Novi Sad J. Math.
Vol. 43, No. 2, 2013, 139-149

LOCAL CLOSURE FUNCTIONS IN IDEAL TOPOLOGICAL SPACES

Ahmad Al-Omari[1] **and Takashi Noiri**[2]

Abstract. In this paper, $(X, \tau, \mathcal{I})$ denotes an ideal topological space. Analogously to the local function [2], we define an operator $\Gamma(A)(\mathcal{I}, \tau)$ called the local closure function of A with respect to $\mathcal{I}$ and τ as follows: $\Gamma(A)(\mathcal{I}, \tau) = \{x \in X : A \cap Cl(U) \notin \mathcal{I}$ for every $U \in \tau(x)\}$. We investigate properties of $\Gamma(A)(\mathcal{I}, \tau)$. Moreover, by using $\Gamma(A)(\mathcal{I}, \tau)$, we introduce an operator $\Psi_\Gamma : \mathcal{P}(X) \to \tau$ satisfying $\Psi_\Gamma(A) = X - \Gamma(X - A)$ for each $A \in \mathcal{P}(X)$. We set $\sigma = \{A \subseteq X : A \subseteq \Psi_\Gamma(A)\}$ and $\sigma_0 = \{A \subseteq X : A \subseteq Int(Cl(\Psi_\Gamma(A)))\}$ and show that $\tau_\theta \subseteq \sigma \subseteq \sigma_0$.

AMS Mathematics Subject Classification (2010): 54A05, 54C10

Key words and phrases: ideal topological space, local function, local closure function

1. Introduction and preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties:

1. $A \in \mathcal{I}$ and $B \subseteq A$ implies that $B \in \mathcal{I}$.

2. $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every open set U containing $x\}$ is called the local function of A with respect to $\mathcal{I}$ and τ (see [1], [2]). We simply write A^* instead of $A^*(\mathcal{I}, \tau)$ in case there is no chance for confusion. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$, finer than τ, generating by the base $\beta(\mathcal{I}, \tau) = \{U - J : U \in \tau$ and $J \in \mathcal{I}\}$. It is known in Example 3.6 of [2] that $\beta(\mathcal{I}, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(\mathcal{I})$ is denoted by τ^*. Recall that A is said to be $*$-dense in itself (resp. τ^*-closed, $*$-perfect) if $A \subseteq A^*$ (resp. $A^* \subseteq A$, $A = A^*$). For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of A in (X, τ^*), respectively. In 1968, Veličko [6] introduced the class of θ-open sets. A set A is said to be θ-open [6] if every point of A

[1]Al al-Bayt University, Faculty of Sciences, Department of Mathematics P.O. Box 130095, Mafraq 25113, Jordan, e-mail: omarimutah1@yahoo.com
[2]2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken, 869-5142 Japan, e-mail: t.noiri@nifty.com

has an open neighborhood whose closure is contained in A. The θ-interior [6] of A in X is the union of all θ-open subsets of A and is denoted by $Int_\theta(A)$. Naturally, the complement of a θ-open set is said to be θ-closed. Equivalently, $Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \phi$ for every $U \in \tau(x)\}$ and a set A is θ-closed if and only if $A = Cl_\theta(A)$. Note that all θ-open sets form a topology on X which is coarser than τ, and is denoted by τ_θ and that a space (X, τ) is regular if and only if $\tau = \tau_\theta$. Note also that the θ-closure of a given set need not be a θ-closed set.

In this paper, analogously to the local function $A^*(\mathcal{I}, \tau)$, we define an operator $\Gamma(A)(\mathcal{I}, \tau)$ called the local closure function of A with respect to $\mathcal{I}$ and τ as follows: $\Gamma(A)(\mathcal{I}, \tau) = \{x \in X : A \cap Cl(U) \notin \mathcal{I}$ for every $U \in \tau(x)\}$. We investigate properties of $\Gamma(A)(\mathcal{I}, \tau)$. Moreover, we introduce an operator $\Psi_\Gamma : \mathcal{P}(X) \to \tau$ satisfying $\Psi_\Gamma(A) = X - \Gamma(X - A)$ for each $A \in \mathcal{P}(X)$. We set $\sigma = \{A \subseteq X : A \subseteq \Psi_\Gamma(A)\}$ and $\sigma_0 = \{A \subseteq X : A \subseteq Int(Cl(\Psi_\Gamma(A)))\}$ and show that $\tau_\theta \subseteq \sigma \subseteq \sigma_0$.

2. Local closure functions

Definition 2.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For a subset A of X, we define the following operator: $\Gamma(A)(\mathcal{I}, \tau) = \{x \in X : A \cap Cl(U) \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. In case there is no confusion $\Gamma(A)(\mathcal{I}, \tau)$ is briefly denoted by $\Gamma(A)$ and is called the local closure function of A with respect to $\mathcal{I}$ and τ.

Lemma 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $A^*(\mathcal{I}, \tau) \subseteq \Gamma(A)(\mathcal{I}, \tau)$ for every subset A of X.*

Proof. Let $x \in A^*(\mathcal{I}, \tau)$. Then, $A \cap U \notin \mathcal{I}$ for every open set U containing x. Since $A \cap U \subseteq A \cap Cl(U)$, we have $A \cap Cl(U) \notin \mathcal{I}$ and hence $x \in \Gamma(A)(\mathcal{I}, \tau)$. □

Example 2.3. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$, and $\mathcal{I} = \{\phi, \{c\}\}$. Let $A = \{b, c, d\}$. Then $\Gamma(A) = \{a, b, c, d\}$ and $A^* = \{b, d\}$.

Example 2.4. Let (X, τ) be the real numbers with the left-ray topology, i.e. $\tau = \{(-\infty, a) : a \in X\} \cup \{X, \phi\}$. Let $\mathcal{I}_f$ be the ideal of all finite subsets of X. Let $A = [0, 1]$. Then $\Gamma(A) = \{x \in X : A \cap Cl(U) = A \notin \mathcal{I}_f$ for every $U \in \tau(x)\} = X$ and $-1 \notin A^*$ which shows $A^* \subset \Gamma(A)$.

Lemma 2.5. *Let (X, τ) be a topological space and A be a subset of X. Then*

1. *If A is open, then $Cl(A) = Cl_\theta(A)$.*
2. *If A is closed, then $Int(A) = Int_\theta(A)$.*

Theorem 2.6. *Let (X, τ) be a topological space, $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X, and let A and B be subsets of X. Then the following properties hold:*

1. *If $A \subseteq B$, then $\Gamma(A) \subseteq \Gamma(B)$.*

2. *If* $\mathcal{I} \subseteq \mathcal{J}$*, then* $\Gamma(A)(\mathcal{I}) \supseteq \Gamma(A)(\mathcal{J})$.

3. $\Gamma(A) = Cl(\Gamma(A)) \subseteq Cl_\theta(A)$ *and* $\Gamma(A)$ *is closed.*

4. *If* $A \subseteq \Gamma(A)$ *and* $\Gamma(A)$ *is open, then* $\Gamma(A) = Cl_\theta(A)$.

5. *If* $A \in \mathcal{I}$*, then* $\Gamma(A) = \emptyset$.

Proof. (1) Suppose that $x \notin \Gamma(B)$. Then there exists $U \in \tau(x)$ such that $B \cap Cl(U) \in \mathcal{I}$. Since $A \cap Cl(U) \subseteq B \cap Cl(U)$, $A \cap Cl(U) \in \mathcal{I}$. Hence $x \notin \Gamma(A)$. Thus $X \setminus \Gamma(B) \subseteq X \setminus \Gamma(A)$ or $\Gamma(A) \subseteq \Gamma(B)$.
(2) Suppose that $x \notin \Gamma(A)(\mathcal{I})$. There exists $U \in \tau(x)$ such that $A \cap Cl(U) \in \mathcal{I}$. Since $\mathcal{I} \subseteq \mathcal{J}$, $A \cap Cl(U) \in \mathcal{J}$ and $x \notin \Gamma(A)(\mathcal{J})$. Therefore, $\Gamma(A)(\mathcal{J}) \subseteq \Gamma(A)(\mathcal{I})$.
(3) We have $\Gamma(A) \subseteq Cl(\Gamma(A))$ in general. Let $x \in Cl(\Gamma(A))$. Then $\Gamma(A) \cap U \neq \emptyset$ for every $U \in \tau(x)$. Therefore, there exists some $y \in \Gamma(A) \cap U$ and $U \in \tau(y)$. Since $y \in \Gamma(A)$, $A \cap Cl(U) \notin \mathcal{I}$ and hence $x \in \Gamma(A)$. Hence we have $Cl(\Gamma(A)) \subseteq \Gamma(A)$ and hence $\Gamma(A) = Cl(\Gamma(A))$. Again, let $x \in Cl(\Gamma(A)) = \Gamma(A)$, then $A \cap Cl(U) \notin \mathcal{I}$ for every $U \in \tau(x)$. This implies $A \cap Cl(U) \neq \emptyset$ for every $U \in \tau(x)$. Therefore, $x \in Cl_\theta(A)$. This shows that $\Gamma(A)(\mathcal{I}) = Cl(\Gamma(A)) \subseteq Cl_\theta(A)$.
(4) For any subset A of X, by (3) we have $\Gamma(A) = Cl(\Gamma(A)) \subseteq Cl_\theta(A)$. Since $A \subseteq \Gamma(A)$ and $\Gamma(A)$ is open, by Lemma 2.5, $Cl_\theta(A) \subseteq Cl_\theta(\Gamma(A)) = Cl(\Gamma(A)) = \Gamma(A) \subseteq Cl_\theta(A)$ and hence $\Gamma(A) = Cl_\theta(A)$.
(5) Suppose that $x \in \Gamma(A)$. Then for any $U \in \tau(x)$, $A \cap Cl(U) \notin \mathcal{I}$. But since $A \in \mathcal{I}$, $A \cap Cl(U) \in \mathcal{I}$ for every $U \in \tau(x)$. This is a contradiction. Hence $\Gamma(A) = \emptyset$. □

Lemma 2.7. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. If* $U \in \tau_\theta$*, then* $U \cap \Gamma(A) = U \cap \Gamma(U \cap A) \subseteq \Gamma(U \cap A)$ *for any subset* A *of* X.

Proof. Suppose that $U \in \tau_\theta$ and $x \in U \cap \Gamma(A)$. Then $x \in U$ and $x \in \Gamma(A)$. Since $U \in \tau_\theta$, then there exists $W \in \tau$ such that $x \in W \subseteq Cl(W) \subseteq U$. Let V be any open set containing x. Then $V \cap W \in \tau(x)$ and $Cl(V \cap W) \cap A \notin \mathcal{I}$ and hence $Cl(V) \cap (U \cap A) \notin \mathcal{I}$. This shows that $x \in \Gamma(U \cap A)$ and hence we obtain $U \cap \Gamma(A) \subseteq \Gamma(U \cap A)$. Moreover, $U \cap \Gamma(A) \subseteq U \cap \Gamma(U \cap A)$ and by Theorem 2.6 $\Gamma(U \cap A) \subseteq \Gamma(A)$ and $U \cap \Gamma(U \cap A) \subseteq U \cap \Gamma(A)$. Therefore, $U \cap \Gamma(A) = U \cap \Gamma(U \cap A)$. □

Theorem 2.8. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* A*,* B *any subsets of* X*. Then the following properties hold:*

1. $\Gamma(\emptyset) = \emptyset$.

2. $\Gamma(A) \cup \Gamma(B) = \Gamma(A \cup B)$.

Proof. (1) The proof is obvious.
(2) It follows from Theorem 2.6 that $\Gamma(A \cup B) \supseteq \Gamma(A) \cup \Gamma(B)$. To prove the reverse inclusion, let $x \notin \Gamma(A) \cup \Gamma(B)$. Then x belongs neither to $\Gamma(A)$ nor to $\Gamma(B)$. Therefore there exist U_x, $V_x \in \tau(x)$ such that $Cl(U_x) \cap A \in \mathcal{I}$ and

$Cl(V_x) \cap B \in \mathcal{I}$. Since $\mathcal{I}$ is additive, $(Cl(U_x) \cap A) \cup (Cl(V_x) \cap B) \in \mathcal{I}$. Moreover, since $\mathcal{I}$ is hereditary and

$$\begin{aligned}(Cl(U_x) \cap A) \cup (Cl(V_x) \cap B) &= [(Cl(U_x) \cap A) \cup Cl(V_x)] \cap [(Cl(U_x) \cap A) \cup B] \\ &= (Cl(U_x) \cup Cl(V_x)) \cap (A \cup Cl(V_x)) \cap (Cl(U_x) \cup B) \cap (A \cup B) \\ &\supseteq Cl(U_x \cap V_x) \cap (A \cup B),\end{aligned}$$

$Cl(U_x \cap V_x) \cap (A \cup B) \in \mathcal{I}$. Since $U_x \cap V_x \in \tau(x)$, $x \notin \Gamma(A \cup B)$. Hence $(X \setminus \Gamma(A)) \cap (X \setminus \Gamma(B) \subseteq X \setminus \Gamma(A \cup B)$ or $\Gamma(A \cup B) \subseteq \Gamma(A) \cup \Gamma(B)$. Hence we obtain $\Gamma(A) \cup \Gamma(B) = \Gamma(A \cup B)$. □

Lemma 2.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be subsets of X. Then $\Gamma(A) - \Gamma(B) = \Gamma(A - B) - \Gamma(B)$.*

Proof. We have by Theorem 2.8 $\Gamma(A) = \Gamma[(A - B) \cup (A \cap B)] = \Gamma(A - B) \cup \Gamma(A \cap B) \subseteq \Gamma(A - B) \cup \Gamma(B)$. Thus $\Gamma(A) - \Gamma(B) \subseteq \Gamma(A - B) - \Gamma(B)$. By Theorem 2.6, $\Gamma(A - B) \subseteq \Gamma(A)$ and hence $\Gamma(A - B) - \Gamma(B) \subseteq \Gamma(A) - \Gamma(B)$. Hence $\Gamma(A) - \Gamma(B) = \Gamma(A - B) - \Gamma(B)$. □

Corollary 2.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be subsets of X with $B \in \mathcal{I}$. Then $\Gamma(A \cup B) = \Gamma(A) = \Gamma(A - B)$.*

Proof. Since $B \in \mathcal{I}$, by Theorem 2.6 $\Gamma(B) = \emptyset$. By Lemma 2.9, $\Gamma(A) = \Gamma(A - B)$ and by Theorem 2.8 $\Gamma(A \cup B) = \Gamma(A) \cup \Gamma(B) = \Gamma(A)$ □

3. Closure compatibility of topological spaces

Definition 3.1. [4] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say the τ is compatible with the ideal $\mathcal{I}$, denoted $\tau \sim \mathcal{I}$, if the following holds for every $A \subseteq X$, if for every $x \in A$ there exists $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Definition 3.2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say the τ is closure compatible with the ideal $\mathcal{I}$, denoted $\tau \sim_\Gamma \mathcal{I}$, if the following holds for every $A \subseteq X$, if for every $x \in A$ there exists $U \in \tau(x)$ such that $Cl(U) \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Remark 3.3. If τ is compatible with $\mathcal{I}$, then τ is closure compatible with $\mathcal{I}$.

Theorem 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, the following properties are equivalent:*

1. *$\tau \sim_\Gamma \mathcal{I}$;*
2. *If a subset A of X has a cover of open sets each of whose closure intersection with A is in $\mathcal{I}$, then $A \in \mathcal{I}$;*
3. *For every $A \subseteq X$, $A \cap \Gamma(A) = \emptyset$ implies that $A \in \mathcal{I}$;*

4. *For every* $A \subseteq X$, $A - \Gamma(A) \in \mathcal{I}$;
5. *For every* $A \subseteq X$, *if* A *contains no nonempty subset* B *with* $B \subseteq \Gamma(B)$, *then* $A \in \mathcal{I}$.

Proof. (1) $\Rightarrow$ (2): The proof is obvious.
(2) $\Rightarrow$ (3): Let $A \subseteq X$ and $x \in A$. Then $x \notin \Gamma(A)$ and there exists $V_x \in \tau(x)$ such that $Cl(V_x) \cap A \in \mathcal{I}$. Therefore, we have $A \subseteq \cup\{V_x : x \in A\}$ and $V_x \in \tau(x)$ and by (2) $A \in \mathcal{I}$.
(3) $\Rightarrow$ (4): For any $A \subseteq X$, $A - \Gamma(A) \subseteq A$ and $(A - \Gamma(A)) \cap \Gamma(A - \Gamma(A)) \subseteq (A - \Gamma(A)) \cap \Gamma(A) = \emptyset$. By (3), $A - \Gamma(A) \in \mathcal{I}$.
(4) $\Rightarrow$ (5): By (4), for every $A \subseteq X$, $A - \Gamma(A) \in \mathcal{I}$. Let $A - \Gamma(A) = J \in \mathcal{I}$, then $A = J \cup (A \cap \Gamma(A))$ and by Theorem 2.8(2) and Theorem 2.6(5), $\Gamma(A) = \Gamma(J) \cup \Gamma(A \cap \Gamma(A)) = \Gamma(A \cap \Gamma(A))$. Therefore, we have $A \cap \Gamma(A) = A \cap \Gamma(A \cap \Gamma(A)) \subseteq \Gamma(A \cap \Gamma(A))$ and $A \cap \Gamma(A) \subseteq A$. By the assumption $A \cap \Gamma(A) = \emptyset$ and hence $A = A - \Gamma(A) \in \mathcal{I}$.
(5) $\Rightarrow$ (1): Let $A \subseteq X$ and assume that for every $x \in A$, there exists $U \in \tau(x)$ such that $Cl(U) \cap A \in \mathcal{I}$. Then $A \cap \Gamma(A) = \emptyset$. Suppose that A contains B such that $B \subseteq \Gamma(B)$. Then $B = B \cap \Gamma(B) \subseteq A \cap \Gamma(A) = \emptyset$. Therefore, A contains no nonempty subset B with $B \subseteq \Gamma(B)$. Hence $A \in \mathcal{I}$.

□

Theorem 3.5. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. If* τ *is closure compatible with* $\mathcal{I}$, *then the following equivalent properties hold:*

1. *For every* $A \subseteq X$, $A \cap \Gamma(A) = \emptyset$ *implies that* $\Gamma(A) = \emptyset$;
2. *For every* $A \subseteq X$, $\Gamma(A - \Gamma(A)) = \emptyset$;
3. *For every* $A \subseteq X$, $\Gamma(A \cap \Gamma(A)) = \Gamma(A)$.

Proof. First, we show that (1) holds if τ is closure compatible with $\mathcal{I}$. Let A be any subset of X and $A \cap \Gamma(A) = \emptyset$. By Theorem 3.4, $A \in \mathcal{I}$ and by Theorem 2.6 (5) $\Gamma(A) = \emptyset$.
(1) $\Rightarrow$ (2): Assume that for every $A \subseteq X$, $A \cap \Gamma(A) = \emptyset$ implies that $\Gamma(A) = \emptyset$. Let $B = A - \Gamma(A)$, then

$$\begin{aligned} B \cap \Gamma(B) &= (A - \Gamma(A)) \cap \Gamma(A - \Gamma(A)) \\ &= (A \cap (X - \Gamma(A))) \cap \Gamma(A \cap (X - \Gamma(A))) \\ &\subseteq [A \cap (X - \Gamma(A))] \cap [\Gamma(A) \cap (\Gamma(X - \Gamma(A)))] = \emptyset. \end{aligned}$$

By (1), we have $\Gamma(B) = \emptyset$. Hence $\Gamma(A - \Gamma(A)) = \emptyset$.
(2) $\Rightarrow$ (3): Assume for every $A \subseteq X$, $\Gamma(A - \Gamma(A)) = \emptyset$.

$$\begin{aligned} A &= (A - \Gamma(A)) \cup (A \cap \Gamma(A)) \\ \Gamma(A) &= \Gamma[(A - \Gamma(A)) \cup (A \cap \Gamma(A))] \\ &= \Gamma(A - \Gamma(A)) \cup \Gamma(A \cap \Gamma(A)) \\ &= \Gamma(A \cap \Gamma(A)). \end{aligned}$$

(3) $\Rightarrow$ (1): Assume for every $A \subseteq X$, $A \cap \Gamma(A) = \emptyset$ and $\Gamma(A \cap \Gamma(A)) = \Gamma(A)$. This implies that $\emptyset = \Gamma(\emptyset) = \Gamma(A)$. □

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then the following properties are equivalent:*

1. $Cl(\tau) \cap \mathcal{I} = \emptyset$, *where* $Cl(\tau) = \{Cl(V) : V \in \tau\}$;
2. *If* $I \in \mathcal{I}$, *then* $Int_\theta(I) = \emptyset$;
3. *For every clopen* G, $G \subseteq \Gamma(G)$;
4. $X = \Gamma(X)$.

Proof. (1) $\Rightarrow$ (2): Let $Cl(\tau) \cap \mathcal{I} = \emptyset$ and $I \in \mathcal{I}$. Suppose that $x \in Int_\theta(I)$. Then there exists $U \in \tau$ such that $x \in U \subseteq Cl(U) \subseteq I$. Since $I \in \mathcal{I}$ and hence $\emptyset \neq \{x\} \subseteq Cl(U) \in Cl(\tau) \cap \mathcal{I}$. This is contrary to $Cl(\tau) \cap \mathcal{I} = \emptyset$. Therefore, $Int_\theta(I) = \emptyset$.
(2)$\Rightarrow$ (3): Let $x \in G$. Assume $x \notin \Gamma(G)$, then there exists $U_x \in \tau(x)$ such that $G \cap Cl(U_x) \in \mathcal{I}$ and hence $G \cap U_x \in \mathcal{I}$. Since G is clopen, by (2) and Lemma 2.5, $x \in G \cap U_x = Int(G \cap U_x) \subseteq Int(G \cap Cl(U_x)) = Int_\theta(G \cap Cl(U_x)) = \emptyset$. This is a contradiction. Hence $x \in \Gamma(G)$ and $G \subseteq \Gamma(G)$.
(3)$\Rightarrow$ (4): Since X is clopen, then $X = \Gamma(X)$.
(4)$\Rightarrow$ (1): $X = \Gamma(X) = \{x \in X : Cl(U) \cap X = Cl(U) \notin \mathcal{I}$ for each open set U containing $x\}$. Hence $Cl(\tau) \cap \mathcal{I} = \emptyset$.

□

Theorem 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, τ be closure compatible with $\mathcal{I}$. Then for every $G \in \tau_\theta$ and any subset A of X, $Cl(\Gamma(G \cap A)) = \Gamma(G \cap A) \subseteq \Gamma(G \cap \Gamma(A)) \subseteq Cl_\theta(G \cap \Gamma(A))$.*

Proof. By Theorem 3.5(3) and Theorem 2.6, we have $\Gamma(G \cap A) = \Gamma((G \cap A) \cap \Gamma(G \cap A)) \subseteq \Gamma(G \cap \Gamma(A))$. Moreover, by Theorem 2.6, $Cl(\Gamma(G \cap A)) = \Gamma(G \cap A) \subseteq \Gamma(G \cap \Gamma(A)) \subseteq Cl_\theta(G \cap \Gamma(A))$. □

4. Ψ_Γ-operator

Definition 4.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. An operator $\Psi_\Gamma : \mathcal{P}(X) \to \tau$ is defined as follows: for every $A \in X$, $\Psi_\Gamma(A) = \{x \in X :$ there exists $U \in \tau(x)$ such that $Cl(U) - A \in \mathcal{I}\}$ and observe that $\Psi_\Gamma(A) = X - \Gamma(X - A)$.

Several basic facts concerning the behavior of the operator Ψ_Γ are included in the following theorem.

Theorem 4.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties hold:*

1. *If* $A \subseteq X$, *then* $\Psi_\Gamma(A)$ *is open.*
2. *If* $A \subseteq B$, *then* $\Psi_\Gamma(A) \subseteq \Psi_\Gamma(B)$.
3. *If* $A, B \in \mathcal{P}(X)$, *then* $\Psi_\Gamma(A \cap B) = \Psi_\Gamma(A) \cap \Psi_\Gamma(B)$.
4. *If* $A \subseteq X$, *then* $\Psi_\Gamma(A) = \Psi_\Gamma(\Psi_\Gamma(A))$ *if and only if* $\Gamma(X - A) = \Gamma(\Gamma(X - A))$.

5. *If* $A \in \mathcal{I}$*, then* $\Psi_\Gamma(A) = X - \Gamma(X)$.
6. *If* $A \subseteq X$*,* $I \in \mathcal{I}$*, then* $\Psi_\Gamma(A - I) = \Psi_\Gamma(A)$.
7. *If* $A \subseteq X$*,* $I \in \mathcal{I}$*, then* $\Psi_\Gamma(A \cup I) = \Psi_\Gamma(A)$.
8. *If* $(A - B) \cup (B - A) \in \mathcal{I}$*, then* $\Psi_\Gamma(A) = \Psi_\Gamma(B)$.

Proof. (1) This follows from Theorem 2.6 (3).
(2) This follows from Theorem 2.6 (1).
(3)

$$\begin{aligned}\Psi_\Gamma(A \cap B) =& X - \Gamma(X - (A \cap B))\\ =& X - \Gamma[(X - A) \cup (X - B)]\\ =& X - [\Gamma(X - A) \cup \Gamma(X - B)]\\ =& [X - \Gamma(X - A) \cap [X - \Gamma(X - B)]\\ =& \Psi_\Gamma(A) \cap \Psi_\Gamma(B).\end{aligned}$$

(4) This follows from the facts:

1. $\Psi_\Gamma(A) = X - \Gamma(X - A)$.
2. $\Psi_\Gamma(\Psi_\Gamma(A)) = X - \Gamma[X - (X - \Gamma(X - A))] = X - \Gamma(\Gamma(X - A))$.

(5) By Corollary 2.10 we obtain that $\Gamma(X - A) = \Gamma(X)$ if $A \in \mathcal{I}$.
(6) This follows from Corollary 2.10 and $\Psi_\Gamma(A - I) = X - \Gamma[X - (A - I)] = X - \Gamma[(X - A) \cup I] = X - \Gamma(X - A) = \Psi_\Gamma(A)$.
(7) This follows from Corollary 2.10 and $\Psi_\Gamma(A \cup I) = X - \Gamma[X - (A \cup I)] = X - \Gamma[(X - A) - I] = X - \Gamma(X - A) = \Psi_\Gamma(A)$.
(8) Assume $(A - B) \cup (B - A) \in \mathcal{I}$. Let $A - B = I$ and $B - A = J$. Observe that $I, J \in \mathcal{I}$ by heredity. Also observe that $B = (A - I) \cup J$. Thus $\Psi_\Gamma(A) = \Psi_\Gamma(A - I) = \Psi[(A - I) \cup J] = \Psi_\Gamma(B)$ by (6) and (7). □

Corollary 4.3. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. Then* $U \subseteq \Psi_\Gamma(U)$ *for every* θ*-open set* $U \subseteq X$.

Proof. We know that $\Psi_\Gamma(U) = X - \Gamma(X - U)$. Now $\Gamma(X - U) \subseteq Cl_\theta(X - U) = X - U$, since $X - U$ is θ-closed. Therefore, $U = X - (X - U) \subseteq X - \Gamma(X - U) = \Psi_\Gamma(U)$. □

Now we give an example of a set A which is not θ-open but satisfies $A \subseteq \Psi_\Gamma(A)$.

Example 4.4. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$, and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Let $A = \{a\}$. Then $\Psi_\Gamma(\{a\}) = X - \Gamma(X - \{a\}) = X - \Gamma(\{b, c, d\}) = X - \{b, d\} = \{a, c\}$. Therefore, $A \subseteq \Psi_\Gamma(A)$, but A is not θ-open.

Example 4.5. Let (X, τ) be the real numbers with the left-ray topology, i.e. $\tau = \{(-\infty, a) : a \in X\} \cup \{X, \phi\}$. Let $\mathcal{I}_f$ be the ideal of all finite subsets of X. Let $A = X - \{0, 1\}$. Then $\Psi_\Gamma(\{A\}) = X - \Gamma(\{0, 1\}) = X$. Therefore, $A \subseteq \Psi_\Gamma(A)$, but A is not θ-open.

Theorem 4.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following properties hold:*

1. $\Psi_\Gamma(A) = \cup\{U \in \tau : Cl(U) - A \in \mathcal{I}\}$.

2. $\Psi_\Gamma(A) \supseteq \cup\{U \in \tau : (Cl(U) - A) \cup (A - Cl(U)) \in \mathcal{I}\}$.

Proof. (1) This follows immediately from the definition of Ψ_Γ-operator.
(2) Since $\mathcal{I}$ is heredity, it is obvious that $\cup\{U \in \tau : (Cl(U) - A) \cup (A - Cl(U)) \in \mathcal{I}\} \subseteq \cup\{U \in \tau : Cl(U) - A \in \mathcal{I}\} = \Psi_\Gamma(A)$ for every $A \subseteq X$. □

Theorem 4.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\sigma = \{A \subseteq X : A \subseteq \Psi_\Gamma(A)\}$. Then σ is a topology for X.*

Proof. Let $\sigma = \{A \subseteq X : A \subseteq \Psi_\Gamma(A)\}$. Since $\phi \in \mathcal{I}$, by Theorem 2.6(5) $\Gamma(\phi) = \phi$ and $\Psi_\Gamma(X) = X - \Gamma(X - X) = X - \Gamma(\phi) = X$. Moreover, $\Psi_\Gamma(\phi) = X - \Gamma(X - \phi) = X - X = \phi$. Therefore, we obtain that $\phi \subseteq \Psi_\Gamma(\phi)$ and $X \subseteq \Psi_\Gamma(X) = X$, and thus ϕ and $X \in \sigma$. Now if $A, B \in \sigma$, then by Theorem 4.2 $A \cap B \subseteq \Psi_\Gamma(A) \cap \Psi_\Gamma(B) = \Psi_\Gamma(A \cap B)$ which implies that $A \cap B \in \sigma$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $A_\alpha \subseteq \Psi_\Gamma(A_\alpha) \subseteq \Psi_\Gamma(\cup A_\alpha)$ for every α and hence $\cup A_\alpha \subseteq \Psi_\Gamma(\cup A_\alpha)$. This shows that σ is a topology. □

Lemma 4.8. *If either $A \in \tau$ or $B \in \tau$, then $Int(Cl(A \cap B)) = Int(Cl(A)) \cap Int(Cl(B))$.*

Proof. This is an immediate consequence of Lemma 3.5 of [5]. □

Theorem 4.9. *Let $\sigma_0 = \{A \subseteq X : A \subseteq Int(Cl(\Psi_\Gamma(A)))\}$, then σ_0 is a topology for X.*

Proof. By Theorem 4.2, for any subset A of X, $\Psi_\Gamma(A)$ is open and $\sigma \subset \sigma_0$. Therefore, $\emptyset$, $X \in \sigma_0$. Let A, $B \in \sigma_0$. Then by Lemma 4.8 and Theorem 4.2, we have $A \cap B \subset Int(Cl(\Psi_\Gamma(A))) \cap Int(Cl(\Psi_\Gamma(B))) = Int(Cl(\Psi_\Gamma(A) \cap \Psi_\Gamma(B))) = Int(Cl(\Psi_\Gamma(A \cap B)))$. Therefore, $A \cap B \in \sigma_0$. Let $A_\alpha \in \sigma_0$ for each $\alpha \in \Delta$. By Theorem 4.2, for each $\alpha \in \Delta$, $A_\alpha \subseteq Int(Cl(\Psi_\Gamma(A_\alpha))) \subseteq Int(Cl(\Psi_\Gamma(\cup A_\alpha)))$ and hence $\cup A_\alpha \subset Int(Cl(\Psi_\Gamma(\cup A_\alpha)))$. Hence $\cup A_\alpha \in \sigma_0$. This shows that σ_0 is a topology for X. □

By Theorem 4.2 and Corollary 4.3 the following relations holds:

$$\begin{array}{ccc} \theta\text{-open} & \longrightarrow & \text{open} \\ \downarrow & & \\ \sigma\text{-open} & \longrightarrow & \sigma_0\text{-open} \end{array}$$

Remark 4.10. 1. In Example 4.4, A is σ-open but it is not open. Therefore, every σ_0-open set is not open.

2. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\phi, \{a\}\}$ be an ideal on X. We observe that $\{a\}$ is open but it is not σ_0-open sets, since $\Psi_\Gamma(\{a\}) = X - \Gamma(\{b, c\}) = X - X = \phi$. Also, $\{c\}$ is not open but it is σ-open set, since $\Psi_\Gamma(\{c\}) = X - \Gamma(\{a, b\}) = X - \{b\} = \{a, c\}$.

3. **Question**: Is there an example which shows that $\sigma \subsetneqq \sigma_0$?.

Theorem 4.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\tau \sim_\Gamma \mathcal{I}$ if and only if $\Psi_\Gamma(A) - A \in \mathcal{I}$ for every $A \subseteq X$.*

Proof. Necessity. Assume $\tau \sim_\Gamma \mathcal{I}$ and let $A \subseteq X$. Observe that $x \in \Psi_\Gamma(A) - A$ if and only if $x \notin A$ and $x \notin \Gamma(X - A)$ if and only if $x \notin A$ and there exists $U_x \in \tau(x)$ such that $Cl(U_x) - A \in \mathcal{I}$ if and only if there exists $U_x \in \tau(x)$ such that $x \in Cl(U_x) - A \in \mathcal{I}$. Now, for each $x \in \Psi_\Gamma(A) - A$ and $U_x \in \tau(x)$, $Cl(U_x) \cap (\Psi_\Gamma(A) - A) \in \mathcal{I}$ by heredity and hence $\Psi_\Gamma(A) - A \in \mathcal{I}$ by assumption that $\tau \sim_\Gamma \mathcal{I}$.

Sufficiency. Let $A \subseteq X$ and assume that for each $x \in A$ there exists $U_x \in \tau(x)$ such that $Cl(U_x) \cap A \in \mathcal{I}$. Observe that $\Psi_\Gamma(X - A) - (X - A) = A - \Gamma(A) = \{x :$ there exists $U_x \in \tau(x)$ such that $x \in Cl(U_x) \cap A \in \mathcal{I}\}$. Thus we have $A \subseteq \Psi_\Gamma(X - A) - (X - A) \in \mathcal{I}$ and hence $A \in \mathcal{I}$ by heredity of $\mathcal{I}$. □

Proposition 4.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_\Gamma \mathcal{I}$, $A \subseteq X$. If N is a nonempty open subset of $\Gamma(A) \cap \Psi_\Gamma(A)$, then $N - A \in \mathcal{I}$ and $Cl(N) \cap A \notin \mathcal{I}$.*

Proof. If $N \subseteq \Gamma(A) \cap \Psi_\Gamma(A)$, then $N - A \subseteq \Psi_\Gamma(A) - A \in \mathcal{I}$ by Theorem 4.11 and hence $N - A \in \mathcal{I}$ by heredity. Since $N \in \tau - \{\phi\}$ and $N \subseteq \Gamma(A)$, we have $Cl(N) \cap A \notin \mathcal{I}$ by the definition of $\Gamma(A)$. □

In [3], Newcomb defines $A = B$ [mod $\mathcal{I}$] if $(A - B) \cup (B - A) \in \mathcal{I}$ and observes that $=$ [mod $\mathcal{I}$] is an equivalence relation. By Theorem 4.2 (8), we have that if $A = B$ [mod $\mathcal{I}$], then $\Psi_\Gamma(A) = \Psi_\Gamma(B)$.

Definition 4.13. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called a Baire set with respect to τ and $\mathcal{I}$, denoted $A \in \mathcal{B}_r(X, \tau, \mathcal{I})$, if there exists a θ-open set U such that $A = U$ [mod $\mathcal{I}$].

Lemma 4.14. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_\Gamma \mathcal{I}$. If U, $V \in \tau_\theta$ and $\Psi_\Gamma(U) = \Psi_\Gamma(V)$, then $U = V$ [mod $\mathcal{I}$].*

Proof. Since $U \in \tau_\theta$, by Corollary 4.3 we have $U \subseteq \Psi_\Gamma(U)$ and hence $U - V \subseteq \Psi_\Gamma(U) - V = \Psi_\Gamma(V) - V \in \mathcal{I}$ by Theorem 4.11. Therefore, $U - V \in \mathcal{I}$. Similarly, $V - U \in \mathcal{I}$. Now, $(U - V) \cup (V - U) \in \mathcal{I}$ by additivity. Hence $U = V$ [mod $\mathcal{I}$]. □

Theorem 4.15. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_\Gamma \mathcal{I}$. If A, $B \in \mathcal{B}_r(X, \tau, \mathcal{I})$, and $\Psi_\Gamma(A) = \Psi_\Gamma(B)$, then $A = B$ [mod $\mathcal{I}$].*

Proof. Let $U, V \in \tau_\theta$ be such that $A = U$ [mod $\mathcal{I}$] and $B = V$ [mod $\mathcal{I}$]. Now $\Psi_\Gamma(A) = \Psi_\Gamma(U)$ and $\Psi_\Gamma(B) = \Psi_\Gamma(V)$ by Theorem 4.2(8). Since $\Psi_\Gamma(A) = \Psi_\Gamma(B)$ implies that $\Psi_\Gamma(U) = \Psi_\Gamma(V)$ and hence $U = V$ [mod $\mathcal{I}$] by Lemma 4.14. Hence $A = B$ [mod $\mathcal{I}$] by transitivity. □

Proposition 4.16. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space.*

1. *If $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then there exists $A \in \tau_\theta - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$].*
2. *Let $Cl(\tau) \cap \mathcal{I} = \phi$, then $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ if and only if there exists $A \in \tau_\theta - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$].*

Proof. (1) Assume $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then $B \in \mathcal{B}_r(X, \tau, \mathcal{I})$. Hence there exists $A \in \tau_\theta$ such that $B = A$ [mod $\mathcal{I}$]. If $A = \phi$, then we have $B = \phi$ [mod $\mathcal{I}$]. This implies that $B \in \mathcal{I}$ which is a contradiction.
(2) Assume there exists $A \in \tau_\theta - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$], hence by Definition 4.13, $B \in \mathcal{B}_r(X, \tau, \mathcal{I})$. Then $A = (B - J) \cup I$, where $J = B - A$, $I = A - B \in \mathcal{I}$. If $B \in \mathcal{I}$, then $A \in \mathcal{I}$ by heredity and additivity. Since $A \in \tau_\theta - \{\phi\}$, $A \neq \phi$ and there exists $U \in \tau$ such that $\phi \neq U \subseteq Cl(U) \subseteq A$. Since $A \in \mathcal{I}$, $Cl(U) \in \mathcal{I}$ and hence $Cl(U) \in Cl(\tau) \cap \mathcal{I}$. This contradicts that $Cl(\tau) \cap \mathcal{I} = \phi$. □

Proposition 4.17. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \cap \mathcal{I} = \phi$. If $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then $\Psi_\Gamma(B) \cap Int_\theta(\Gamma(B)) \neq \phi$.*

Proof. Assume $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then by Proposition 4.16(1), there exists $A \in \tau_\theta - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$]. By Theorem 3.6 and Lemma 2.7, $A = A \cap X = A \cap \Gamma(X) \subseteq \Gamma(A \cap X) = \Gamma(A)$. This implies that $\phi \neq A \subseteq \Gamma(A) = \Gamma((B - J) \cup I) = \Gamma(B)$, where $J = B - A, I = A - B \in \mathcal{I}$ by Corollary 2.10. Since $A \in \tau_\theta$, $A \subseteq Int_\theta(\Gamma(B))$. Also, $\phi \neq A \subseteq \Psi_\Gamma(A) = \Psi_\Gamma(B)$ by Corollary 4.3 and Theorem 4.2(8). Consequently, we obtain $A \subseteq \Psi_\Gamma(B) \cap Int_\theta(\Gamma(B))$. □

Given an ideal topological space $(X, \tau, \mathcal{I})$, let $\mathcal{U}(X, \tau, \mathcal{I})$ denote $\{A \subseteq X :$ there exists $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.

Proposition 4.18. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \cap \mathcal{I} = \phi$. If $\tau = \tau_\theta$, then the following statements are equivalent:*

1. *$A \in \mathcal{U}(X, \tau, \mathcal{I})$;*
2. *$\Psi_\Gamma(A) \cap Int_\theta(\Gamma(A)) \neq \phi$;*
3. *$\Psi_\Gamma(A) \cap \Gamma(A) \neq \phi$;*
4. *$\Psi_\Gamma(A) \neq \phi$;*
5. *$Int_*(A) \neq \phi$;*
6. *There exists $N \in \tau - \{\phi\}$ such that $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): Let $B \in \mathcal{B}_r(X,\tau,\mathcal{I})-\mathcal{I}$ such that $B \subseteq A$. Then $Int_\theta(\Gamma(B)) \subseteq Int_\theta(\Gamma(A))$ and $\Psi_\Gamma(B) \subseteq \Psi_\Gamma(A)$ and hence $Int_\theta(\Gamma(B))\cap\Psi_\Gamma(B) \subseteq Int_\theta(\Gamma(A))\cap \Psi_\Gamma(A)$. By Proposition 4.17, we have $\Psi_\Gamma(A) \cap Int_\theta(\Gamma(A)) \neq \phi$.
(2) $\Rightarrow$ (3): The proof is obvious.
(3) $\Rightarrow$ (4): The proof is obvious.
(4) $\Rightarrow$ (5): If $\Psi_\Gamma(A) \neq \phi$, then there exists $U \in \tau - \{\phi\}$ such that $U - A \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U = (U - A) \cup (U \cap A)$, we have $U \cap A \notin \mathcal{I}$. By Theorem 4.2, $\phi \neq (U \cap A) \subseteq \Psi_\Gamma(U) \cap A = \Psi_\Gamma((U - A) \cup (U \cap A)) \cap A = \Psi_\Gamma(U \cap A) \cap A \subseteq \Psi_\Gamma(A) \cap A = Int_*(A)$. Hence $Int_*(A) \neq \phi$.
(5) $\Rightarrow$ (6): If $Int_*(A) \neq \phi$, then by Theorem 3.1 of [2] there exists $N \in \tau-\{\phi\}$ and $I \in \mathcal{I}$ such that $\phi \neq N-I \subseteq A$. We have $N-A \in \mathcal{I}$, $N = (N-A)\cup(N\cap A)$ and $N \notin \mathcal{I}$. This implies that $N \cap A \notin \mathcal{I}$.
(6) $\Rightarrow$ (1): Let $B = N \cap A \notin \mathcal{I}$ with $N \in \tau_\theta - \{\phi\}$ and $N - A \in \mathcal{I}$. Then $B \in \mathcal{B}_r(X,\tau,\mathcal{I}) - \mathcal{I}$ since $B \notin \mathcal{I}$ and $(B - N) \cup (N - B) = N - A \in \mathcal{I}$. □

Theorem 4.19. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space with $\tau \sim_\Gamma \mathcal{I}$, where $Cl(\tau) \cap \mathcal{I} = \phi$. Then for $A \subseteq X$, $\Psi_\Gamma(A) \subseteq \Gamma(A)$.*

Proof. Suppose $x \in \Psi_\Gamma(A)$ and $x \notin \Gamma(A)$. Then there exists a nonempty neighborhood $U_x \in \tau(x)$ such that $Cl(U_x) \cap A \in \mathcal{I}$. Since $x \in \Psi_\Gamma(A)$, by Theorem 4.6 $x \in \cup\{U \in \tau : Cl(U) - A \in \mathcal{I}\}$ and there exists $V \in \tau(x)$ and $Cl(V) - A \in \mathcal{I}$. Now we have $U_x \cap V \in \tau(x)$, $Cl(U_x \cap V) \cap A \in \mathcal{I}$ and $Cl(U_x \cap V) - A \in \mathcal{I}$ by heredity. Hence by finite additivity we have $Cl(U_x \cap V) \cap A) \cup (Cl(U_x \cap V) - A) = Cl(U_x \cap V) \in \mathcal{I}$. Since $(U_x \cap V) \in \tau(x)$, this is contrary to $Cl(\tau) \cap \mathcal{I} = \phi$. Therefore, $x \in \Gamma(A)$. This implies that $\Psi_\Gamma(A) \subseteq \Gamma(A)$. □

Acknowledgement

The authors wish to thank the referee for useful comments and suggestions.

References

[1] Al-omari, A., Noiri, T., A topology via $\mathcal{M}$-local functions in ideal m-spaces. Questions and Answers in General Topology, 30 (2012), 105-114.

[2] Janković, D., Hamlet, T. R., New topologies from old via ideals. Amer. Math. Monthly, 97 (1990), 295-310.

[3] Newcomb, R. L., Topologies which are compact modulo an ideal. Ph. D. Dissertation, Univ. of Cal. at Santa Barbara, 1967.

[4] Njåstad, O., Remarks on topologies defined by local properties. Anh. Norske Vid.-Akad. Oslo (N. S.), 8 (1966), 1-6.

[5] Noiri, T., On α-continuous functions. Casopis Pest. Mat. 109 (1984), 118-126.

[6] Veličko, N. V., H-closed topological spaces. Amer. Math. Soc. Transl. 78 (1968), 103-118.

Received by the editors March 1, 2013