8]). Most of the properties which provide the existence of fixed point and common fixed point are of linear contractive type conditions. On the other hand, there are many generalizations ([15], [19]) of commutativity for the functions defined on spaces with non-deterministic distances (probabilistic metric spaces, fuzzy metric spaces, etc.) which have an important role in the statements providing the existence of a common fixed point.

The results in fixed point theory including nonlinear type contractive conditions were given by D.W. Boyd and J.S.W. Wong [2], S.N. Ješić and N.A. Babačev [6], D. O'Regan and R. Sadaati [14] and recently by S.N. Ješić et al.

Altering distance functions in Menger PM-spaces have been recently considered by B.S. Choudhury and K. Das [3]. Some fixed point results involving altering distances in Menger PM-spaces were given by D. Miheţ in [12].

The purpose of this paper is to prove a common fixed point theorem for four mappings satisfying nonlinear contractive type condition involving altering distances in Menger PM-spaces.

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2. Preliminaries

In the standard notation, let $D+$ be the set of all distribution functions $F : \mathbb{R} \to [0, 1]$, such that F is a nondecreasing, left-continuous mapping, which satisfies $F(0) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$. The space $D+$ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for $D+$ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1. ([16]) A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Examples of t-norm are $T(a, b) = \min\{a, b\}$ and $T(a, b) = ab$.

The t-norms are defined recursevly by $T^1 = T$ and

$$T^n(x_1, \dots, x_{n+1}) = T(T^{n-1}(x_1, \dots, x_n), x_{n+1}).$$

for $n \geq 2$ and $x_i \in [0, 1]$ for all $i \in \{1, \dots, n + 1\}$.

Definition 2.2. A *Menger probabilistic metric space* (briefly, Menger PM-space) is a triple $(X, \mathcal{F}, T)$ where X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x, y), the following conditions hold:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t + s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $s, t \geq 0$.

Remark 2.3. [17] Every metric space is a PM-space. Let (X, d) be a metric space and $T(a, b) = \min\{a, b\}$ is a continuous t-norm. Define $F_{x,y}(t) = \varepsilon_0(t - d(x, y))$ for all $x, y \in X$ and $t > 0$. The triple $(X, \mathcal{F}, T)$ is a PM-space induced by the metric d.

Definition 2.4. Let $(X, \mathscr{F}, T)$ be a Menger PM-space.

(1) A sequence $\{x_n\}_n$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer N such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}_n$ in X is called Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$ there a exists positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \geq N$.

(3) A Menger PM-space is said to be complete if every Cauchy sequence in X is convergent to a point in X.

The (ε, λ)-topology ([16]) in the Menger PM-space $(X, \mathcal{F}, T)$ is introduced by the family of neighbourhoods $\mathcal{N}_x$ of a point $x \in X$ given by

$$\mathcal{N}_x = \{N_x(\varepsilon, \lambda) : \ \varepsilon > 0, \lambda \in (0, 1)\}$$

where

$$N_x(\varepsilon, \lambda) = \{y \in X : \ F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

The (ε, λ)-topology is a Hausdorff topology. In this topology the function f is continuous in $x_0 \in X$ if and only if for every sequence $x_n \to x_0$ it holds that $f(x_n) \to f(x_0)$.

The following lemma is proved by B. Schweizer and A. Sklar.

Lemma 2.5. [16] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the function $\mathcal{F}$ is lower semi-continuous for every fixed $t > 0$, i.e. for every fixed $t > 0$ and every two convergent sequences $\{x_n\}, \{y_n\} \subseteq X$ such that $x_n \to x, y_n \to y$ it follows that*

$$\liminf_{n \to \infty} F_{x_n, y_n}(t) = F_{x,y}(t).$$

Definition 2.6. Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $A \subseteq X$. Closure of the set A is the smallest closed set containing A, denoted by $\overline{A}$.

Obviously, having in mind the Hausdorff topology and the definition of converging sequences we have that the next remark holds.

Remark 2.7. $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ in A such that $x_n \to x$.

Definition 2.8. [4] Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $A \subseteq X$. The probabilistic diameter of set A is given by

$$\delta_A(t) = \inf_{x,y \in A} \sup_{\varepsilon < t} F_{x,y}(\varepsilon).$$

The diameter of the set A is defined by

$$\delta_A = \sup_{t > 0} \inf_{x,y \in A} \sup_{\varepsilon < t} F_{x,y}(\varepsilon).$$

If there exists $\lambda \in (0, 1)$ such that $\delta_A = 1 - \lambda$, the set A will be called probabilistic semi-bounded. If $\delta_A = 1$, the set A will be called probabilistic bounded.

Lemma 2.9. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. A set $A \subseteq X$ is probabilistic bounded if and only if for each $\lambda \in (0, 1)$ there exists $t > 0$ such that $F_{x,y}(t) > 1 - \lambda$ for all $x, y \in A$.*

Proof. The proof follows from the definitions of $\sup A$ and $\inf A$ of non-empty sets. □

It is not difficult to see that every metrically bounded set is also probabilistic bounded if it is considered in an induced PM-space.

H. Sherwood has proved the following theorem.

Theorem 2.10. [18] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $\{F_n\}$ a nested sequence of nonempty, closed subsets of X such that $\delta_{F_n} \to \varepsilon_0$ as $n \to \infty$. Then, there is exactly one point $x_0 \in F_n$, for every $n \in \mathbb{N}$.*

It is easy to show that the following lemma is satisfied.

Lemma 2.11. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. A collection $\{F_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero i.e. for each $r \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $F_{x,y}(t) > 1 - r$ for all $x, y \in F_{n_0}$ if and only if $\delta_{F_n} \to \varepsilon_0$ as $n \to \infty$.*

Lemma 2.12. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space with the continuous t-norm T which satisfies $T(a, a) \geq a$ for every $a \in [0,1]$. Then, for every $x, y, z \in X$ and all $t > 0$ holds*

$$F_{x,y}(2t) \geq \min\{F_{x,z}(t), F_{y,z}(t)\}. \tag{1}$$

Proof. For every t-norm T which satisfies $T(a, a) \geq a$, for every $a, b \in [0,1]$ it holds $T(a, b) \geq T(\min\{a, b\}, \min\{a, b\}) \geq \min\{a, b\}$. From the previous, property (PM3) and the fact that T is nondecreasing we have that for every $x, y, z \in X$ and all $t > 0$ holds $F_{x,y}(2t) \geq \min\{F_{x,z}(t), F_{y,z}(t)\}$. □

Khan et al. in [10] introduced the concept of altering distance functions that alter the distance between two points in metric spaces. Recently, B.S. Choudhury and K. Das extended this concept to the probabilistic fixed point theory in [3], and proved a fixed point theorem for the t-norm $T = \min$. D. Miheţ proved some fixed point results that generalize the results given in [3], considering continuous t-norm.

Definition 2.13. [3] A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a Φ-function if the following conditions hold:

(i) $\phi(t) = 0$ if and only if $t = 0$;
(ii) ϕ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$;
(iii) ϕ is left-continuous in $(0, \infty)$;
(iv) ϕ is continuous at 0.

The class of all ϕ-functions will be denoted by Φ.

Lemma 2.14. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Let $\phi : [0, \infty) \to [0, \infty)$ be a Φ-function. Then, the following statement holds.*

If for $x, y \in X, 0 < c < 1$, we have that $F_{x,y}(t) \geq F_{x,y}(\phi(s/c))$ for all $t > 0$ and some $s > 0$ such that $t > \phi(s) > 0$, then $x = y$.

Proof. Since $\phi(0) = 0$ and ϕ is continuous in 0, there exists $s > 0$ such that $t > \phi(s) > 0$. From the fact that ϕ is strictly increasing, and since $c \in (0,1)$, by induction we get that $F_{x,y}(t) \geq F_{x,y}(\phi(s)) \geq F_{x,y}(\phi(s/c)) \geq \cdots \geq F_{x,y}(\phi(s/c^n))$. Taking $\liminf$ as $n \to \infty$ we get $F_{x,y}(t) \geq 1$, i.e. $x = y$. □

In fixed point theory, a very important role is played by the generalizations of commutativity. The concept of compatible mappings was introduced by

G. Jungck ([9]) and S.N. Mishra ([13]). There are many generalizations of compatibility in different senses. Recently, B. Singh et al. introduced the concept of weak compatibility in [19].

Definition 2.15. [13] Let $(X, \mathcal{F}, T)$ be a Menger PM-space and S and R self-mappings on X. We say that the mappings S and R are compatible if

$$\liminf_{n\to\infty} F_{SRx_n,RSx_n}(t) = 1 \quad \text{for every} \quad t > 0, \tag{2}$$

holds whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Rx_n = z \in X$ holds.

Definition 2.16. [19] Let $(X, \mathcal{F}, T)$ be a Menger PM-space and S and R self-mappings on X. We say that the mappings S and R are weakly compatible if for some $z \in X$ holds that $Sz = Rz$ then $SRz = RSz$.

It is easy to see that the class of compatible mappings is broader than the class of commuting mappings. Indeed, every pair of commuting mappings is also compatible, while the converse is not true ([19]). Also, every pair of compatible mappings is weakly compatible, as the following remark shows.

Remark 2.17. Let S and R be compatible mappings on a Menger PM-space $(X, \mathcal{F}, T)$. Then, the following holds:

If for some $z \in X$ we have $Sz = Rz$ then $SRz = RSz$.

Proof. This follows directly from Definition 2.15 taking $x_n = z$ for every $n \in \mathbb{N}$ for some point $z \in X$. □

Examples of compatible and weak compatible mappings can be found in [9], [13] and [19].

3. Main results

Lemma 3.1. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space with continuous t-norm T which satisfies $T(a, a) \geq a$ for every $a \in [0, 1]$ and S and R compatible self-mappings on X and let Sx_n and Rx_n converge to some point $z \in X$ for a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X. If S is continuous, then $\lim\limits_{n\to\infty} RSx_n = Sz$.*

Proof. Let $\lambda \in (0, 1)$ and $t > 0$ be arbitrary. Since S and R are compatible, it follows that $F_{RSx_n,SRx_n}(t) > 1 - \lambda$ for $n \in \mathbb{N}$ large enough. Also, Sx_n and Rx_n converge to z, so $F_{Rx_n,z}(t) > 1 - \lambda$ and $F_{Sx_n,z}(t) > 1 - \lambda$. From Lemma 2.12 and the continuity of S it follows that

$$F_{RSx_n,Sz}(2t) \geq \min\{F_{RSx_n,SRx_n}(t), F_{SRx_n,Sz}(t)\} \geq \min\{1 - \lambda, 1 - \lambda\} = 1 - \lambda$$

holds. Since $\lambda \in (0, 1)$ is arbitrary, we get that $\liminf\limits_{n\to\infty} F_{RSx_n,Sz}(t) = 1$, i.e. $\lim\limits_{n\to\infty} RSx_n = Sz$. □

Theorem 3.2. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space with continuous t-norm T which satisfies $T(a,a) \geq a$ for every $a \in [0,1]$, let $c \in (0,1)$ be fixed and let A, B, S and R be self-mappings on X and there exists $x_0 \in X$ such that $\mathcal{O}(A, x_0) = \{A^n x_0, n \in \mathbb{N} \cup \{0\}\}$ and $\mathcal{O}(B, x_0) = \{B^n x_0, n \in \mathbb{N} \cup \{0\}\}$ are probabilistic bounded sets. Let the following conditions hold:*

(a) $A(X) \subseteq R(X), B(X) \subseteq S(X)$,
(b) One of the mappings A and S is continuous,
(c) The pair $\{A, S\}$ is compatible and $\{B, R\}$ is weakly compatible,
(d) There exists a Φ-function ϕ such that

$$F_{Ax,By}(\phi(t)) \geq F_{Sx,Ry}(\phi(t/c)), \tag{3}$$

for every $t > 0$ and all $x, y \in X$. Then A, B, S and R have a unique common fixed point.

Proof. From *(a)* it follows that, for x_0 there exists $x_1 \in X$ such that $A(x_0) = R(x_1)$ and for such a point x_1 there exists $x_2 \in X$ such that $B(x_1) = S(x_2)$. By induction we can construct the following sequence $\{z_n\}_{n \in \mathbb{N}}$

$$\begin{cases} z_{2n-1} = Rx_{2n-1} = Ax_{2n-2} \\ z_{2n} = Sx_{2n} = Bx_{2n-1} \end{cases}. \tag{4}$$

Let us consider a nested sequence of non-empty, closed sets defined by

$$F_n = \overline{\{z_n, z_{n+1}, \ldots\}}, \ n \in \mathbb{N}.$$

We now prove that the family $\{F_n\}_{n \in \mathbb{N}}$ has the probabilistic diameter zero.

Let $\lambda \in (0,1)$ and $t > 0$ be arbitrary. From $F_k \subseteq \overline{\mathcal{O}(A, x_0)} \cup \overline{\mathcal{O}(B, x_0)}$, it follows that F_k is a probabilistic bounded set for arbitrary $k \in \mathbb{N}$.

Let $x, y \in F_k$ be arbitrary. There are sequences $\{z_{n(i)}\}$, $\{z_{n(j)}\}$ in F_k ($n(i), n(j) \geq n$, $i, j \in \mathbb{N}$) such that $\lim\limits_{i \to \infty} z_{n(i)} = x$ and $\lim\limits_{j \to \infty} z_{n(j)} = y$.

Case I. Let us assume that $n(i) \in 2\mathbb{N} - 1$ and $n(j) \in 2\mathbb{N}$ or vice-versa, for large enough $i, j \in \mathbb{N}$ i.e. $z_{n(i)} = Ax_{n(i)-1}$ and $z_{n(j)} = Bx_{n(j)-1}$.

Since $\phi(0) = 0$ and ϕ is continuous in 0, there exists $r > 0$ such that $t > \phi(r) > 0$. From (3) and the fact that F is nondecreasing, it follows that

$$\begin{aligned} F_{z_{n(i)}, z_{n(j)}}(t) &= F_{Ax_{n(i)-1}, Bx_{n(j)-1}}(t) \geq F_{Ax_{n(i)-1}, Bx_{n(j)-1}}(\phi(r)) \\ &\geq F_{Sx_{n(i)-1}, Rx_{n(j)-1}}(\phi(r/c)) = F_{Ax_{n(j)-2}, Bx_{n(i)-2}}(\phi(r/c)) \\ &= F_{z_{n(i)-1}, z_{n(j)-1}}(\phi(r/c)). \end{aligned}$$

By induction, for $m \in \mathbb{N}$, we get that

$$F_{z_{n(i)}, z_{n(j)}}(t) \geq F_{z_{n(i)-m}, z_{n(j)-m}}(\phi(r/c^m)).$$

Since $\{z_{n(i)-m}\}, \{z_{n(j)-m}\}$ are sequences in F_k we have that

$$F_{z_{n(i)-m}, z_{n(j)-m}}(\phi(r/c^m)) \geq \delta_{F_k}(\phi(r/c^m))$$

holds. Since F_k is probabilistic bounded and ϕ is a Φ-function, letting $m \to \infty$ we get

$$F_{z_{n(i)},z_{n(j)}}(t) \geq \delta_{F_k}(\phi(r/c^m)) \to \epsilon_0.$$

It follows that

$$(5) \qquad F_{z_{n(i)},z_{n(j)}}(t) > 1 - \lambda, \quad \text{for } n(i) \in 2\mathbb{N} - 1, n(j) \in 2\mathbb{N}, \text{ or vice-versa.}$$

Case II. Let us assume that both $n(i)$ and $n(j)$ are from the set $2\mathbb{N} - 1$ and let $n(l) \geq k$ be an arbitrary positive integer and $n(l) \in 2\mathbb{N}$.

Analoguously as in Case I, by replacing t with $\frac{t}{2}$, we show that

$$F_{Ax_{n(j)-1},Bx_{n(l)-1}}(t/2) > 1 - \lambda \ \text{ and } \ F_{Ax_{n(i)-1},Bx_{n(l)-1}}(t/2) > 1 - \lambda.$$

From Lemma 2.12 and the previous, we conclude that

$$\begin{aligned} F_{z_{n(i)},z_{n(j)}}(t) &= F_{Ax_{n(i)-1},Ax_{n(j)-1}}(t) \\ &\geq \min\left\{F_{Ax_{n(i)-1},Bx_{n(l)-1}}(t/2), F_{Ax_{n(j)-1},Bx_{n(l)-1}}(t/2)\right\} \\ &\geq \min\{1-\lambda, 1-\lambda\} = 1 - \lambda \end{aligned}$$

holds, i.e.

$$(6) \qquad F_{z_{n(i)},z_{n(j)}}(t) \geq 1 - \lambda, \qquad \text{for } n(i), n(j) \in 2\mathbb{N} - 1.$$

Similarly we can prove that (6) holds for $n(i), n(j) \in 2\mathbb{N}$.

Finally, from (5) and (6) we conclude that

$$F_{z_{n(i)},z_{n(j)}}(t) \geq 1 - \lambda$$

holds for every $i, j \in \mathbb{N}$. Taking liminf when $i, j \to \infty$ and applying Lemma 2.5 we get that $F_{x,y}(t) > 1 - \lambda$ for every $x, y \in F_k$. From Lemma 2.11 it follows that the collection $\{F_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero.

Applying Theorem 2.10 we conclude that this collection has a non-empty intersection that consists of exactly one point z. Since the collection $\{F_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero and $z \in F_n$ for every $n \in \mathbb{N}$, then for every $\lambda \in (0,1)$ and for all $t > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have that $F_{z_n,z}(t) > 1 - \lambda$. From this it follows that for every $\lambda \in (0,1)$ we have that $\liminf_{n\to\infty} F_{z_n,z}(t) > 1 - \lambda$. Taking $\lambda \to 0$ we get that

$$\liminf_{n\to\infty} F_{z_n,z}(t) = 1$$

i.e. $\lim_{n\to\infty} z_n = z$. From the definition of the sequences $\{Ax_{2n-2}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{R_{2n-1}\}$ it follows that every one of these sequences converges to z.

We shall prove that z is a common fixed point of the mappings A, B, S and R. Let us first assume that S is continuous. Then we have that $\lim_{n\to\infty} SSx_{2n} =$

Sz. From the compatibility of the pair $\{A, S\}$ and from Lemma 3.1 it follows that $\lim_{n\to\infty} ASx_{2n} = Sz$.

From the properties of the function ϕ it follows that there exists $r > 0$ such that $t > \phi(r) > 0$. Using the condition (3) we get that the following inequality holds:

$$F_{ASx_{2n},Bx_{2n-1}}(t) \geq F_{ASx_{2n},Bx_{2n-1}}(\phi(r)) \geq F_{SSx_{2n},Rx_{2n-1}}(\phi(r/c)).$$

Taking $\liminf$ as $n \to \infty$ we get that

$$F_{Sz,z}(t) \geq F_{Sz,z}(\phi(r/c)).$$

From Lemma 2.14 it follows that $Sz = z$. Using condition (3) again, we get that

$$F_{Az,Bx_{2n-1}}(t) \geq F_{Az,Bx_{2n-1}}(\phi(r)) \geq F_{Sz,Rx_{2n-1}}(\phi(r/c))$$

and taking $\liminf$ as $n \to \infty$ we get that

$$F_{Az,z}(t) \geq F_{Sz,z}(\phi(r/c)) = F_{z,z}(\phi(r/c)) = 1.$$

This means that $Az = z$. Since $A(X) \subseteq R(X)$, there exists a point $u \in X$ such that $z = Az = Ru$ and we have that

$$F_{z,Bu}(t) \geq F_{z,Bu}(\phi(r)) = F_{Az,Bu}(\phi(r)) \geq F_{Sz,Ru}(\phi(r/c)) = F_{z,z}(\phi(r/c)) = 1,$$

which means that $Bu = z$. From the weak compatibility of the pair $\{B, R\}$ it follows that $Rz = RBu = BRu = Bz$. Also, from (3) it follows that

$$F_{Ax_{2n},Bz}(t) \geq F_{Ax_{2n},Bz}(\phi(r)) \geq F_{Sx_{2n},Rz}(\phi(r/c)).$$

Taking $\liminf$ when $n \to \infty$ and from Lemma 2.14, we get that $Bz = z$. Thus, z is a common fixed point of the mappings A, B, S and R.

Now, let us assume that A is a continuous mapping. Then we have that $\lim_{n\to\infty} AAx_{2n} = Az$. From the compatibility of the pair $\{A, S\}$ and Lemma 3.1 it follows that $\lim_{n\to\infty} SAx_{2n} = Az$. Using the condition (3) we get that

$$F_{AAx_{2n},Bx_{2n-1}}(t) \geq F_{AAx_{2n},Bx_{2n-1}}(\phi(r)) \geq F_{SAx_{2n},Rx_{2n-1}}(\phi(r/c)).$$

Taking $\liminf$ as $n \to \infty$ we get that

$$F_{Az,z}(t) \geq F_{Az,z}(\phi(r/c)).$$

From Lemma 2.14 it follows that $Az = z$. Since $A(X) \subseteq R(X)$, there exists a point $v \in X$ such that $z = Az = Rv$. From (3) we have that

$$F_{AAx_{2n},Bv}(t) \geq F_{AAx_{2n},Bv}(\phi(r)) \geq F_{SAx_{2n},Rv}(\phi(r/c)).$$

Taking $\liminf$ as $n \to \infty$ we get that

$$F_{Az,Bv}(t) \geq F_{Az,Rv}(\phi(r/c)) = F_{z,z}(\phi(r/c)) = 1,$$

which means that $z = Bv$. Since the pair $\{B, R\}$ is weakly compatible we have that $Rz = RBv = BRv = Bz$. Also, using condition (3) we have

$$F_{Ax_{2n},Bz}(t) \geq F_{Ax_{2n},Bz}(\phi(r)) \geq F_{Sx_{2n},Rz}(\phi(r/c)).$$

Taking lim inf as $n \to \infty$ we get that

$$F_{z,Bz}(t) \geq F_{z,Bz}(\phi(r)) \geq F_{z,Rz}(\phi(r/c)) = F_{z,Bz}(\phi(r/c)).$$

This means that $z = Bz = Rz$. Since $B(X) \subseteq S(X)$, there exists a point $w \in X$ such that $z = Bz = Sw$. From (3) it follows that

$$F_{Aw,z}(t) \geq F_{Aw,z}(\phi(r)) = F_{Aw,Bz}\phi(r)) \geq F_{Sw,Rz}(\phi(r/c)) = F_{z,z}(\phi(r/c)) = 1,$$

i.e. $Aw = z$. Since the pair $\{A, S\}$ is compatible and $z = Aw = Sw$, from Remark 2.17 we have that $Az = ASw = SAw = Sz$. Thus, z is a common fixed point for the mappings A, B, S and R.

Let us now show that z is a unique common fixed point. Let us assume that there exists another common fixed point y. From (3) it follows that

$$F_{z,y}(t) \geq F_{z,y}(\phi(r)) = F_{Az,By}(\phi(r)) \geq F_{Sz,Ry}(\phi(r/c)) = F_{z,y}(\phi(r/c)).$$

Finally, from Lemma 2.14 it follows that $z = y$.

□

Example 3.3. Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space induced by a metric $d(x, y) = |x - y|$ on $X = [0, +\infty) \subset \mathbb{R}$ given in Remark 2.3. Let $\phi(t) = t$, $t > 0$, $c = \frac{1}{2}$ and

$$Ax = \frac{x}{1+x}, \qquad Sx = 2x,$$

$$Bx = \begin{cases} \frac{x}{1+x}, & x \in [0,1] \\ 0, & x > 1 \end{cases}, \qquad Rx = \begin{cases} 2x, & x \in [0,1] \\ 0, & x > 1 \end{cases}$$

Note that ϕ is a Φ-function. We shall prove that all the conditions of Theorem 3.2 are satisfied. First notice that $A(X) = [0, 1) \subset [0, 2] = R(X)$ and $B(X) = [0, \frac{1}{2}) \subset [0, +\infty) = S(X)$. The sets $A(X)$ and $B(X)$ are metrically bounded, i.e. probabilistic bounded as subsets of the Menger PM-space. Because $ASx = \frac{2x}{1+2x}$ and $SAx = \frac{2x}{1+x}$ we conclude that A and S are not commuting.

We now prove that they are compatible mappings. Note that

$$F_{ASx,SAx}(t) = \varepsilon_0\left(t - \frac{2x^2}{(1+2x)(1+x)}\right) \quad \text{and} \quad F_{Ax,Sx}(t) = \varepsilon_0\left(t - \frac{2x^2+x}{1+x}\right)$$

Since $\frac{2x^2}{(1+x)(1+2x)} \leq \frac{x+2x^2}{1+x}$ holds for all $x \geq 0$ we get

$$F_{ASx,SAx}(t) \geq F_{Sx,Ax}(t)$$

for all $x, t \geq 0$. For a sequence $\{x_n\}$ in $[0, +\infty)$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, from the previous inequality it follows that $\liminf_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1$.

Now we prove that the mappings B and R are weakly compatible. If $Bz = Rz$ then $z = 0$ or $z > 1$. In the case when $z = 0$ we get $RB(0) = BR(0) = 0$. On the other hand, if $z > 1$ then $RB(z) = R(0) = 0$ and $BR(z) = B(0) = 0$, i.e. the condition $RBz = BRz$ from Definition 2.16 is satisfied.

We now prove that the condition (3) is satisfied, too. Note that for all $x, y \in X$ we have that $\frac{1}{(1+x)(1+y)} \leq 1$.

a) For $x, y \in [0, 1]$ we get

$$F_{Ax,By}(t) = \varepsilon_0 \left(t - \frac{|x-y|}{(1+x)(1+y)} \right) \geq \varepsilon_0(2t - 2|x-y|) = F_{Sx,Ry}(2t).$$

b) For $x > 1$ and $y > 1$ we get

$$F_{Ax,By}(t) = \varepsilon_0 \left(t - \frac{x}{1+x} \right) \geq \varepsilon_0(2t - 2x) = F_{Sx,Ry}(2t).$$

c) If $x \in [0, 1]$ and $y > 1$ the proof is reduced to b). If $x > 1$ and $y \in [0, 1]$ the proof is reduced to a).

From the above we conclude that condition (3) is satisfied. We get that all the mappings have a unique common fixed point. It is easy to see that this point is $x = 0$.

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