

# A NOTE ON THE EXISTENCE AND GROWTH OF MILD SOLUTIONS OF ABSTRACT CAUCHY PROBLEMS FOR GENERATORS OF INTEGRATED $C$ -SEMIGROUPS AND COSINE FUNCTIONS

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**Abstract.** In this note we analyze the existence and growth of mild solutions of abstract Cauchy problems for generators of integrated  $C$ -semigroups and cosine functions in sequentially complete locally convex spaces.

*AMS Mathematics Subject Classification* (2010): 47D06, 47D60, 47D62, 47D99

*Key words and phrases:* integrated  $C$ -semigroups, integrated  $C$ -cosine functions, mild solutions, fractional powers of operators

## 1. Introduction and preliminaries

The theory of fractional powers of operators has an extensive and long history, so that it would be really difficult to mention here all relevant references on this subject. Complex powers of various types of  $C$ -sectorial operators, in the setting of sequentially complete locally convex spaces, has been recently analyzed in a series of papers by C. Chen, M. Li and the author of this paper [1]-[3]. Our intention here is to incorporate some of results obtained in the above-mentioned papers in the study of existence and growth of mild solutions of abstract Cauchy problems involving generators of integrated  $C$ -semigroups and cosine functions. In order to do that, we shall follow the method proposed by J.M.A.M. van Neerven and B. Straub in [11] (cf. also [4] and [12] for some pioneering results in this direction, and the paper [8] in which the assertions of [11, Theorem 1.1-Theorem 1.2] has been generalized to generators with not necessarily dense domain).

Throughout the paper, we use the standard notation. By  $E$  we denote a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short; the abbreviation  $\otimes$  stands for the fundamental system of seminorms which defines the topology of  $E$ . By  $L(E)$  we denote the space which consists of all continuous linear mappings from  $E$  into  $E$ . The domain, range and resolvent set of a closed linear operator  $A$  on  $E$  are denoted by  $D(A)$ ,  $R(A)$  and  $\rho(A)$ , respectively. Let  $C \in L(E)$  be injective. Then the  $C$ -resolvent set of  $A$ ,  $\rho_C(A)$  in short, is defined by  $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\}$ . We shall always assume that  $C^{-1}AC = A$ . The notions of  $C$ -nonnegative,  $C$ -positive and  $C$ -sectorial operators are taken in the sense of [1].

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Given  $s \in \mathbb{R}$  in advance, set  $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ . The Gamma function is denoted by  $\Gamma(\cdot)$  and the principal branch is always used to take the powers. Set  $0^\alpha := 0$  and  $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$  ( $\alpha > 0$ ,  $t > 0$ ). If  $\gamma \in (0, \pi]$  and  $d \in (0, 1]$ , then we define  $\Sigma_\gamma := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg \lambda| < \gamma\}$ ,  $B_d := \{z \in \mathbb{C} : |z| \leq d\}$  and  $\Sigma(\gamma, d) := \overline{\Sigma_\gamma \cup B_d}$ .

For the sake of convenience, we shall repeat the following definitions of exponentially equicontinuous integrated  $C$ -semigroups and cosine functions in SCLCSs ([7], [6], [14]).

**Definition 1.1.** Suppose  $\alpha > 0$  and  $A$  is a closed linear operator on  $E$ . If there exists a strongly continuous operator family  $(S^\alpha(t))_{t \geq 0}$  ( $S^\alpha(t) \in L(E)$ ,  $t \geq 0$ ) such that:

- (i)  $S^\alpha(t)A \subseteq AS^\alpha(t)$ ,  $t \geq 0$ ,
- (ii)  $S^\alpha(t)C = CS^\alpha(t)$ ,  $t \geq 0$  and
- (iii) for all  $x \in E$  and  $t \geq 0$ :  $\int_0^t S^\alpha(s)x ds \in D(A)$  and

$$A \int_0^t S^\alpha(s)x ds = S^\alpha(t)x - g_{\alpha+1}(t)Cx,$$

then it is said that  $A$  is a subgenerator of a (global)  $\alpha$ -times integrated  $C$ -semigroup  $(S^\alpha(t))_{t \geq 0}$ . It is said that  $(S^\alpha(t))_{t \geq 0}$  is an exponentially equicontinuous  $\alpha$ -times integrated  $C$ -semigroup with a subgenerator  $A$  if, in addition, there exists  $\omega \in \mathbb{R}$  such that the family  $\{e^{-\omega t}S^\alpha(t) : t \geq 0\}$  is equicontinuous.

**Definition 1.2.** Suppose  $\alpha > 0$  and  $A$  is a closed linear operator on  $E$ . If there exists a strongly continuous operator family  $(C^\alpha(t))_{t \geq 0}$  ( $C^\alpha(t) \in L(E)$ ,  $t \geq 0$ ) such that:

- (i)  $C^\alpha(t)A \subseteq AC^\alpha(t)$ ,  $t \geq 0$ ,
- (ii)  $C^\alpha(t)C = CC^\alpha(t)$ ,  $t \geq 0$  and
- (iii) for all  $x \in E$  and  $t \geq 0$ :  $\int_0^t (t-s)C^\alpha(s)x ds \in D(A)$  and

$$A \int_0^t (t-s)C^\alpha(s)x ds = C^\alpha(t)x - g_{\alpha+1}(t)Cx,$$

then it is said that  $A$  is a subgenerator of a (global)  $\alpha$ -times integrated  $C$ -cosine function  $(C^\alpha(t))_{t \geq 0}$ . It is said that  $(C^\alpha(t))_{t \geq 0}$  is an exponentially equicontinuous  $\alpha$ -times integrated  $C$ -cosine function with a subgenerator  $A$  if, in addition, there exists  $\omega \in \mathbb{R}$  such that the family  $\{e^{-\omega t}C^\alpha(t) : t \geq 0\}$  is equicontinuous.

The integral generator of  $(S^\alpha(t))_{t \geq 0}$ , resp.  $(C^\alpha(t))_{t \geq 0}$ , is defined by

$$\hat{A} := \left\{ (x, y) \in E \times E : S^\alpha(t)x - g_{\alpha+1}(t)Cx = \int_0^t S^\alpha(s)y \, ds, \, t \geq 0 \right\}, \text{ resp.},$$

$$\hat{A} := \left\{ (x, y) \in E \times E : C^\alpha(t)x - g_{\alpha+1}(t)Cx = \int_0^t (t-s)C^\alpha(s)y \, ds, \, t \geq 0 \right\}.$$

Recall that  $\hat{A}$  is the maximal subgenerator of  $(S^\alpha(t))_{t \geq 0}$ , resp.  $(C^\alpha(t))_{t \geq 0}$ , with respect to the set inclusion and that  $C^{-1}\hat{A}C = \hat{A}$ .

We need the following useful lemma (cf. [7, Theorem 2.1.11]).

**Lemma 1.3.** *Suppose  $\alpha > 0$  and  $A$  is a closed linear operator on  $E$ . Then the following assertions are equivalent:*

(i)  *$A$  is a subgenerator of an  $\alpha$ -times integrated  $C$ -cosine function  $(C^\alpha(t))_{t \geq 0}$  in  $E$ .*

(ii) *The operator  $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$  is a subgenerator of an  $(\alpha+1)$ -times integrated  $C$ -semigroup  $(S^{\alpha+1}(t))_{t \geq 0}$  in  $E \times E$ , where  $\mathcal{C} := \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ .*

In this case:

$$S^{\alpha+1}(t) = \begin{pmatrix} \int_0^t C^\alpha(s) \, ds & \int_0^t (t-s)C^\alpha(s) \, ds \\ C^\alpha(t) - g_{\alpha+1}(t)C & \int_0^t C^\alpha(s) \, ds \end{pmatrix}, \quad t \geq 0,$$

and the integral generators of  $(C^\alpha(t))_{t \geq 0}$  and  $(S^{\alpha+1}(t))_{t \geq 0}$ , denoted respectively by  $B$  and  $\mathcal{B}$ , satisfy  $\mathcal{B} = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ . Furthermore, the integral generator of  $(C^\alpha(t))_{t \geq 0}$ , resp.  $(S^{\alpha+1}(t))_{t \geq 0}$ , is  $C^{-1}AC$ , resp.  $C^{-1}\mathcal{A}C \equiv \begin{pmatrix} 0 & I \\ C^{-1}AC & 0 \end{pmatrix}$ .

## 2. Existence and growth of mild solutions of operators generating fractionally integrated $C$ -semigroups and cosine functions

Recall that the function  $u(\cdot, x_0)$  is a mild solution of the abstract Cauchy problem

$$(ACP_1) : u'(t, x_0) = Au(t, x_0), \, t \geq 0, \, u(0, x_0) = x_0, \text{ resp.},$$

$$(ACP_2) : u''(t, x_0, y_0) = Au(t, x_0, y_0), \, t \geq 0, \, u(0, x_0, y_0) = x_0, \, u'(0, x_0, y_0) = y_0,$$

iff the mapping  $t \mapsto u(t, x_0)$ ,  $t \geq 0$  is continuous,  $\int_0^t u(s, x_0) \, ds \in D(A)$  and  $A \int_0^t u(s, x_0) \, ds = u(t, x_0) - x_0$ ,  $t \geq 0$ , resp., the mapping  $t \mapsto u(t, x_0, y_0)$ ,  $t \geq 0$  is continuous,  $\int_0^t (t-s)u(s, x_0, y_0) \, ds \in D(A)$  and  $A \int_0^t (t-s)u(s, x_0, y_0) \, ds = u(t, x_0, y_0) - x_0 - ty_0$ ,  $t \geq 0$ .

Suppose  $\alpha \geq 0$  and  $A$  is the integral generator of a global  $\alpha$ -times integrated  $C$ -semigroup  $(S^\alpha(t))_{t \geq 0}$  satisfying that there exists  $\omega \geq 0$  such that the family  $\{e^{-\omega t}S^\alpha(t) : t \geq 0\}$  is equicontinuous. Let  $\sigma \in (0, 1]$  be fixed. Then  $C^{-1}AC = A$  and, for every  $\gamma \in (0, \frac{\pi}{2})$ , there exists  $d \in (0, 1]$  such that

$\Sigma(\gamma, d) \subseteq \rho(A - \omega - \sigma)$  and that the family  $\{(1 + |\lambda|)^{1-\alpha}(\lambda - (A - \omega - \sigma))^{-1}C : \lambda \in \Sigma(\gamma, d)\}$  is equicontinuous. Set  $A_{\omega+\sigma} := -(\omega + \sigma - A)$  and, after that,  $C_\alpha := (-A_{\omega+\sigma})^{-1-\lfloor\alpha\rfloor}C^2$ . Then  $C_\alpha^{-1}A_{\omega+\sigma}C_\alpha = A_{\omega+\sigma}$  and it is not difficult to prove that the operator  $-A_{\omega+\sigma}$  is  $C_\alpha$ -sectorial of angle  $\pi/2$  and that the condition [1, (H)] holds with  $d = \sigma/2$ . Therefore, for every  $z \in \mathbb{C}$ , we can construct the power  $(-A_{\omega+\sigma})_z$  following the method proposed in [1], with the operator  $C$  replaced by  $C_\alpha$ . Then, for every  $z \in \mathbb{C}$ , the power  $(-A_{\omega+\sigma})_z$  coincides with that constructed in [2]; see [2, Remark 2.13(i)]. The following properties of powers will be used henceforth (cf. [1]-[2] for more details):

- (P0) For every  $k \in \mathbb{Z}$ , we have  $(-A_{\omega+\sigma})_k = C_\alpha^{-1}(-A_{\omega+\sigma})^k C_\alpha$ , where  $(-A_{\omega+\sigma})^k$  denotes the usual power of the operator  $-A_{\omega+\sigma}$  and  $(-A_{\omega+\sigma})^0 := 1$  (the identity operator on  $E$ ).
- (P1) For every  $z \in \mathbb{C}$ , the operator  $(-A_{\omega+\sigma})_z$  is injective and the following equality holds:

$$(-A_{\omega+\sigma})_{-z} = \left( (-A_{\omega+\sigma})_z \right)^{-1} = \left( (-A_{\omega+\sigma})^{-1} \right)_z.$$

- (P2) Let  $z_1, z_2 \in \mathbb{C}$ . Then  $(-A_{\omega+\sigma})_{z_1}(-A_{\omega+\sigma})_{z_2} \subseteq (-A_{\omega+\sigma})_{z_1+z_2}$ , and for every  $x \in D((-A_{\omega+\sigma})_{z_1+z_2}) \cap D((-A_{\omega+\sigma})_{z_2})$ , one has  $(-A_{\omega+\sigma})_{z_2}x \in D((-A_{\omega+\sigma})_{z_1})$  and  $(-A_{\omega+\sigma})_{z_1}(-A_{\omega+\sigma})_{z_2}x = (-A_{\omega+\sigma})_{z_1+z_2}x$ . Furthermore, the supposition  $(-A_{\omega+\sigma})_{z_1} \in L(E)$  implies  $(-A_{\omega+\sigma})_{z_1}(-A_{\omega+\sigma})_{z_2} = (-A_{\omega+\sigma})_{z_1+z_2}$ .

- (P3) If  $0 < \Re z < 1$ , then

$$(-A_{\omega+\sigma})_{-z}C_\alpha x = \frac{\sin z\pi}{\pi} \int_0^\infty \lambda^{-z}(\lambda - A_{\omega+\sigma})^{-1}C_\alpha x d\lambda, \quad x \in E.$$

- (P4) If  $C = 1$ , then  $(-A_{\omega+\sigma})_z \in L(E)$  for any  $z \in \mathbb{C}$  with  $\Re z < -\alpha$ .

**Theorem 2.1.** *Let  $\alpha \in (0, \infty) \setminus \mathbb{N}$ , let  $\omega \geq 0$ , and let  $A$  be the integral generator of an  $\alpha$ -times integrated  $C$ -semigroup  $(S^\alpha(t))_{t \geq 0}$  satisfying that the family  $\{e^{-\omega t}S^\alpha(t) : t \geq 0\}$  is equicontinuous. Suppose  $\epsilon > 0$ ,  $\lfloor\alpha\rfloor = \lfloor\alpha + \epsilon\rfloor$ ,  $x'_0 \in D((-A_{\omega+\sigma})_{\alpha+\epsilon}) \cap D((-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor})$  and  $x_0 = Cx'_0$ . Then the abstract Cauchy problem  $(ACP_1)$  has a unique mild solution, denoted by  $u(\cdot, x_0)$ , and for every  $\varepsilon > 0$ , the set  $\{e^{-(\omega+\sigma+\varepsilon)t}u(t, x_0) : t \geq 0\}$  is bounded. If, in addition,  $A_{\omega+\sigma}x'_0 \in D((-A_{\omega+\sigma})_{\alpha+\epsilon}) \cap D((-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor})$ , then the solution is classical.*

*Proof.* Set  $x''_0 := (-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}x'_0$ . Denote by  $(S^\alpha_{\omega+\sigma}(t))_{t \geq 0}$  the  $\alpha$ -times integrated  $C$ -semigroup generated by  $A_{\omega+\sigma}$  (cf. [6, Theorem 4.2(ii)-(b)]). Then, for every  $\beta > \alpha$ ,  $((S^\beta_{\omega+\sigma}(t) \equiv (g_{\beta-\alpha} * S^\alpha_{\omega+\sigma}))_{t \geq 0})$  is the  $\beta$ -times integrated  $C$ -semigroup generated by  $A_{\omega+\sigma}$ . Furthermore, it is not difficult to prove that the following representation formula holds:

$$S^\beta_{\omega+\sigma}(t)x = \int_0^\infty e^{-(\omega+\sigma)(t-s)}S^\beta(t-s)x dg_{\omega+\sigma,\beta}, \quad x \in E, \quad t \geq 0,$$

where

$$g_{\omega+\sigma,\beta}(s) := \chi_{(0,\infty)}(s) + \sum_{k=1}^{\infty} \beta(\beta-1) \cdots (\beta-k+1)(\omega+\sigma)^k s^k / k!^2, \quad s \geq 0;$$

cf. [11, Proposition 3.3]. Since, by (P1),

$x'_0 \in D((-A_{\omega+\sigma})_{\alpha+\epsilon}) = R((-A_{\omega+\sigma})_{-\alpha-\epsilon})$ , we have the existence of an element  $z_0 \in E$  such that  $x''_0 = (-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}(-A_{\omega+\sigma})_{-\alpha-\epsilon}z_0$ . Keeping in mind (P0) and (P2), as well as [2, Lemma 1.4], the above implies  $x''_0 = C_{\alpha}^{-1}(-A_{\omega+\sigma})_{-\lfloor\alpha+\epsilon\rfloor}C_{\alpha}z_0$  and  $(-A_{\omega+\sigma})^{\lfloor\alpha+\epsilon\rfloor}Cx''_0 = Cz_0$ . Define now, for every  $t \geq 0$ ,

$$S_{\omega+\sigma}^{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}(t)x''_0 := (-1)^{\lfloor\alpha+\epsilon\rfloor}S_{\omega+\sigma}^{\alpha+\epsilon}(t)z_0 + \sum_{i=0}^{\lfloor\alpha+\epsilon\rfloor-1} g_{\alpha+\epsilon-i}(t)A_{\omega+\sigma}^{\lfloor\alpha+\epsilon\rfloor-1-i}Cx''_0.$$

Then [7, Proposition 2.3.3(i)] implies that, for every  $t \geq 0$ ,

$$S_{\omega+\sigma}^{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}(t)x''_0 = C_{\alpha}^{-1} \frac{d^{\lfloor\alpha+\epsilon\rfloor}}{dt^{\lfloor\alpha+\epsilon\rfloor}} S_{\omega+\sigma}^{\alpha+\epsilon}(t)C_{\alpha}x''_0.$$

We shall prove that the mild solution in (i)-(ii) is given by the formula

$$u(t, x_0) := e^{(\omega+\sigma)t}v_{\omega+\sigma}(t, x''_0), \quad t \geq 0,$$

where

(2.1)

$$v_{\omega+\sigma}(t, x''_0) := \Gamma_{\alpha,\epsilon} \int_0^{\infty} \frac{ds}{s-1} \left( s^{\lfloor\alpha+\epsilon\rfloor-\alpha-\epsilon} S_{\omega+\sigma}^{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}(t) - \frac{1}{s} S_{\omega+\sigma}^{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}\left(\frac{t}{s}\right) \right) x''_0,$$

and  $\Gamma_{\alpha,\epsilon} := \frac{\sin(\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor)\pi}{\pi}$ , see [11, Sections 3-4] and [8, Theorem 4.1]. First of all, notice that the convergence of the singular integral appearing in (2.1), written as the sum of corresponding integrals along the intervals  $(0, 1/2)$ ,  $(1/2, 2)$  and  $(2, \infty)$ , comes out from the following:

Suppose that the operator family  $\{(1+t^{\gamma})^{-1}e^{-\omega t}S^{\alpha}(t) : t \geq 0\}$  is equicontinuous. Put  $\delta := 2^{-1} \min(\epsilon, \alpha + \epsilon - \lfloor\alpha + \epsilon\rfloor)$ . Then the computation given in the proofs of [11, Lemma 4.1-Lemma 4.2] shows that there exists  $c_{\alpha,\epsilon,\gamma,\omega} > 0$  such that, for every  $p \in \mathfrak{B}$ , there exist  $r_p \in \mathfrak{B}$ ,  $c_p > 0$  and  $c_{p,\omega,\gamma,\epsilon,\sigma} > 0$  such that:

$$\begin{aligned} p\left(S_{\omega+\sigma}^{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}(t)x''_0\right) &\leq c_p \frac{\sigma^{\min(-\lfloor\alpha+\epsilon\rfloor, \alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor-\gamma-1)}}{\ln\left(1 + \frac{\sigma}{4\omega+2\sigma}\right)} t^{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor-1} \\ &\quad \times \left[ r_p(z_0) + \sum_{i=0}^{\lfloor\alpha+\epsilon\rfloor-1} p(A^i Cx''_0) \right], \quad t \geq 0, \end{aligned}$$

and

$$\begin{aligned}
 & p\left(S_{\omega+\sigma}^{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}(t)x_0'' - S_{\omega+\sigma}^{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}(\tau)x_0''\right) \\
 & \leq c_p c_{\alpha,\epsilon,\gamma,\omega}(t-s)^\delta \left[ r_p(z_0) + \sum_{i=0}^{\lfloor\alpha+\epsilon\rfloor-1} p(A^i C x_0'') \right] \\
 & \times \frac{\sigma^{\min(-\lfloor\alpha+\epsilon\rfloor, \alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor-\gamma-\epsilon-1)}}{\ln\left(1 + \frac{\sigma}{4\omega+2\sigma}\right)}, \quad 0 \leq \tau \leq t < \infty.
 \end{aligned}$$

Similarly as in the proofs of [11, Lemma 4.3-Lemma 4.4] we obtain that the mapping  $t \mapsto v_{\omega+\sigma}(t, x_0'')$ ,  $t \geq 0$  is continuous, and that the equicontinuity of the operator family  $\{(1+t^\gamma)^{-1}S^\alpha(t) : t \geq 0\}$  implies that there exists  $c_{\alpha,\epsilon,\gamma} > 0$  such that, for every  $p \in \mathfrak{O}$ , there exist  $r_p \in \mathfrak{O}$  and  $c_p > 0$  so that:

$$\begin{aligned}
 & p\left(v_\sigma(t, x_0'')\right) \leq c_p c_{\alpha,\epsilon,\gamma} \sigma^{\min(-\lfloor\alpha+\epsilon\rfloor, \alpha-\lfloor\alpha+\epsilon\rfloor-\gamma-1)} \\
 (2.2) \quad & \times \left[ r_p(z_0) + \sum_{i=0}^{\lfloor\alpha+\epsilon\rfloor-1} p(A^i C x_0'') \right] t^{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor-1}, \quad t \geq 2.
 \end{aligned}$$

Since  $\int_0^\infty e^{-\lambda t} S_{\omega+\sigma}^{\alpha+\epsilon}(t)x dt = \lambda^{-\alpha-\epsilon}(\lambda - A_{\omega+\sigma})^{-1}Cx$  for all  $x \in E$  and  $\lambda > 0$ , it is not difficult to prove, with the help of proof of [11, Lemma 6.1] and the property (P3) of powers that, for every  $\lambda > 0$ ,

$$\int_0^\infty e^{-\lambda t} CC_\alpha v_{\omega+\sigma}(t, x_0'') dt = (\lambda - A_{\omega+\sigma})^{-1} C^2 (-A_{\omega+\sigma})_{\lfloor\alpha+\epsilon\rfloor-\alpha-\epsilon} C_\alpha x_0''.$$

Using the resolvent equation and the previous equality, we immediately obtain that:

$$\begin{aligned}
 & A_{\omega+\sigma} \int_0^\infty e^{-\lambda t} \int_0^t CC_\alpha v_{\omega+\sigma}(s, x_0'') ds dt \\
 & = CC_\alpha \left[ \int_0^\infty e^{-\lambda t} v_{\omega+\sigma}(t, x_0'') dt - \frac{x_0}{\lambda} \right], \quad \lambda > 0.
 \end{aligned}$$

Taking into account the Laplace transformability of the function  $t \mapsto v_{\omega+\sigma}(t, x_0'')$ ,  $t \geq 0$  (this follows from its continuity and the estimate (2.2)) and the equality  $(CC_\alpha)^{-1}A_{\omega+\sigma}CC_\alpha = A_{\omega+\sigma}$ , we get that

$$\begin{aligned}
 & A_{\omega+\sigma} \int_0^\infty e^{-\lambda t} \int_0^t v_{\omega+\sigma}(s, x_0'') ds dt \\
 & = \int_0^\infty e^{-\lambda t} v_{\omega+\sigma}(t, x_0'') dt - \frac{x_0}{\lambda}, \quad \lambda > 0.
 \end{aligned}$$

The previous equality in combination with [14, Theorem 1.1.10] implies that

$$A_{\omega+\sigma} \int_0^t v_{\omega+\sigma}(s, x_0'') ds = v_{\omega+\sigma}(t, x_0'') - x_0, \quad t \geq 0.$$

Hence, the mapping  $t \mapsto u(t, x_0)$ ,  $t \geq 0$  is the mild solution in (i)-(ii); the uniqueness is a simple consequence of Lyubich type theorem [6, Theorem 4.2(i)]. If, in addition,  $A_{\omega+\sigma}x_0' \in D((-A_{\omega+\sigma})_{\alpha+\epsilon}) \cap D((-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor})$ , then (P2) implies that the terms

$A_{\omega+\sigma}(-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}x_0''$  and  $(-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}A_{\omega+\sigma}x_0''$  are well defined and equal each other. As a simple consequence, we have that

$$A_{\omega+\sigma}v_{\omega+\sigma}(t, x_0'') = v_{\omega+\sigma}(t, A_{\omega+\sigma}x_0''), \quad t \geq 0$$

and that the constructed mild solution, for such an initial value  $x_0$ , is classical in fact.  $\square$

Before proceeding further, we would like to recommend for the interested reader the paper [13], and Section 7 of [11], for more details concerning the exponential type of constructed classical solutions.

*Remark 2.2.* Suppose  $C = 1$ .

(i) Then the assumption  $x_0' \in D((-A_{\omega+\sigma})_{\alpha+\epsilon})$  implies by (P1)-(P2) that  $x_0' \in D((-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor})$  and  $(-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor}x_0' \in D(A^{\lfloor\alpha+\epsilon\rfloor})$ . Using the same properties of powers, it is checked at once that the assumption  $x_0' \in D((-A_{\omega+\sigma})_{1+\alpha+\epsilon})$  implies  $x_0' \in D((-A_{\omega+\sigma}) \cap D((-A_{\omega+\sigma})_{\alpha+\epsilon}) \cap D((-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor})$  as well as  $-A_{\omega+\sigma}x_0' \in D((-A_{\omega+\sigma})_{\alpha+\epsilon}) \cap D((-A_{\omega+\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor})$ .

(ii) It is worth noting that, for every  $z \in \mathbb{C}$  with  $|\Re z| > \alpha$ , the domain of power  $(-A_{\omega+\sigma})_z$  does not depend on the particular choice of number  $\sigma \in (0, 1]$ . In order to better explain this, suppose that  $0 < \sigma_1 < \sigma_2 \leq 1$ . Then the operator  $-A_{\sigma_1, \sigma_2} \equiv -A_{\sigma_2}(-A_{\sigma_1})^{-1}$  belongs to  $L(E)$  and the computation given in the proof of [11, Lemma 5.2] shows that the operator  $-A_{\sigma_1, \sigma_2}$  is positive, so that the power  $(-A_{\sigma_1, \sigma_2})_z$  can be constructed in the usual way (see e.g. [9] and [2]). Having in mind that  $(-A_{\omega+\sigma})_z \in L(E)$ , provided  $\Re z < -\alpha$ , it is straightforward to verify that the following equalities hold, for every  $z \in \mathbb{C}$  with  $\Re z < -\alpha$ ,

$$(2.3) \quad (-A_{\sigma_2})_z x = (-A_{\sigma_1, \sigma_2})_z (-A_{\sigma_1})_z x = (-A_{\sigma_1})_z (-A_{\sigma_1, \sigma_2})_z x, \quad x \in E.$$

If  $\Re z > \alpha$ , then one can use the equality  $D((-A_{\sigma_2})_z) = R((-A_{\sigma_2})_{-z})$  and (2.3) in order to see that  $D((-A_{\sigma_2})_z) \subseteq D((-A_{\sigma_1})_z)$ . The converse inclusion can be proved in a similar fashion, so that  $D((-A_{\sigma_2})_z) = D((-A_{\sigma_1})_z)$  for  $\Re z > \alpha$ . Therefore, the supposition  $x_0' \in D((-A_{\sigma})_{\alpha+\epsilon})$  implies  $x_0' \in D((-A_1)_{\alpha+\epsilon})$  and, in this case, (2.3) holds with  $\sigma_2 = \sigma$ ,  $\sigma_1 = \sigma$ ,  $z = \alpha + \epsilon$  and  $x_0' = x$ . This simply implies that, for  $0 \leq j \leq \lfloor\alpha + \epsilon\rfloor$ ,

$$(2.4) \quad A^j C (-A_{\sigma})_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor} x_0' = (-A_{1, \sigma})_{\lfloor\alpha+\epsilon\rfloor - (\alpha+\epsilon)} A^j C (-A_1)_{\alpha+\epsilon-\lfloor\alpha+\epsilon\rfloor} x_0'.$$

- (iii) Consider the situation of Theorem 2.1 with  $\omega = 0$ . Using again the computation given in the proof of [11, Lemma 5.2], we get that the family  $\{\sigma^{-\min(0, \alpha - \gamma)}(A_{\sigma, 1})_{[\alpha + \epsilon] - \alpha - \epsilon} : 0 < \sigma \leq 1/2\} \subseteq L(E)$  is equicontinuous. Combining this with the proof of [11, Theorem 1.2], and using also (2.4), we have that the set  $\{(1 + t)^{-\max(\alpha - 1 + \epsilon, \gamma + \epsilon, 2\gamma - \alpha + \epsilon)}u(t, x_0) : t \geq 0\}$  is bounded - this is certainly the fact that cannot be so easily reformulated in the case of general operator  $C \neq 1$ .
- (iv) The assertion of [8, Theorem 4.2] continues to hold, with appropriate technical modifications in the setting of sequentially complete locally convex spaces.

Notice that Theorem 2.1 and Remark 2.2 taken together provide a proper extension of [8, Theorem 4.1]. As an application, we can simply state results concerning the growth of mild solutions of abstract Cauchy problems for elliptic differential operators acting on  $E_l$ -type spaces (cf. [14], [7] and apply the result stated in Remark 2.2(iii)); we can also prove an extension of [5, Theorem 3.7] for such operators.

Suppose now that the operator  $A$  is the integral generator of an  $\alpha$ -times integrated cosine function  $(C^\alpha(t))_{t \geq 0}$  satisfying that the family  $\{e^{-\omega t}C^\alpha(t) : t \geq 0\}$  is equicontinuous for some  $\omega \geq 0$ . Then we know from Lemma 1.3 that the operator  $\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$  is the integral generator of an  $(\alpha + 1)$ -times integrated  $\mathcal{C}$ -semigroup  $(S^{\alpha+1}(t))_{t \geq 0}$  in  $E \times E$ , where  $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ . Therefore, for any  $\sigma \in (0, 1]$  given in advance, the operator  $-\mathcal{A}_{\omega+\sigma} \equiv \mathcal{A} - \omega - \sigma$  is  $\mathcal{C}_\alpha$ -sectorial of angle  $\pi/2$ , with  $\mathcal{C}_\alpha$  being defined by  $\mathcal{C}_\alpha := (-\mathcal{A}_{\omega+\sigma})^{-1 - \lfloor \alpha \rfloor} \mathcal{C}^2$ . Therefore, we can construct the power  $(-\mathcal{A}_{\omega+\sigma})_z$  for any  $z \in \mathcal{C}$ . Keeping in mind the representation formula for  $(S^{\alpha+1}(t))_{t \geq 0}$ , given in the formulation of the above-mentioned lemma, it is not difficult to prove that the following theorem holds.

**Theorem 2.3.** *Let  $\alpha \in (0, \infty) \setminus \mathbb{N}$ , let  $\epsilon > 0$  such that  $\lfloor \alpha \rfloor = \lfloor \alpha + \epsilon \rfloor$ , and let  $\sigma \in (0, 1]$ . Suppose that  $A$  is the integral generator of an  $\alpha$ -times integrated cosine function  $(C^\alpha(t))_{t \geq 0}$  satisfying that the family  $\{e^{-\omega t}C^\alpha(t) : t \geq 0\}$  is equicontinuous for some  $\omega \geq 0$ . Then, for every  $(x_0, y_0) \in D((-\mathcal{A}_{\omega+\sigma})_{\alpha+\epsilon+1}) \cap D((-\mathcal{A}_{\omega+\sigma})_{\alpha+\epsilon-\lfloor \alpha + \epsilon \rfloor})$ , the abstract Cauchy problem  $(ACP_2)$  has a unique mild solution, denoted by  $u(t, x_0, y_0)$ , and for every  $\varepsilon > 0$ , the set  $\{e^{-(\omega+\sigma+\varepsilon)t}u(t, x_0, y_0) : t \geq 0\}$  is bounded. If, in addition,  $\mathcal{A}_{\omega+\sigma}x'_0 \in D((-\mathcal{A}_{\omega+\sigma})_{\alpha+\epsilon+1}) \cap D((-\mathcal{A}_{\omega+\sigma})_{\alpha+\epsilon-\lfloor \alpha + \epsilon \rfloor})$ , then the solution is classical.*

*Remark 2.4.* Suppose that  $C = 1$  and that the family  $\{(1 + t^\gamma)^{-1}C^\alpha(t) : t \geq 0\}$  is equicontinuous for some  $\gamma \geq 0$ . By the foregoing, we have that, for every  $(x_0, y_0) \in D((-\mathcal{A}_{\omega+\sigma})_{\alpha+\epsilon+1}) \cap D((-\mathcal{A}_{\omega+\sigma})_{\alpha+\epsilon-\lfloor \alpha + \epsilon \rfloor})$ , the set  $\{(1 + t)^{-\max(\alpha+\epsilon, \max(\alpha, \gamma+2)+\epsilon, 2\max(\alpha, \gamma+2)-(\alpha+1)+\epsilon)}u(t, x_0, y_0) : t \geq 0\}$  is bounded.

## Acknowledgement

The author is partially supported by Grant 174024 of the Ministry of Education, Science and Technological Development of the Republic of Serbia.



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*Received by the editors March 14, 2013*