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SEMI-SYMMETRIC NON-METRIC CONNECTION IN A P-SASAKIAN MANIFOLD

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Abstract. The present paper deals with the study of a Para-Sasakian manifold admitting a semi-symmetric non-metric connection whose conharmonic curvature tensor satisfies certain curvature conditions.

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1. Introduction

In [16], Takahashi introduced the notion of locally ϕ-symmetric Sasakian manifolds as a weaker version of local symmetry of such manifolds. In respect of contact geometry, the notion of ϕ-symmetry was introduced and studied by Boeckx, Buecken and Vanhecke [4] with several examples. In [5], De studied the notion of ϕ-symmetry with several examples for Kenmotsu manifolds. In 1977, Adati and Matsumoto defined Para-Sasakian and Special Para-Sasakian manifolds [1], which are special classes of an almost paracontact manifold introduced by Sato [15]. Para-Sasakian manifolds have been studied by Tarafdar and De [17], De and Pathak [9], Matsumoto, Ianus and Mihai [14], Matsumoto [13], and many others.

Hayden [11] introduced semi-symmetric linear connections on a Riemannian manifold. Let M be an n-dimensional Riemannian manifold of class C^∞ endowed with the Riemannian metric g and ∇ be the Levi-Civita connection on (M^n, g).

A linear connection $\bar{\nabla}$ defined on (M^n, g) is said to be semi-symmetric [10] if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

where η is a 1-form and ξ is a vector field defined by

$$\eta(X) = g(X, \xi), \tag{1.2}$$

for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on M^n.

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A semi-symmetric connection $\bar{\nabla}$ is called a semi-symmetric non-metric connection [2] if it further satisfies

$$\bar{\nabla} g \neq 0. \tag{1.3}$$

The semi-symmetric non-metric connections have been studied by several authors such as De and Biswas [6], Biswas, De and Barua [3], De and Kamilya ([7], [8]) and many others.

Let M be a Riemannian manifold of dimension n equipped with two metric tensors g and $\bar{g}$. If a transformation of M does not change the angle between two tangent vectors at a point with respect to g and $\bar{g}$, then such a transformation is said to be a conformal transformation of the metrics on the Riemannian manifold. Under conformal transformation, the length of the curves are changed but the angles made by curves remain the same.

Let us consider a Riemannian manifold M with two metric tensors g and $\bar{g}$ such that they are related by

$$\bar{g}(X,Y) = e^{2\sigma} g(X,Y), \tag{1.4}$$

where σ is a real function on M.

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In general, a harmonic function is not transformed into a harmonic function. The condition under which a harmonic function remains invariant have been studied by Ishii [12], who introduced the conharmonic transformation as a subgroup of the conformal transformation (1.4) satisfying the condition

$$\sigma,_{i}^{\;i} + \sigma,_{i}\, \sigma,^{i} = 0, \tag{1.5}$$

where comma denotes the covariant differentiation with respect to the metric g.

A rank four tensor $\bar{C}$ that remains invariant under conharmonic transformation for an n-dimensional Riemannian manifold M, is given by

$$\begin{aligned} \bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-2}[&g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y \\ &+ \bar{S}(Y,Z)X - \bar{S}(X,Z)Y], \end{aligned} \tag{1.6}$$

where $\bar{R}$ and $\bar{S}$ are the curvature tensor and the Ricci tensor with respect to semi-symmetric non-metric connection respectively and $\bar{S}(Y,Z) = g(\bar{Q}Y,Z)$.

Taking the inner product of (1.6) with W, we have

$$\begin{aligned} \tilde{\bar{C}}(X,Y,Z,W) = \tilde{\bar{R}}(X,Y,Z,W) - \frac{1}{n-2}[&g(Y,Z)\bar{S}(X,W) - g(X,Z)\bar{S}(Y,W) \\ &+ \bar{S}(Y,Z)g(X,W) - \bar{S}(X,Z)g(Y,W)], \end{aligned} \tag{1.7}$$

where $\tilde{\bar{C}}$ and $\bar{C}$ are the conharmonic curvature tensor of type $(0,4)$ and $(1,3)$ with respect to the semi-symmetric non-metric connection respectively, and $\tilde{\bar{C}}(X,Y,Z,W) = g(\bar{C}(X,Y)Z,W)$, $\tilde{\bar{R}}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$.

The present paper is organized as follows: Section 2 is equipped with some prerequisites about P-Sasakian manifolds. In section 3, we study the semi-symmetric non-metric connection on P-Sasakian manifolds. Section 4 of the paper establishes the relation of the curvature tensor between the Levi-Civita connection and the semi-symmetric non-metric connection of a P-Sasakian manifold. Section 5 deals with ξ-conharmonicly flat P-Sasakian manifolds with respect to the semi-symmetric non-metric connection. Finally, we investigate globally ϕ-conharmonicly symmetric P-Sasakian manifolds with respect to the semi-symmetric non-metric connection.

2. P-Sasakian manifolds

An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X, \tag{2.4}$$

for any vector fields X, Y on M.

In addition, if (ϕ, ξ, η, g), satisfy the equations

$$d\eta = 0, \quad \nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold.

It is known ([1], [15]) that in a P-Sasakian manifold the following relations hold :

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{2.9}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

where R and S are the curvature tensor and the Ricci tensor of the Levi-Civita connection respectively.

3. Semi-symmetric non-metric connection

Let M be an n-dimensional Riemannian manifold with Riemannian metric g. If $\bar{\nabla}$ is the semi-symmetric non-metric connection of a Riemannian manifold M, a linear connection $\bar{\nabla}$ is given by [2]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \tag{3.1}$$

Then $\bar{R}$ and R are related by [2]

$$\bar{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{3.2}$$

for all vector fields X, Y, Z on M, where α is a $(0,2)$ tensor field denoted by

$$\alpha(X,Z) = (\nabla_X \eta)(Z) - \eta(X)\eta(Z). \tag{3.3}$$

From (3.1) yields

$$(\bar{\nabla}_W g)(X,Y) = -\eta(X)g(Y,W) - \eta(Y)g(X,W) \neq 0. \tag{3.4}$$

4. Curvature tensor of a P-Sasakian manifold with respect to the semi-symmetric non-metric connection

Using (2.4) in (3.3), we get

$$\alpha(X,Y) = g(X,\phi Y) - \eta(X)\eta(Y). \tag{4.1}$$

Again, using (4.1) in (3.2), we have

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ g(X,\phi Z)Y - \eta(X)\eta(Z)Y \\ &- g(Y,\phi Z)X + \eta(Y)\eta(Z)X.\end{aligned} \tag{4.2}$$

Taking the inner product of (4.2) with W, it follows that

$$\begin{aligned}\tilde{\bar{R}}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) &+ g(X,\phi Z)g(Y,W) - \eta(X)\eta(Z)g(Y,W) \\ &- g(Y,\phi Z)g(X,W) + \eta(Y)\eta(Z)g(X,W),\end{aligned} \tag{4.3}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Let $\{e_1, ..., e_n\}$ be a local orthonormal basis of vector fields in M. Then, by putting $X = W = e_i$ in (4.3) and taking summation over i, $1 \leq i \leq n$ and also using (2.1), we obtain

$$\bar{S}(Y,Z) = S(Y,Z) - (n-1)g(Y,\phi Z) + (n-1)\eta(Y)\eta(Z). \tag{4.4}$$

Let $\{e_1, ..., e_n\}$ be a local orthonormal basis of vector fields in M. Then, by putting $Y = Z = e_i$ in (4.4) and taking summation over i, $1 \leq i \leq n$ and also using (2.1), we have

$$\bar{r} = r - (n-1)\beta + n - 1, \tag{4.5}$$

where $\bar{r}$ and r are the scalar curvature with respect to the semi-symmetric non-metric connection and the Levi-Civita connection respectively, and $\beta = trace\phi$.

From (4.4) yields

$$\bar{Q}Y = QY - (n-1)\phi Y + (n-1)\eta(Y)\xi, \tag{4.6}$$

where

$$S(Y,Z) = g(QY,Z). \tag{4.7}$$

Again, putting $Z = \xi$ in (4.4) and using (2.1) and (2.11), we get

$$\bar{S}(Y,\xi) = 0. \tag{4.8}$$

Combining (3.1) and (2.4), it follows that

$$(\bar{\nabla}_W \eta)(Z) = g(W,\phi Z) - \eta(Z)\eta(W). \tag{4.9}$$

Again combining (3.1) and (2.6), we obtain

$$\begin{aligned}(\bar{\nabla}_W \phi)(X) = -g(X,W)\xi - \eta(X)W + 2\eta(X)\eta(W)\xi \\ -\eta(X)\phi W.\end{aligned} \tag{4.10}$$

Combining (3.1) and (2.5) yields

$$\bar{\nabla}_W \xi = \phi W + W. \tag{4.11}$$

From the above discussion we can state the following theorem:

Theorem 4.1. *For a P-Sasakian manifold M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$*

(i) The curvature tensor $\bar{R}$ is given by (4.2),
(ii) The Ricci tensor $\bar{S}$ is given by (4.4),
(iii) The scalar curvature $\bar{r}$ is given by (4.5),
(iv) The Ricci tensor $\bar{S}$ is symmetric,
(v) $\bar{S}(Y,\xi) = 0$,
(vi) $(\bar{\nabla}_W \eta)(Z) = g(W,\phi Z) - \eta(Z)\eta(W)$,
(vii) $(\bar{\nabla}_W \phi)(X) = -g(X,W)\xi - \eta(X)W + 2\eta(X)\eta(W)\xi - \eta(X)\phi W$,
(viii) $\bar{\nabla}_W \xi = \phi W + W$.

5. ξ-conharmonicly flat P-Sasakian manifolds with respect to the semi-symmetric non-metric connection

Using (4.2) in (1.6), we get

$$\bar{C}(X,Y)Z = R(X,Y)Z + g(X,\phi Z)Y - \eta(X)\eta(Z)Y$$
$$-g(Y,\phi Z)X + \eta(Y)\eta(Z)X - \frac{1}{n-2}[g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y$$
$$+\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \tag{5.1}$$

Using (4.4) and (4.6) in (5.1), we have

$$\bar{C}(X,Y)Z = C(X,Y)Z + \frac{1}{n-2}[g(Y,\phi Z)X - g(X,\phi Z)Y -$$
$$\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y] - \frac{n-1}{n-2}[g(Y,Z)\eta(X)\xi$$
$$-g(X,Z)\eta(Y)\xi - g(Y,Z)\phi X + g(X,Z)\phi Y], \tag{5.2}$$

where

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY$$
$$+S(Y,Z)X - S(X,Z)Y], \tag{5.3}$$

be the conharmonic curvature tensor with respect to Levi-Civita connection.

Putting $Z = \xi$ in (5.2) and using (2.1), it follows that

$$\bar{C}(X,Y)\xi = C(X,Y)\xi + \frac{1}{n-2}[\eta(X)Y - \eta(Y)X]$$
$$+\frac{n-1}{n-2}[\eta(Y)\phi X - \eta(X)\phi Y]. \tag{5.4}$$

Suppose X and Y are orthogonal to ξ, then from (5.4), we obtain

$$\bar{C}(X,Y)\xi = C(X,Y)\xi. \tag{5.5}$$

In view of the above discussion we can state the following theorem :

Theorem 5.1. *An n-dimensional P-Sasakian manifold is ξ-conharmonicly flat with respect to the semi-symmetric non-metric connection if and only if the manifold is also ξ-conharmonicly flat with respect to the Levi-Civita connection provided the vector fields X and Y are horizontal vector fields.*

6. Globally ϕ-conharmonicly symmetric P-Sasakian manifolds with respect to the semi-symmetric non-metric connection

Definition 6.1. *A P-Sasakian manifold M with respect to the semi-symmetric non-metric connection is called to be globally ϕ-conharmonicly symmetric if*

$$\phi^2((\bar{\nabla}_W \bar{C})(X,Y)Z) = 0, \tag{6.1}$$

for all vector fields $X, Y, Z, W \in \chi(M)$.

Combining (5.3) and (2.7), we get

$$\eta(C(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) - \frac{1}{n-2}[g(Y,Z)\eta(QX) - g(X,Z)\eta(QY) + S(Y,Z)\eta(X) - S(X,Z)\eta(Y)]. \tag{6.2}$$

Combining (4.7) and (2.11) yields

$$\eta(QX) = -(n-1)\eta(X). \tag{6.3}$$

Moreover, combining (3.1), (6.2) and (6.3) and taking X, Y, Z, W are orthogonal to ξ, it follows that

$$(\bar{\nabla}_W C)(X,Y)Z = \bar{\nabla}_W C(X,Y)Z - C(\bar{\nabla}_W X, Y)Z - C(X, \bar{\nabla}_W Y)Z - C(X,Y)\bar{\nabla}_W Z = (\nabla_W C)(X,Y)Z. \tag{6.4}$$

Taking covariant differentiation of (5.2) with respect to W and also taking X, Y, Z, W are orthogonal to ξ and using (3.4), (4.9), (4.10), (4.11) and (6.4), we have

$$(\bar{\nabla}_W \bar{C})(X,Y)Z = (\nabla_W C)(X,Y)Z - \frac{n-1}{n-2}[g(Y,Z)g(X,\phi W)\xi - g(X,Z)g(Y,\phi W)\xi + g(X,W)g(Y,Z)\xi - g(X,Z)g(Y,W)\xi]. \tag{6.5}$$

Now, applying ϕ^2 on both sides of (6.5) and using (2.1), it follows that

$$\phi^2((\bar{\nabla}_W \bar{C})(X,Y)Z) = \phi^2((\nabla_W C)(X,Y)Z). \tag{6.6}$$

Thus we can state the following theorem :

Theorem 6.1. *An n-dimensional P-Sasakian manifold is globally ϕ-conharmonicly symmetric with respect to the semi-symmetric non-metric connection if and only if the manifold is also globally ϕ-conharmonicly symmetric with respect to the Levi-Civita connection provided the vector fields X, Y, Z, W are orthogonal to ξ.*

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