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SOME ABELIAN AND TAUBERIAN RESULTS FOR THE SHORT-TIME FOURIER TRANSFORM

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Abstract. In this paper we provide some Abelian and Tauberian type results relating the boundary asymptotic behavior of the short-time Fourier transform with the quasiasymptotic behavior of tempered distributions.

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1. Introduction

The quasiasymptotic behavior (quasiasymptotics) was introduced by Zavialov as a result of his investigations in quantum field theory, and further developed by him, Vladimirov and Drožinov [17, and references therein], as well as Pilipović and his coworkers [9, 15, 16]. It has shown to be a very effective tool in the asymptotic analysis of various integral transforms and Abelian and Tauberian theory [7, 8, 9, 10, 12, 13, 14, 17].

The short-time Fourier transform [5] is an optimal analytic tool that conveys the information which frequency occurs at which instant of a signal and, in combination with moderate weights [6], is used to define modulation spaces [5, 2, 4, 3].

In this paper we provided some Abelian and Tauberian type results relating the quasiasymptotics at the origin and infinity of tempered distributions with the asymptotics of the short-time Fourier transform.

The paper is organized as follows. In Section 2 we give a brief summary to time-frequency analysis tools using mostly [5] as reference. Then we recall the basics of quasiasymptotic analysis of distributions. Section 3 connects the boundary asymptotic behavior of the short-time Fourier transform through the Abelian theorems and Tauberian characterizations of the quasiasymptotic behavior of tempered distributions.

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2. Preliminaries and notations

2.1. Spaces of Functions and Distributions

The Schwartz spaces of test functions and distributions over the space $\mathbb{R}^n$ are denoted by $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$, respectively; the space of rapidly decreasing smooth functions and its dual, the space of tempered distributions, are denoted by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, respectively, [11]. We have

$$\mathcal{D}(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R}) \hookrightarrow \mathcal{S}'(\mathbb{R}) \hookrightarrow \mathcal{D}'(\mathbb{R}),$$

where "$\mathcal{A} \hookrightarrow \mathcal{B}$" means that $\mathcal{A}$ is a dense subset of $\mathcal{B}$ and that the inclusion mapping is continuous.

The spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ play a particularly important role in various applications since Fourier transform is a topological isomorphism between $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, and extends to a continuous linear transform from $\mathcal{S}'(\mathbb{R})$ onto itself.

2.2. Short-time Fourier transform

The translation and modulation operators, T and M are defined by

$$T_x f(\cdot) = f(\cdot - x) \quad \text{and} \quad M_\omega f(\cdot) = e^{2\pi i\omega\cdot} f(\cdot), \ x, \omega \in \mathbb{R}.$$

The *short-time Fourier transform* (STFT) of a function $f \in L^2(\mathbb{R})$ with respect to a window function $g \in L^2(\mathbb{R})$ is defined as

$$V_g f(x, \omega) := \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t-x)} e^{-2\pi i\omega t}\, dt, \ x, \omega \in \mathbb{R} \tag{2.1}$$

and it holds $\|V_g f\|_2 = \|f\|_2 \|g\|_2$. Given an analysis window g and a synthesis window γ such that $\langle g, \gamma\rangle \neq 0$, for any f it holds

$$f = \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \langle f, M_\omega T_x g\rangle \, M_\omega T_x \gamma \, d\omega dx. \tag{2.2}$$

Whenever the generalized inner product in (2.1) is well-defined, the definition of $V_g f$ can be generalized to larger classes, for instance: $f \in \mathcal{S}'(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$; in fact, it is enough that g and f belong to time-frequency shift-invariant, mutually dual spaces.

It is obvious that for $g \in \mathcal{S}(\mathbb{R})$ the set

$$\{M_\omega T_x g \ : \ (x, \omega) \in K\} \tag{2.3}$$

is compact in $\mathcal{S}(\mathbb{R})$, where K is a compact subset of $\mathbb{R}^2$.

Note that for each used window $g \in \mathcal{S}(\mathbb{R})\backslash\{0\}$ and $f \in \mathcal{S}'(\mathbb{R})$ there exist constants $C > 0$ and $N \geq 0$ such that

$$|V_g f(x, \omega)| \leq C(1 + |x| + |\omega|)^N \ \text{ for all } \ x, \omega \in \mathbb{R}. \tag{2.4}$$

It is also known that if $f, g \in \mathcal{S}(\mathbb{R})$ then for all $n \geq 0$, there exists a constant $C_n > 0$ such that

$$|V_g f(x, \omega)| \leq C_n(1 + |x| + |\omega|)^{-n} \ \text{ for all } \ x, \omega \in \mathbb{R}. \tag{2.5}$$

In the proof of our results we use the relations (2.6) and (2.7) regarding the use of an adapted STFT window. In particular, we apply dilation to adapt the window (or any function), and we use the notation

$$f_\varepsilon(x) = f(\varepsilon x), \quad \varepsilon > 0.$$

It turns out that dilating the window is equivalent to the inverse dilation of the function of interest.

$$V_g f_\varepsilon(x,\omega) = \frac{1}{\varepsilon} V_{g_{1/\varepsilon}} f(\varepsilon x, \omega/\varepsilon). \tag{2.6}$$

Indeed, using the substitution $t = y/\varepsilon$ we have

$$\begin{aligned} V_g f_\varepsilon(x,\omega) = \langle f_\varepsilon, M_\omega T_x g\rangle &= \int_R f_\varepsilon(t)\overline{g(t-x)}e^{-2\pi i\omega t}dt \\ &= \frac{1}{\varepsilon}\int_R f(y)\overline{g\left(\frac{y-\varepsilon x}{\varepsilon}\right)}e^{-2\pi i \frac{\omega}{\varepsilon} y}dy \\ &= \frac{1}{\varepsilon}\langle f, M_{\omega/\varepsilon}T_{\varepsilon x} g_{1/\varepsilon}\rangle = \frac{1}{\varepsilon}V_{g_{1/\varepsilon}} f(\varepsilon x, \omega/\varepsilon). \end{aligned}$$

We will also prove the following relation

$$\varepsilon V_g f_\varepsilon(\frac{x_0}{\varepsilon} + x, \varepsilon^2\omega) = V_{g_{1/\varepsilon}} f(x_0 + \varepsilon x, \varepsilon\omega),\ x_0 \in \mathbb{R}. \tag{2.7}$$

Indeed, using the substitution $y = t/\varepsilon$ we obtain

$$\begin{aligned} V_{g_{1/\varepsilon}} f(x_0+\varepsilon x, \varepsilon\omega) &= \langle f, M_{\varepsilon\omega}T_{x_0+\varepsilon x} g_{1/\varepsilon}\rangle \\ &= \int_{\mathbb{R}} f(t)\overline{g(\frac{t-x_0-\varepsilon x}{\varepsilon})}e^{-2\pi i\omega\varepsilon t}dt \\ &= \varepsilon\int_{\mathbb{R}} f(\varepsilon y)\overline{g(y-\frac{x_0}{\varepsilon}-x)}e^{-2\pi i\omega\varepsilon^2 y}dy \\ &= \varepsilon\langle f_\varepsilon, M_{\varepsilon^2\omega}T_{\frac{x_0}{\varepsilon}+x} g\rangle = \varepsilon V_g f_\varepsilon(\frac{x_0}{\varepsilon}+x, \varepsilon^2\omega). \end{aligned}$$

2.3. Quasiasymptotic behavior

We will measure the behavior of a distribution by comparison with Karamata regularly varying functions [1], that is, the so-called quasiasymptotic behavior of distributions [15, 16, 9, 17].

A measurable real-valued function, defined and positive on an interval $(0, A]$ (resp. $[A, \infty)$), $A > 0$, is called *a slowly varying function* at the origin (resp. at infinity), if

$$\lim_{\varepsilon\to 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1 \quad (\text{ resp. } \lim_{\lambda\to\infty}\frac{L(a\lambda)}{L(\lambda)} = 1) \quad \text{for each } a > 0.$$

Let L be a slowly varying function at the origin. We say that the distribution $f \in \mathcal{S}'(\mathbb{R})$ has *quasiasymptotic behavior* (*quasiasymptotics*) of degree $\alpha \in \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ with respect to L if there exists $u \in \mathcal{S}'(\mathbb{R})$ such that for each $\varphi \in \mathcal{S}(\mathbb{R})$

$$\lim_{\varepsilon\to 0^+} \langle \frac{f(x_0+\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)},\ \varphi(x)\rangle = \langle u(x), \varphi(x)\rangle. \tag{2.8}$$

We will use the following convenient notation for the quasiasymptotic behavior,

$$f(x_0+\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) u(x) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathcal{S}'(\mathbb{R})\ ,$$

which should always be interpreted in the weak topology of $\mathcal{S}'(\mathbb{R})$, i.e., in the sense of (2.8).

One can prove that u cannot have an arbitrary form; indeed, it must be homogeneous with degree of homogeneity α, i.e., $u(ax) = a^\alpha u(x)$, for all $a \in \mathbb{R}_+$ [9, 17]. We remark that all homogeneous distributions on the real line are explicitly known; indeed, they are linear combinations of either x_+^α and x_-^α, if $\alpha \notin \mathbb{Z}_-$, or $\delta^{(k-1)}(x)$ and x^{-k}, if $\alpha = -k \in \mathbb{Z}_-$. It can also be shown ([15], Theorem 6.1) that if (2.8) holds just for each $\varphi \in \mathcal{D}(\mathbb{R})$, then it must hold for each $\varphi \in \mathcal{S}(\mathbb{R})$; therefore, the quasiasymptotic behavior at finite points is a local property. The quasiasymptotics of distributions at infinity with respect to a slowly varying function L at infinity is defined in a similar manner, and the notation $f(\lambda x) \sim \lambda^\alpha L(\lambda) u(x)$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$ will be used in this case.

We may also consider quasiasymptotics in other distribution spaces. The relation $f(x_0+\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) u(x)$ as $\varepsilon \to 0^+$ in $\mathcal{A}'(\mathbb{R})$ means that (2.8) is satisfied just for each $\varphi \in \mathcal{A}(\mathbb{R})$; and analogously for quasiasymptotics at infinity in $\mathcal{A}'(\mathbb{R})$.

3. Main results

Our main goal in this paper is to provide Abelian and Tauberian type results relating asymptotics of STFT and the quasiasymptotic behavior of tempered distributions.

Theorem 3.1. *Let L be a slowly varying function at the origin, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R})$. Suppose that*

$$f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) u(x) \quad \textit{as} \quad \varepsilon \to 0^+ \quad \textit{in} \quad \mathcal{S}'(\mathbb{R}).$$

Then for its STFT with respect to window $g \in \mathcal{S}(\mathbb{R})\backslash\{0\}$ we have

$$V_{g_{1/\varepsilon}} f(\varepsilon x, \omega/\varepsilon) \sim \varepsilon^{\alpha+1} L(\varepsilon) V_g u(x,\omega) \quad \textit{as} \quad \varepsilon \to 0^+.$$

uniformly for x, ω in compact subsets of $\mathbb{R}$.

Proof. By relation (2.6) we have

$$\begin{aligned} \frac{V_{g_{1/\varepsilon}} f(\varepsilon x, \omega/\varepsilon)}{\varepsilon^{\alpha+1} L(\varepsilon)} &= \frac{\varepsilon V_g f_\varepsilon(x,\omega)}{\varepsilon^{\alpha+1} L(\varepsilon)} = \Big\langle \frac{f_\varepsilon(t)}{\varepsilon^\alpha L(\varepsilon)}, M_\omega T_x g(t) \Big\rangle \\ &= \Big\langle \frac{f(\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)},\ M_\omega T_x g(t) \Big\rangle. \end{aligned}$$

Using the compactness of the set given by (2.3) and the Banach-Steinhaus theorem we obtain

$$\lim_{\varepsilon\to 0^+}\frac{V_{g_{1/\varepsilon}}f(\varepsilon x,\omega/\varepsilon)}{\varepsilon^{\alpha+1}L(\varepsilon)} = \lim_{\varepsilon\to 0^+}\langle\frac{f(\varepsilon t)}{\varepsilon^{\alpha}L(\varepsilon)},M_\omega T_x g(t)\rangle = \langle u(t),M_\omega T_x g(t)\rangle = V_g u(x,\omega),$$

uniformly for x,ω in compact subsets of $\mathbb{R}$. □

Remark 3.1. Let $f,g_1,g_2\in\mathcal{S}(\mathbb{R})\backslash\{0\}$ and

$$g_1(\varepsilon x)\sim\varepsilon^{\alpha}L(\varepsilon)g_2(x)\quad\text{as}\quad\varepsilon\to 0^+\quad\text{in}\quad\mathcal{S}'(\mathbb{R}). \tag{3.9}$$

According to Theorem 3.1 it follows

$$V_{f_{1/\varepsilon}}g_1(\varepsilon x,\omega/\varepsilon)\sim\varepsilon^{\alpha+1}L(\varepsilon)V_f g_2(x,\omega)\ \text{ as }\ \varepsilon\to 0^+.$$

By relation $V_g f(x,\omega)=e^{-2\pi i x\omega}\overline{V_f g(x,\omega)}$, $x,\omega\in\mathbb{R}$ we obtain

$$e^{-2\pi i x\omega}\overline{V_{g_1}f_{1/\varepsilon}(\varepsilon x,\frac{\omega}{\varepsilon})}\sim\varepsilon^{\alpha+1}L(\varepsilon)e^{-2\pi i x\omega}\overline{V_{g_2}f(x,\omega)}\ \text{ as }\quad\varepsilon\to 0^+,$$

i.e.

$$V_{g_1}f_{1/\varepsilon}(\varepsilon x,\omega/\varepsilon)\sim\varepsilon^{\alpha+1}L(\varepsilon)V_{g_2}f(x,\omega)\quad\text{as}\quad\varepsilon\to 0^+.$$

This is an expected result, given that the choice of STFT window is causing no significant change in the quality of the STFT; that is, two windows with the same quasiasymptotic property result with STFTs with related quasiasymptotics.

Theorem 3.2. *Let L be a slowly varying function at the origin, $\alpha\in\mathbb{R}$ and $f\in\mathcal{S}'(\mathbb{R})$, $g\in\mathcal{S}(\mathbb{R})\backslash\{0\}$. The following two conditions:*

(i) the limits

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{\alpha+1}L(\varepsilon)}V_{g_{1/\varepsilon}}f(\varepsilon x,\omega/\varepsilon)=M_{x,\omega}<\infty, \tag{3.10}$$

exist for every $x,\omega\in\mathbb{R}$, and

(ii) there exist $C>0$ and $N\geq 0$ such that

$$\frac{|V_{g_{1/\varepsilon}}f(\varepsilon x,\omega/\varepsilon)|}{\varepsilon^{\alpha+1}L(\varepsilon)}<C(1+|x|+|\omega|)^N, \tag{3.11}$$

for all $x,\omega\in\mathbb{R}$ and $0<\varepsilon\leq 1$, are necessary and sufficient conditions for the existence of a homogeneous distribution u such that

$$f(\varepsilon x)\sim\varepsilon^{\alpha}L(\varepsilon)u(x)\quad\textit{as}\quad\varepsilon\to 0^+\quad\textit{in}\quad\mathcal{S}'(\mathbb{R}). \tag{3.12}$$

Proof. (3.10) and (3.11) imply that the function given by $J(x,\omega) = M_{x,\omega}$, $x,\omega \in \mathbb{R}$ is measurable and satisfies the estimate

$$|J(x,\omega)| = |M_{x,\omega}| \leq C(1+|x|+|\omega|)^N,$$

for all $x,\omega \in \mathbb{R}$ and some constant $C > 0$. Moreover, by relation (2.6) and the inversion formula we obtain

$$\lim_{\varepsilon\to 0^+} \left\langle \frac{f(\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t) \right\rangle = \frac{1}{\langle \gamma, g\rangle} \lim_{\varepsilon\to 0^+} \int\int_{\mathbb{R}^2} \frac{V_{g_{1/\varepsilon}} f(\varepsilon x, \omega/\varepsilon)}{\varepsilon^{\alpha+1}L(\varepsilon)} \overline{V_\gamma \varphi(x,\omega)} d\omega dx,$$

where γ is the synthesis window for g such that $\langle g,\gamma\rangle \neq 0$. Because of (3.10) and (3.11) we can use the Lebesque dominated convergence theorem

$$\lim_{\varepsilon\to 0^+} \left\langle \frac{f(\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t) \right\rangle = \frac{1}{\langle \gamma, g\rangle} \int\int_{\mathbb{R}^2} J(x,\omega) \overline{V_\gamma \varphi(x,\omega)} d\omega dx.$$

Observe that the last integral converges absolutely because $|J(x,\omega)| = O((1+|x|+|\omega|)^N)$ for some $N > 0$ and $|V_\gamma\varphi(x,\omega)| = O((1+|x|+|\omega|)^{-n})$ for all $n \geq 0$, whenever $\varphi,\gamma \in \mathcal{S}(\mathbb{R})$ [[5], Theorem 11.2.5]. It follows that the limit $\lim_{\varepsilon\to 0^+} \langle \frac{f(\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t)\rangle$ exists for each $\varphi \in \mathcal{S}(\mathbb{R})$. So, we conclude that f has quasiasymptotic behavior at the origin in $\mathcal{S}'(\mathbb{R})$.

We now prove the converse. If (3.12) holds, then (3.10) follows from the Abelian type result given in Theorem 3.1. Also, from (2.6), (3.12) and (2.4) it follows that there exist constants C_1, $C_2 > 0$ and $N \geq 0$ such that

$$\begin{aligned} \frac{|V_{g_{1/\varepsilon}} f(\varepsilon x, \omega/\varepsilon)|}{\varepsilon^{\alpha+1}L(\varepsilon)} &= \frac{|V_g f_\varepsilon(x,\omega)|}{\varepsilon^\alpha L(\varepsilon)} = \frac{|\langle f(\varepsilon t), M_\omega T_x g(t)\rangle|}{\varepsilon^\alpha L(\varepsilon)} \\ &< C_1 |\langle u, M_\omega T_x g\rangle| = C_1 |V_g u(x,\omega)| \\ &\leq C_2 (1+|x|+|\omega|)^N. \end{aligned}$$

□

Remark 3.2. Clearly, the STFT $V_g u(x,\omega)$ in Theorem 3.2 is given by the limits (3.10).

A similar assertion as previous theorem holds for quasiasymptotics at infinity.

Theorem 3.3. *Let L be a slowly varying function at infinity, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R})\backslash\{0\}$ The following two conditions:*

(i) the limits

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^{\alpha+1}L(\lambda)} V_{g_{1/\lambda}} f(\lambda x, \omega/\lambda) = M_{x,\omega} < \infty,$$

exist for every $x,\omega \in \mathbb{R}$, and

(ii) there exist $C > 0$ and $N \geq 0$ such that

$$\frac{|V_{g_{1/\lambda}} f(\lambda x, \omega/\lambda)|}{\lambda^{\alpha+1}L(\lambda)} < C(1+|x|+|\omega|)^N,$$

for all $x,\omega\in\mathbb{R}$ and $\lambda\geq 1$, are necessary and sufficient conditions for existence of a homogeneous distribution u such that

$$f(\lambda x)\sim\lambda^{\alpha}L(\lambda)u(x)\quad as\quad \lambda\to\infty\quad in\quad \mathcal{S}'(\mathbb{R}).$$

Remark 3.3. The same consideration of Remark 3.2 applies to the case of infinity by analogy.

Theorem 3.4. *Let L be a slowly varying function at the origin, $\alpha\in\mathbb{R}$, $x_0\in\mathbb{R}$ and $f\in\mathcal{S}'(\mathbb{R})$. Suppose that*

$$f(x_0+\varepsilon x)\sim\varepsilon^{\alpha}L(\varepsilon)u(x)\quad as\quad \varepsilon\to 0^{+}\quad in\quad \mathcal{S}'(\mathbb{R}).$$

Then for its STFT with respect to window $g\in\mathcal{S}(\mathbb{R})\backslash\{0\}$ we have

$$V_{g_{1/\varepsilon}}f(x_0+\varepsilon x,\varepsilon\omega)\sim\varepsilon^{\alpha+1}L(\varepsilon)V_g u(x,0)\quad as\quad \varepsilon\to 0^{+},$$

uniformly for x,ω in compact subsets of $\mathbb{R}$.

Proof. Using the substitution $t-x_0=\varepsilon y$ we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^{+}}\frac{V_{g_{1/\varepsilon}}f(x_0+\varepsilon x,\varepsilon\omega)}{\varepsilon^{\alpha+1}L(\varepsilon)} &= \lim_{\varepsilon\to 0^{+}}\frac{1}{\varepsilon^{\alpha+1}L(\varepsilon)}\langle f,M_{\varepsilon\omega}T_{x_0+\varepsilon x}g_{1/\varepsilon}\rangle\\
&= \lim_{\varepsilon\to 0^{+}}\frac{1}{\varepsilon^{\alpha+1}L(\varepsilon)}\Big\langle f(t),g(\frac{t-x_0-\varepsilon x}{\varepsilon})e^{2\pi i\varepsilon\omega t}\Big\rangle\\
&= \lim_{\varepsilon\to 0^{+}}\frac{1}{\varepsilon^{\alpha}L(\varepsilon)}\Big\langle f(x_0+\varepsilon y),g(y-x)e^{2\pi i\varepsilon\omega(x_0+\varepsilon y)}\Big\rangle\\
&= \lim_{\varepsilon\to 0^{+}}\frac{1}{\varepsilon^{\alpha}L(\varepsilon)}\Big\langle f(x_0+\varepsilon y),M_0T_xg(y)e^{2\pi i\varepsilon\omega(x_0+\varepsilon y)}\Big\rangle.
\end{aligned}$$

In view of (3.4), the Banach-Steinhaus theorem and the compactness of the set given by (2.3) we have

$$\lim_{\varepsilon\to 0^{+}}\frac{V_{g_{1/\varepsilon}}f(x_0+\varepsilon x,\varepsilon\omega)}{\varepsilon^{\alpha+1}L(\varepsilon)}=\langle u(y),M_0T_xg(y)\rangle=V_g u(x,0).$$

□

We now investigate the inverse (Tauberian) theorem related to Theorem 3.4.

Theorem 3.5. *Let L be a slowly varying function at the origin, $\alpha\in\mathbb{R}$, $x_0\in\mathbb{R}$, and $f\in\mathcal{S}'(\mathbb{R})$, $g\in\mathcal{S}(\mathbb{R})\backslash\{0\}$. Suppose that the limits*

$$\lim_{\varepsilon\to 0^{+}}\frac{1}{\varepsilon^{\alpha-1}L(\varepsilon)}V_{g_{1/\varepsilon}}f(x_0+\varepsilon x,\varepsilon\omega)=M_{x,\omega}<\infty, \tag{3.13}$$

exists for every $x,\omega\in\mathbb{R}$, and there exist $C>0,N\geq 0$ and $M>1$ such that

$$\frac{|V_{g_{1/\varepsilon}}f(x_0+\varepsilon x,\varepsilon\omega)|}{\varepsilon^{\alpha-1}L(\varepsilon)}<C\frac{(1+|x|)^N}{(1+|\omega|)^M}, \tag{3.14}$$

for all $x,\omega\in\mathbb{R}$ and $0<\varepsilon\leq 1$. Then, there exists a homogeneous distribution u such that

$$f(x_0+\varepsilon x)\sim\varepsilon^{\alpha}L(\varepsilon)u(x)\quad as\quad \varepsilon\to 0^{+}\quad in\quad \mathcal{S}'(\mathbb{R}). \tag{3.15}$$

Proof. (3.13) and (3.14) imply that the function $M_{x,\omega} = J(x,\omega)$ satisfies the estimate

$$|J(x,\omega)| = |M_{x,\omega}| \leq C\frac{(1+|x|)^N}{(1+|\omega|)^M},$$

for every $x,\omega \in \mathbb{R}$ and for some constants $C > 0, N \geq 0$ and $M > 1$. Let $\varphi \in \mathcal{S}(\mathbb{R})$ and $\gamma \in \mathcal{S}(\mathbb{R})\backslash\{0\}$ be a synthesis window for g such that $\langle g,\gamma\rangle \neq 0$. By inversion formula (2.2) and the substitution $\omega = \varepsilon^2\omega_1,\ t = t_1 - \dfrac{x_0}{\varepsilon}$ we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\left\langle \frac{f(x_0+\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t)\right\rangle\\
=\ & \frac{1}{\langle\gamma,g\rangle}\lim_{\varepsilon\to 0^+}\int\int_{\mathbb{R}^2}\frac{\langle f(x_0+\varepsilon t), M_\omega T_x g(t)\rangle}{\varepsilon^\alpha L(\varepsilon)}\langle M_\omega T_x\gamma,\varphi\rangle d\omega dx\\
=\ & \frac{1}{\langle\gamma,g\rangle}\lim_{\varepsilon\to 0^+}\int\int_{\mathbb{R}^2}\frac{\langle f(\varepsilon t_1), M_{\varepsilon^2\omega_1} T_x g(t_1-\frac{x_0}{\varepsilon})\rangle}{\varepsilon^{\alpha-2} L(\varepsilon)}\langle M_{\varepsilon^2\omega_1} T_x\gamma,\varphi\rangle d\omega_1 dx\\
=\ & \frac{1}{\langle\gamma,g\rangle}\lim_{\varepsilon\to 0^+}\int\int_{\mathbb{R}^2}\frac{\langle f(\varepsilon t_1), M_{\varepsilon^2\omega_1} T_{x+\frac{x_0}{\varepsilon}} g(t_1)\rangle}{\varepsilon^{\alpha-2} L(\varepsilon)}\langle M_{\varepsilon^2\omega_1} T_x\gamma,\varphi\rangle d\omega_1 dx\\
=\ & \frac{1}{\langle\gamma,g\rangle}\lim_{\varepsilon\to 0^+}\int\int_{\mathbb{R}^2}\frac{V_g f_\varepsilon(x+\frac{x_0}{\varepsilon},\varepsilon^2\omega_1)}{\varepsilon^{\alpha-2} L(\varepsilon)}\overline{V_\gamma\varphi(x,\varepsilon^2\omega_1)}d\omega_1 dx.
\end{aligned}$$

By relation (2.7) we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\left\langle \frac{f(x_0+\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t)\right\rangle\\
=\ & \frac{1}{\langle\gamma,g\rangle}\lim_{\varepsilon\to 0^+}\int\int_{\mathbb{R}^2}\frac{V_{g_{1/\varepsilon}} f(x_0+\varepsilon x,\varepsilon\omega_1)}{\varepsilon^{\alpha-1} L(\varepsilon)}\overline{V_\gamma\varphi(x,\varepsilon^2\omega_1)}d\omega_1 dx.
\end{aligned}$$

Because of (3.13), (3.14) and (2.4) we can use the Lebesque dominated convergence theorem

$$\lim_{\varepsilon\to 0^+}\left\langle \frac{f(x_0+\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t)\right\rangle = \frac{1}{\langle\gamma,g\rangle}\int\int_{\mathbb{R}^2} J(x,\omega)\overline{V_\gamma\varphi(x,0)}d\omega dx.$$

Observe that the last integral converges absolutely because $|J(x,\omega)| = O((1+|x|)^N(1+|\omega|)^{-M})$ for some $N \geq 0, M > 1$, and $|V_\gamma\varphi(x,0)| = O((1+|x|)^{-n})$ for all $n \geq 0$, whenever $\varphi \in \mathcal{S}(\mathbb{R})$. It follows that the limit $\lim_{\varepsilon\to 0^+}\left\langle \dfrac{f(x_0+\varepsilon t)}{\varepsilon^{\alpha+1} L(\varepsilon)}, \varphi(t)\right\rangle$ exists for each $\varphi \in \mathcal{S}(\mathbb{R})$. So, we conclude that f has quasiasymptotic behavior in $\mathcal{S}'(\mathbb{R})$. □

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