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ALMOST PSEUDO-VALUATION MAP AND PSEUDO-ALMOST VALUATION MAP

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Abstract. Recently author (with A. Taouti) have introduced pseudo-valuation maps and discussed pseudo-valuation domains through these maps. In continuation we also introduced P-Krull domains with the help of defined maps as well. In this note we generalize a pseudo-valuation map v in the form of almost pseudo-valuation map and a pseudo-almost valuation map η . Furthermore, we construct and discuss an almost pseudo-valuation domain and a pseudo-almost valuation domain through the defined maps. Moreover, a few relationships between both integral domains through the defined maps have been proved.

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1. Introduction and Preliminaries

There are numerous studies on PVDs through various aspects. In [8], the group of divisibility of semi-valuation domains has discussed on the basis of a semi-valuation map. Further, [7] deals with the group of divisibility of quasi-local domains, for example see [7, Proposition 3.10]. An integral domain R is said to be a pseudo-valuation domain (PVD) if every prime ideal of R is a strongly prime [5, Definition, p. 2]. A prime ideal P of R is called strongly prime if $xy \in P$, where x, $y \in K$, then $x \in P$ or $y \in P$ (alternatively P is strongly prime if and only if $x^{-1}P \subset P$, whenever $x \in K\backslash R$ [5, Definition, p. 2]. Every valuation domain is a PVD [5, Proposition 1.1] but converse is not true. A quasi-local domain (R, M) is a PVD if and only if $x^{-1}M \subset M$ whenever $x \in K\backslash R$ [5, Theorem 1.4].

By [6, p. 12], an integral domain D with the quotient field K, is said to be a valuation domain if it satisfies either of the (equivalent) conditions: (i) For any two elements $x, y \in D$, either x divides y or y divides x. (ii) For any element $x \in K$, either $x \in D$ or $x^{-1} \in D$. When D is a valuation domain, $G(D)$ is merely the value group; and in this case, ideal theoretic properties of D are easily derived from the corresponding properties of $G(D)$, and conversely. D is said to be an almost valuation domain (AVD) if for every $0 \neq x \in K$, there is a positive integer n such that either x^n or $x^{-n} \in D$.

Following [2], D is said to be a pseudo-almost valuation domain (PAVD) if each prime ideal P of D is pseudo-strongly prime ideal (that is if, whenever

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$x, y \in K$ and $xyP \subseteq P$, then there is a positive integer $m \geq 1$ such that either $x^m \in R$ *or* $y^m P \subseteq P$). Equivalently, D is a PAVD if and only if D is quasi-local and for every nonzero element $x \in K$, there is an integer $n \geq 1$ such that either $x^n \in D$ or $ax^{-n} \in D$ for every nonunit $a \in D$.

Following [1], an integral domain D is said to be an almost pseudo-valuation domain (APVD) if each prime ideal P of D is strongly primary ideal, in the sense that $xy \in P$, $x, y \in K$ implies that either $x^n \in P$ for some $n \geq 1$ or $y \in P$. Equivalently, D is an APVD if and only if D is quasi-local with maximal ideal M such that for every nonzero element $x \in K$, either $x^n \in M$ for some positive integer $n \geq 1$ or $ax^{-1} \in M$ for every nonunit $a \in D$.

In general,

$$\begin{array}{ccccc} VD & \Rightarrow & AVD & \Rightarrow & PAVD \\ \Downarrow & & & & \Uparrow \\ PVD & & \Rightarrow & & APVD \end{array}$$

But none of the above implications is reversible.

As every PVD is necessarily quasi-local [5, Cor 1.3] and a quasi-local domain is PVD if and only if its maximal ideal is strongly prime [5, Theorem 1.4].

Author (with T. Shah) has introduced almost pseudo valuation monoids and pseudo almost valuation monoids in [9]. In [12], authors introduced a generalization of valuation maps by using some different conditions. Recently author, (with A. Taouti) introduced pseudo-valuation maps, and discussed pseudo-valuation domains through these maps [10, Theorem 1.4]. Also, author (with T. Shah and A. Taouti) introduced the class of domains (P-Krull domains) through these maps [11].

In this note we continue our study, and first generalize the pseudo-valuation map as an almost pseudo-valuation map υ and pseudo-almost valuation map η, then we constructed almost pseudo-valuation domain and pseudo-almost valuation domain through the defined maps. Finally, we discuss a few relationships between these domains.

2. Almost pseudo-valuation map and pseudo-almost valuation map

Here we consider $K^* = K \backslash \{0\}$ is a field and G a partially ordered group. Now we begin with the following definition.

Definition 1. *Let $\upsilon : K^* \to G$ be an onto map, which has the following properties. For $x, y \in K^*$;*

(a) $\upsilon(xy) = \upsilon(x) + \upsilon(y)$

(b) $\upsilon(x) < \upsilon(y)$ *implies* $\upsilon(x + y) = \upsilon(x)$.

(c) $n\upsilon(x) = ng > 0$, *for* $n \in \mathbb{Z}^+$ *or* $g = \upsilon(x) < n\upsilon(y) = nh$ *such that* $nh > 0$, *where* $g, h \in G$ *and* $h > 0$.

In Definition 1, the map υ is an extended semi-valuation map. No doubt, (b) implies that it is a quasi-local domain as discussed in [4, p. 180]. Moreover, condition (c) plays an important role, hereafter we call υ, the almost pseudo-valuation map. From Definition 1 we see that υ inherit a specific characteristic in G. We manipulate G in the definition below.

Definition 2. *Let $(G, \leq)$ be a partially ordered group. The partial order $\leq$ is an almost total order if for all $g, h \in G$, there exists some fixed positive integer n such that either $g \leq nh$ (and also $ng \leq nh$) or $h \leq ng$ (and also $nh \leq ng$). We will denote such a group by $G^{\#}$.*

Definition 3. *A partially ordered set X is said to be directed if every two elements have both an upper bound and a lower bound. A partially ordered group G whose partial order is directed is called directed group [3, p. 2].*

Remark 1. *The group $G^{\#}$ is directed and not a torsion free.*

Let $R_\upsilon = \{x \in K : \upsilon(x) \geq 0\}$ be a subset of K^*which is related through the map υ to G. We derive the nature of R_υ and we will find that R_υ is a basically APVD.

Proposition 1. *$R_\upsilon = \{x \in K : \upsilon(x) \geq 0\}$ is an almost pseudo-valuation domain.*

Proof. Clearly $1 \in R_\upsilon$ and, by Definition 1(a), R_υ is closed under multiplication. If x, $y \in R_\upsilon$ then $\upsilon(x - y) \geq \upsilon(1) = 0$ since $\upsilon(x) \geq \upsilon(1)$ and $\upsilon(y) \geq \upsilon(1)$. Thus R_υ is a subring of K with identity. The map υ is, no doubt, a group homomorphism and its kernel is $U = \{x \in K : \upsilon(x) = 0\}$ which shows that U is a group of units of R_υ. So R_υ is an integral domain. Definition1(b) shows that R_υ is a quasi local domain. Let M be a maximal ideal of R_υ, furthermore, let $x \in K \backslash R_\upsilon$ so by definition1(c), $x^n \in M$. Thus R_υ is an almost pseudovaluation domain. □

Below we give a crucial proposition for a better utilization of a group $G^{\#}$.

Proposition 2. *Let D be an integral domain with quotient field K and group of divisibility G. The following are equivalent.*

(i) D is an APVD (and hence quasi-local).

(ii) For each $g \in G$, there exist $n \in \mathbb{Z}^+$ such that either $ng > 0$ or $g < nh$ for all $h \in G$.

Proof. (1) ⇒ (2)

Let M be the only maximal ideal in D. Let $g \in G$ such that $g = xU$, where $x \in K^*$. So, the definition implies if $x^n \in M$ for some positive integer $n \geq 1$, then we have $ng = x^n U > 0$ and if $ax^{-1} \in M$ for any nonunit $a \in D$ such that $nh = aU > 0$, then $ax^{-1}U = h - g > 0$. Thus $g < nh$.

(2) ⇒ (1)

We first show that D is quasi-local. If D has two distinct maximal ideals M and N, then choose $x \in M \backslash N$ and $y \in N \backslash M$. Let $ng' = x^n y^{-n}\ U$ and $h = yU$. Clearly, $ng' \not> 0$ and $g' \not< nh$, while $nh > 0$ because if $g' < nh$ then this means that $xy^{-1}U < yU$ implies that $xU < y^2U$ and equivalently $y^2D \subset xD \subset M$ and hence $y \in M$ contradiction to our supposition, therefore $g' \not< nh$. This contradicts the hypothesis, so D must be local. Let $x \in K$ such that $xU = g$ and $ng = x^nU$, if $ng > 0$ this implies that $x^n \in M$ and if $g < nh$ then $xU < nh$. Let a be a nonunit element in D such that $nh = aU$, obviously $nh > 0$ and hence $xU < aU \Rightarrow ax^{-1}U > 0 \Rightarrow ax^{-1} \in M$. Hence, D is APVD. □

Remark 2. *By above Proposition 2(ii), it becomes clear that $G \cong G^{\#}$ (G is isomorphic to $G^{\#}$). Thus $G^{\#}$ is a group of divisibility of an almost pseudo-valuation domain.*

We can discuss all the characteristics of a APVD through the map υ. Now, after defining an almost pseudo-valuation map we define pseudo-almost valuation map. In Definition 4 we consider $K^* = K\backslash\{0\}$ is a field and G, a partially ordered group.

Definition 4. *Let $\eta : K^* \to G$ be an onto map, which has the following properties. For $x, y \in K^*$;*

(a) $\eta(xy) = \eta(x) + \eta(y)$.

(b) $\eta(x) < \eta(y)$ *implies* $\eta(x + y) = \eta(x)$.

(c) $\eta(x^n) = ng > 0$ *or* $\eta(y) = h$ *such that* $ng < h$ *for all* $h \in G$, *where* $h > 0$.

We call η, the pseudo-almost valuation map.

In Definition 4 η inherits a specific property in G, hereafter we denote such a G by $G^{\#\#}$.

Let $R_\eta = \{x \in K : \eta(x) \geq 0\}$ be a subset of K^*which is related through the map η to G. We derive the nature of R_η and we will find that R_η is a basically APVD.

Proposition 3. $R_\eta = \{x \in K : \eta(x) \geq 0\}$ *is a pseudo almost valuation domain.*

Proof. Clearly, $1 \in R_\eta$ and, by definition1(a) R_η is closed under multiplication. If x, $y \in R_\eta$ then $\eta(x - y) \geq \eta(1) = 0$ since $\eta(x) \geq \eta(1)$ and $\eta(y) \geq \eta(1)$. Thus R_η is a subring of K with identity. The map η is no doubt a group homomorphism and its kernel is $U = \{x \in K : \eta(x) = 0\}$, which shows that U is a group of units of R_η. So R_η is an integral domain. Definition 1(b) shows that R_η is a quasi local domain. Let $x \in K\backslash R_\eta$, by Definition 1(c), $x^n \in D$ for $n \geq 1$. Thus R_η is a pseudo almost valuation domain. □

Further we check the validity of our defined map η and group of divisibility $G^{\#\#}$ in Proposition 4.

Proposition 4. *Let D be an integral domain with quotient field K and group of divisibility $G^{\#\#}$, then the following are equivalent*

(i) D is a PAVD.

(ii) For each $g \in G^{\#\#}$, there exist $n \in \mathbb{Z}^+$ and is fixed, such that either $ng > 0$ or $ng < h$ for all $h \in G^{\#\#}$, where $h > 0$.

Proof. (1) $\Longrightarrow$ (2), let $E(D) = \{x \in K \mid x^n \notin D$ for every $n \geq 1\}$ if $x \in E(D)$, then clearly $ng \ngtr 0$. As in D, every prime ideal is a pseudo strongly prime, so $x^{-n}M \subset M$. Then for each $m \in M$, $x^{-n}m \in M$. Let $xU = g$ and $mU = h > 0$, so $(x^{-n}m)U = x^{-n}U + mU = -ng + h > 0$ implies $ng < h$ for each $h > 0$ otherwise $g > nh$, which follows that $x^n \in D$.

(2) $\Longrightarrow$ (1) Let M be the maximal ideal of D, to show D is a PAVD we only need to show M is the pseudo strongly prime ideal. For this let $x \in E(D)$

such that $xU = g \in G$, for each integer $n \geq 1$. So for each $m \in M$, we choose $mU = h' > 0$. Then by the hypothesis $ng < h$ implies that $x^n U < mU$. This implies $mx^{-n}U > 0$ and so $mx^{-n} \in M$. Hence $x^{-n}M \subset M$. So M is a pseudo strongly prime ideal. □

Remark 3. *From Definition 2 and by Proposition 4 it is clear that* $APVD \Longrightarrow PAVD$*. Also, we have* $G^{\#\#} \subset G^{\#}$*.*

Conclusion 1. *This study brings a method by which one can discuss each of the characteristics of an almost pseudo-valuation domain and a pseudo almost valuation domain with the help of maps* υ *and* η*. Furthermore at the base of the maps* υ *and* η *we can construct new integral domains which can be written as an intersection of almost pseudo-valuation overrings and pseudo-almost valuation overrings. Authors have already introduced the domains that can be written as intersection of pseudo-valuation overrings.*

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