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SOME PROPERTIES ON THE POWERS OF n-ARY RELATIONAL SYSTEMS[1]

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Abstract. In the paper, we study the properties of n-ary relational systems with respect to a decomposition of the corresponding index set. It is a common generalization of certain properties of binary and ternary relational systems. For each of the properties, we give sufficient condition under which this property is preserved by powers of relational systems.

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1. Introduction

In [1], G. Birkhoff introduced the operation of a cardinal power of partially ordered sets and showed the validity of the first exponential law for the operation, i.e., the law $(A^B)^C \cong A^{B\times C}$. In [9], J. Šlapal extended the concept of direct (i.e., cardinal) power from partially ordered sets onto n-ary relational systems (cf. also [7]). In [6], M. Novotný and J. Šlapal, introduced and studied a power of n-ary relational systems in which they generalized the direct power investigated in [7, 9]. In [2], we studied the powers of n-ary relational systems, improved and completed the results from [6, 7, 9].

In [8], J. Šlapal extended the fundamental concepts concerning binary and ternary relations to general relations. Therefore, some results are analogous to the well-known ones of the theory of binary relations, but many results are new.

In this paper, we study some properties of relations which were defined in [8] and discuss these properties on n-ary relational systems. We solve the problem of finding sufficient conditions under which some properties of relational systems (from [8]) are preserved by powers (from [6]) of these systems.

2. Preliminaries

Some relevant definitions that are considered to be necessary will be given in further sections. Throughout this paper, n denotes a positive integer. Let

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A and H be nonempty sets and let A^H be a set of all mappings from H into A. A *sequence* in A is any mapping from H into A. The notation for sequences is suitably defined as $(a_h \mid h \in H)$. In case of $H = \{1, \ldots, n\}$, the notation $(a_1, \ldots, a_n)$ is used instead of $(a_h \mid h \in H)$. The set of all mappings from $\{1, \ldots, n\}$ into A is commonly identified with the cartesian power $A^n = \underbrace{A \times \ldots \times A}_{n-terms}$. By a relation R in the general sense, it means a set of mappings $R \subseteq A^H$. The pair (A, R) is called a *relational system* of type H. The sets A and H are the *carrier set* and the *index set* of R, respectively. In case of $H = \{1, \ldots, n\}, R \subseteq A^H$ is called an *n-ary relation* on A and (A, R) is said to be an *n-ary relational system*. If $H = \{1, 2\}, R \subseteq A^H$, then (A, R) is called a *binary relational system*-see e.g. [4, 3, 5].

Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be a pair of n-ary relational systems. A mapping $f : B \to A$ is said to be a *homomorphism* of B into A if $(x_1, \ldots, x_n) \in S$, yields $(f(x_1), \ldots, f(x_n)) \in R$. We denote by $Hom(\mathbf{B}, \mathbf{A})$ the set of all homomorphisms of B into A. If $f : B \to A$ is a bijection and both $f : B \to A$ and $f^{-1} : A \to B$ are homomorphisms, then f is called an *isomorphism* of B onto A. If there exists an isomorphism of B onto A we say that B and A are *isomorphic*, in symbols $B \cong A$.

An n-ary relational system $\mathbf{A} = (A, R)$ is said to be

(i) *reflexive* provided that $(x_1, \ldots, x_n) \in R$ whenever $x_1 = \ldots = x_n$.

(ii) *diagonal* provided that, whenever (x_{ij}) is an $n \times n$-matrix over A, from $(x_{1j}, \ldots, x_{nj}) \in R$ for each $j = 1, \ldots, n$ and $(x_{i1}, \ldots, x_{in}) \in R$ for each $i = 1, \ldots, n$ it follows that $(x_{11}, \ldots, x_{nn}) \in R$.

Let $\mathbf{A} = (A, R)$ and $\mathbf{B} = (B, S)$ be n-ary relational systems. According to [6], we define the *power* $\mathbf{A}^{\mathbf{B}}$ by $\mathbf{A}^{\mathbf{B}} = (Hom(\mathbf{B}, \mathbf{A}), r)$, where for any $f_1, \ldots, f_n \in Hom(\mathbf{B}, \mathbf{A}), (f_1, \ldots, f_n) \in r$ if and only if $(x_1, \ldots, x_n) \in S$ implies $(f_1(x_1), \ldots, f_n(x_n)) \in R$ whenever $x_1, \ldots, x_n \in B$.

Theorem 2.1. [2] *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be n-ary relational systems. If $\mathbf{C}$ is reflexive, then $(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}} \cong \mathbf{A}^{\mathbf{B} \times \mathbf{C}}$, where $\mathbf{B} \times \mathbf{C}$ is the cartesian product of $\mathbf{B}$ and $\mathbf{C}$.*

According to [8], we define the concepts of general relations as following.

Let H be a set with *card* $H \geq 2$ (*card* $H \geq 3$). Then a sequence of three (four) sets $\mathcal{K} = \{K_i\}_{i=1}^{3(4)}$ is called a *b-decomposition (t-decomposition)* of the set H if

(i) $\displaystyle\bigcup_{i=1}^{3(4)} K_i = H$,

(ii) $K_i \cap K_j = \varnothing$ for all $i, j \in \{1, 2, 3\}$ $(i, j \in \{1, 2, 3, 4\}, i \neq j)$,

(iii) $0 < card\ K_1 = card\ K_2$ $(0 < card\ K_1 = card\ K_2 = card\ K_3)$.

Let A, H be sets and let $\mathcal{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of H. If $R, S \subseteq A^H$, then we define the relations

(i) $E_{\mathcal{K}} \subseteq A^H$, called the *diagonal* with regard to $\mathcal{K}$ if:

$$f \in E_{\mathcal{K}} \Longleftrightarrow f(K_1) = f(K_2),$$

(ii) $R_{\mathcal{K}}^{-1} \subseteq A^H$, called the *inversion of* R with regard to $\mathcal{K}$ if:

$$f \in R_{\mathcal{K}}^{-1} \Longleftrightarrow \exists g \in R : f(K_1) = g(K_2), f(K_2) = g(K_1),$$
$$f(K_i) = g(K_i) \text{ for } i = 3 \text{ (for } i = 3,4),$$

(iii) $(RS)_{\mathcal{K}} \subseteq A^H$, called the *composition of* R *and* S with regard to $\mathcal{K}$ if:

$$f \in (RS)_{\mathcal{K}} \Longleftrightarrow \exists g \in R, \exists h \in S : f(K_1) = g(K_1), f(K_2) = h(K_2),$$
$$f(K_i) = g(K_i) = h(K_i) \text{ for } i = 3 \text{ (for } i = 3,4) \text{ and } g(K_2) = h(K_1).$$

Further, we put $R_{\mathcal{K}}^1 = R$ and $R_{\mathcal{K}}^{n+1} = (R_{\mathcal{K}}^n R)_{\mathcal{K}}$ for every $n \in N$. $R_{\mathcal{K}}^n$ is called the *n-th power* of R with regard to $\mathcal{K}$.

Let $R \subseteq A^H$ be a relation with *card* $H \geq 2$ (*card* $H \geq 3$) and let $\mathcal{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of the set H. Then R is called

(i) *reflexive* (*areflexive*) with regard to $\mathcal{K}$ if $E_{\mathcal{K}} \subseteq R$ ($R \cap E_{\mathcal{K}} = \varnothing$),

(ii) *symmetric* (*asymmetric*, *antisymmetric*) with regard to $\mathcal{K}$ if $R_{\mathcal{K}}^{-1} \subseteq R$ ($R \cap R_{\mathcal{K}}^{-1} = \varnothing$, $R \cap R_{\mathcal{K}}^{-1} \subseteq E_{\mathcal{K}}$),

(iii) *transitive* (*intransitive*) with regard to $\mathcal{K}$ if $R_{\mathcal{K}}^2 \subseteq R$ ($R \cap R_{\mathcal{K}}^n = \varnothing$, for every $n \in \mathbb{N}$),

(iv) *regular* with regard to $\mathcal{K}$ if $f \in R$, $g \in A^H$, $f(K_i) = g(K_i)$ for $i = 1,2,3$ (for $i = 1,2,3,4$) $\Rightarrow$ $g \in R$.

Remark 2.2. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be n-ary relational systems with *card* $H = n, n \geq 2$ ($n \geq 3$) and let $\mathcal{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of the set H.

(i) If $\mathbf{A}$ is reflexive with regard to $\mathcal{K}$, then $\mathbf{A}$ is reflexive.

(ii) If $\mathbf{C}$ is reflexive with regard to $\mathcal{K}$, then $(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}} \cong \mathbf{A}^{\mathbf{B} \times \mathbf{C}}$.

Let $R \subseteq A^H$ be a relation with *card* $H \geq 3$ and let $\mathcal{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of the set H. Then we define the relation ${}^1R_{\mathcal{K}} \subseteq A^H$ by

$$f \in {}^1R_{\mathcal{K}} \Leftrightarrow \exists g \in R : f(K_1) = g(K_2), f(K_2) = g(K_3),$$

$$f(K_3) = g(K_1), f(K_4) = g(K_4).$$

Further, we put ${}^{n+1}R_{\mathcal{K}} = {}^1({}^nR_{\mathcal{K}})_{\mathcal{K}}$ for every $n \in N$. ${}^nR_{\mathcal{K}}$ is called the *n-th cyclic transposition* of R with regard to $\mathcal{K}$.

Let $R \subseteq A^H$ be a relation with *card* $H \geq 3$ and let $\mathcal{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of the set H. Then R is called

(i) *cyclic* (*acyclic*, *anticyclic*) with regard to $\mathcal{K}$ if ${}^1R_{\mathcal{K}} \subseteq R$ ($R \cap {}^1R_{\mathcal{K}} = \varnothing$, $f \in R \cap {}^1R_{\mathcal{K}} \Rightarrow f(K_1) = f(K_2) = f(K_3)$),

(ii) *strongly symmetric* (*strongly asymmetric*, *strongly antisymmetric*) with regard to $\mathcal{K}$ if for any permutation (any odd permutation) φ of the set $\{1,2,3\}$ and any mapping $f \in A^H$ for which there exists a mapping $g \in R$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1,2,3$ and $f(K_4) = g(K_4)$, we have $f \in R$ ($f \notin R$, $f \in R \Rightarrow f(K_1) = f(K_2) = f(K_3)$).

3. Main results

In general, if $\mathbf{A}, \mathbf{B}$ are n-ary relational systems and $\mathbf{A}$ has some properties of n-ary relational system with respect to a decomposition, then the power $\mathbf{A^B}$ need not necessarily preserve such properties - see the following example:

Example 3.1. Let $A = \{a,b\}$ and $B = \{x,y\}$. Consider the b-decomposition $\mathcal{K} = \{\{1,2\},\{3,4\},\varnothing\}$ of the index set $H = \{1,2,3,4\}$. Assume $f_i : B \to A$ for $i = 1,2,3$ are maps given by $f_1(x) = a, f_1(y) = a$, $f_2(x) = b, f_2(y) = b$ and $f_3(x) = a, f_3(y) = b$. Let $\mathbf{B_i} = (B, S_i)$ and $\mathbf{A_i} = (A, R_i)$ for $i = 1,2,3$ be 4-ary relational systems of type H, and $\mathbf{A_i}^{\mathbf{B_i}} = (Hom(\mathbf{B_i}, \mathbf{A_i}), r_i)$.

(i) If $R_1 = \{(a,a,a,a),(a,b,a,b),(a,b,b,a),(b,a,a,b),(b,a,b,a),(b,b,b,b)\}$, then $\mathbf{A_1}$ is reflexive with regard to $\mathcal{K}$. Let $S_1 = \{(x,y,y,x)\}$. Then $f_i \in Hom(\mathbf{B_1}, \mathbf{A_1})$ for $i = 1,2,3$. It may be easily seen that $(f_1, f_3, f_1, f_3) \in E_{\mathcal{K}}$ but it is not an element of r_1, so that $\mathbf{A_1}^{\mathbf{B_1}}$ is not reflexive with regard to $\mathcal{K}$.

(ii) If $R_2 = \{(a,a,a,a),(a,a,a,b),(a,a,b,a),(a,b,a,a),(b,a,a,a),(b,b,b,b)\}$, then $\mathbf{A_2}$ is symmetric with regard to $\mathcal{K}$. Let $S_2 = \{(x,y,x,x)\}$. So, $f_i \in Hom(\mathbf{B_2}, \mathbf{A_2})$ for $i = 1,2,3$. Since $(f_3, f_1, f_2, f_3) \in r_2$, there exists a mapping $(f_2, f_3, f_1, f_3) \in (r_2)^{-1}_{\mathcal{K}}$ but $(f_2, f_3, f_1, f_3) \notin r_2$. Therefore, $\mathbf{A_2}^{\mathbf{B_2}}$ is not symmetric with regard to $\mathcal{K}$.

(iii) If $R_3 = \{(a,a,a,a),(a,a,b,b),(a,b,b,a),(a,a,b,a),(a,a,a,b),(b,b,b,b)\}$, then $\mathbf{A_3}$ is transitive with regard to $\mathcal{K}$. Let $S_3 = \{(x,x,y,x)\}$. Thus $f_i \in Hom(\mathbf{B_3}, \mathbf{A_3})$ for $i = 1,2,3$. Since $(f_1, f_2, f_3, f_1), (f_3, f_1, f_3, f_2) \in r_3$, there exists a mapping $(f_1, f_2, f_3, f_2) \in (r_3)^2_{\mathcal{K}}$ but it is not a member of the relation r_3. Hence $\mathbf{A_3}^{\mathbf{B_3}}$ is not transitive with regard to $\mathcal{K}$.

We will give the sufficient conditions for $\mathbf{A}$ and $\mathbf{B}$ which transfer properties with regard to a b-decomposition (t-decomposition) $\mathcal{K}$ of $\mathbf{A}$ to the power of relational systems $\mathbf{A^B}$.

Lemma 3.2. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 2$ ($n \geq 3$) and let $\mathcal{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of the set H. Let $\mathbf{A^B} = (Hom(\mathbf{B}, \mathbf{A}), r)$, $(f_1, \ldots, f_n) \in (Hom(\boldsymbol{B}, \boldsymbol{A}))^H$ and $x \in B$. Then the following statements hold.*

(i) If $(f_1, \ldots, f_n) \in E_{\mathcal{K}}$, then $(f_1(x), \ldots, f_n(x)) \in E_{\mathcal{K}}$.

(ii) If $\boldsymbol{B}$ is reflexive and $(f_1, \ldots, f_n) \in r^{-1}_{\mathcal{K}}$, then $(f_1(x), \ldots, f_n(x)) \in R^{-1}_{\mathcal{K}}$.

(iii) If $\boldsymbol{B}$ is reflexive and $(f_1, \ldots, f_n) \in r^k_{\mathcal{K}}$, then $(f_1(x), \ldots, f_n(x)) \in R^k_{\mathcal{K}}$ where $k \in \mathbb{N}, k \geq 1$.

(iv) If $\boldsymbol{B}$ is reflexive, $\varphi = (f_1, \ldots, f_n) \in r$ and $\psi = (g_1, \ldots, g_n) \in (Hom(\boldsymbol{B}, \boldsymbol{A}))^H$ such that $\varphi(K_i) = \psi(K_i)$, then $f_x(K_i) = g_x(K_i)$, where

$f_x = (f_1(x), \ldots, f_n(x))$ and $g_x = (g_1(x), \ldots, g_n(x))$ for $i = 1, 2, 3$ (for $i = 1, 2, 3, 4$).

Proof. Case 1: $\mathcal{K}$ is a b-decomposition of the set H. Assume $|K_1| = |K_2| = l, |K_3| = m$. Let $\varphi, \psi, \chi \in (Hom(\mathbf{B}, \mathbf{A}))^H$ and $\varphi = (f_1, \ldots, f_n), \psi = (g_1, \ldots, g_n)$, $\chi = (h_1, \ldots, h_n)$. Suppose $f_x = (f_1(x), \ldots, f_n(x)), g_x = (g_1(x), \ldots, g_n(x)), h_x = (h_1(x), \ldots, h_n(x))$ for each $x \in B$.

(i) If $\varphi \in E_{\mathcal{K}}$, then $\varphi(K_1) = \varphi(K_2)$. Thus, there exists a permutation α of $\{1, \ldots, l\}$ such that

$$f_i = f_{l+\alpha(i)} \text{ for each } i = 1, \ldots, l. \tag{3.1}$$

By (3.1), we have $f_i(x) = f_{l+\alpha(i)}(x)$ for each $x \in B$, so we get $f_x(K_1) = f_x(K_2)$. Therefore $f_x \in E_{\mathcal{K}}$.

(ii) If $\varphi \in r_{\mathcal{K}}^{-1}$, then there exists a mapping $\psi \in r$ such that $\varphi(K_1) = \psi(K_2), \varphi(K_2) = \psi(K_1)$ and $\varphi(K_3) = \psi(K_3)$. So, there exist permutations α and β of $\{1, \ldots, l\}$ such that

$$f_i = g_{l+\alpha(i)} \text{ and } g_i = f_{l+\beta(i)} \text{ for each } i = 1, \ldots, l, \tag{3.2}$$

and there exists a permutation γ of $\{1, \ldots, m\}$ such that

$$f_{2l+j} = g_{2l+\gamma(j)} \text{ for each } j = 1, \ldots, m. \tag{3.3}$$

Since $\psi \in r$ and $\mathbf{B}$ is reflexive, $g_x \in R$. By (3.2) and (3.3), we have $f_x(K_1) = g_x(K_2)$, $f_x(K_2) = g_x(K_1)$ and $f_x(K_3) = g_x(K_3)$. Thus $f_x \in R_{\mathcal{K}}^{-1}$.

(iii) By the definition of $\mathbf{A}^{\mathbf{B}}$, the statement holds for $k = 1$. Suppose the statement holds for $k = p, p \in \mathbb{N}, p \geq 1$. Let $\varphi \in r_{\mathcal{K}}^{p+1}$. Then, there exist the mappings $\psi \in r_{\mathcal{K}}^{p}, \chi \in r$ such that $\varphi(K_1) = \psi(K_1), \varphi(K_2) = \chi(K_2), \psi(K_2) = \chi(K_1)$ and $\varphi(K_3) = \psi(K_3) = \chi(K_3)$.

Thus, there exist the permutations α, β and γ of $\{1, \ldots, l\}$ such that

$$f_i = g_{\alpha(i)}, f_{l+i} = h_{l+\beta(i)} \text{ and } h_i = g_{l+\gamma(i)} \text{ for each } i = 1, \ldots, l, \tag{3.4}$$

and there exist the permutations η, μ of $\{1, \ldots, m\}$ such that

$$f_{2l+j} = g_{2l+\eta(j)} = h_{2l+\mu(j)} \text{ for each } j = 1, \ldots, m. \tag{3.5}$$

By the assumption for $k = p, \mathbf{B}$ is reflexive and $\psi \in r_{\mathcal{K}}^{p}$, we have $g_x \in R_{\mathcal{K}}^{p}$. Since $\mathbf{B}$ is reflexive and $\chi \in r$, $h_x \in R$. From (3.4) and (3.5), we get $f_x(K_1) = g_x(K_1), f_x(K_2) = h_x(K_2), g_x(K_2) = h_x(K_1)$ and $f_x(K_3) = g_x(K_3) = h_x(K_3)$. This implies that $f_x \in R_{\mathcal{K}}^{p+1}$. Hence $(f_1(x), \ldots, f_n(x)) \in R_{\mathcal{K}}^{k}$ for each $x \in B$ if $(f_1, \ldots, f_n) \in r_{\mathcal{K}}^{k}$, where $k \in \mathbb{N}, k \geq 1$.

(iv) Let $\varphi \in r$ such that $\varphi(K_i) = \psi(K_i)$ for $i = 1, 2, 3$. Then there exist the permutations α and β of $\{1, \ldots, l\}$ such that

$$f_i = g_{\alpha(i)} \text{ and } f_{l+i} = g_{l+\beta(i)} \text{ for each } i = 1, \ldots, l, \tag{3.6}$$

and there exists a permutation γ of $\{1, \ldots, m\}$ such that

$$f_{2l+j} = g_{2l+\gamma(j)} \text{ for each } j = 1, \ldots, m. \tag{3.7}$$

From (3.6) and (3.7), we get $f_x(K_i) = g_x(K_i)$ for $i = 1, 2, 3$.

Case 2: $\mathcal{K}$ is a t-decomposition of the set H. We can prove the statements by using the same argument as in Case 1.

□

Theorem 3.3. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 2$ $(n \geq 3)$ and let $\mathcal{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of H. Then the following statements hold.*

(i) If $\mathbf{A}$ is both diagonal and reflexive with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is reflexive with regard to $\mathcal{K}$.

(ii) If $\mathbf{B}$ is reflexive and $\mathbf{A}$ is areflexive with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is areflexive with regard to $\mathcal{K}$.

Proof. Suppose $\mathbf{A}^{\mathbf{B}} = (Hom(\mathbf{B}, \mathbf{A}), r)$ and $\varphi = (f_1, \ldots, f_n) \in Hom(\mathbf{B}, \mathbf{A})^H$.

(i) Let $\varphi \in E_{\mathcal{K}}$ and $(x_1, \ldots, x_n) \in S$. By Lemma 3.2(i), we have $(f_1(x_i), \ldots, f_n(x_i)) \in E_{\mathcal{K}}$ for each $i = 1, \ldots, n$. Since $\mathbf{A}$ is reflexive with regard to $\mathcal{K}$, we have also

$$(f_1(x_i), \ldots, f_n(x_i)) \in R \text{ for each } i = 1, \ldots, n. \tag{3.8}$$

As f_i is a homomorphism, we have

$$(f_i(x_1), \ldots, f_i(x_n)) \in R \text{ for each } i = 1, \ldots, n. \tag{3.9}$$

From (3.8), (3.9) and the diagonality of $\mathbf{A}$, we get $(f_1(x_1), \ldots, f_n(x_n)) \in R$, and so $\varphi \in r$. Hence $\mathbf{A}^{\mathbf{B}}$ is reflexive with regard to $\mathcal{K}$.

(ii) Assume $\varphi \in r \cap E_{\mathcal{K}}$. By Lemma 3.2($i$) and the reflexivity of $\mathbf{B}$, we get $(f_1(x), \ldots, f_n(x)) \in R \cap E_{\mathcal{K}}$ for each $x \in B$, which contradicts to the assumption that $\mathbf{A}$ is areflexive with regard to $\mathcal{K}$. Hence $\mathbf{A}^{\mathbf{B}}$ is areflexive with regard to $\mathcal{K}$. □

Theorem 3.4. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 2$ $(n \geq 3)$ and let $\mathcal{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of H. If $\boldsymbol{B}$ is reflexive, then the following statements hold.*

(i) If $\mathbf{A}$ is both diagonal and symmetric (transitive, regular) with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is symmetric (transitive, regular) with regard to $\mathcal{K}$.

(ii) If $\mathbf{A}$ is asymmetric (intransitive) with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is asymmetric (atransitive) with regard to $\mathcal{K}$.

(iii) If $\mathbf{A}$ is antisymmetric with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is antisymmetric with regard to $\mathcal{K}$.

Proof. Since $\mathbf{B}$ is reflexive, $\mathbf{A}$ is diagonal and symmetric (transitive, regular) with regard to $\mathcal{K}$. By using Lemma 3.2(ii) (Lemma 3.2(iii), 3.2(vi)), we can prove in the same manner as Theorem 3.3(i) that $\mathbf{A}^{\mathbf{B}}$ is symmetric (transitive, regular) with regard to $\mathcal{K}$. The proof of (ii) is similar to that of Theorem

3.3(ii) by using Lemma 3.2(ii) for asymmetric property (Lemma 3.2(iii) for intransitive property).

(iii) Assume $|K_1| = |K_2| = l, |K_3| = m$ ($|K_1| = |K_2| = |K_3| = l, |K_4| = m$). Let $\varphi = (f_1, \dots, f_n) \in (Hom(\mathbf{B}, \mathbf{A}))^H$ and $\varphi \in r \cap r_{\mathcal{K}}^{-1}$. Because the reflexivity of $\mathbf{B}$ and Theorem 3.2(ii), we have $(f_1(x), \dots, f_n(x)) \in R \cap R_{\mathcal{K}}^{-1}$ for each $x \in B$.

Since $\mathbf{A}$ is antisymmetric with regard to $\mathcal{K}$,

$$(f_1(x), \dots, f_n(x)) \in E_{\mathcal{K}} \text{ for each } x \in B. \tag{3.10}$$

We will show that $\varphi \in E_{\mathcal{K}}$. Suppose $\varphi \notin E_{\mathcal{K}}$. Then $\varphi(K_1) \neq \varphi(K_2)$, and for any permutation η of $\{1, \dots, l\}$ there exists $i \in \{1, \dots, l\}$ such that $f_i \neq f_{l+\eta(i)}$. Therefore, $f_i(x) \neq f_{l+\eta(i)}(x)$ for some $x \in B$, which contradicts to (3.10). Thus $\varphi \in E_{\mathcal{K}}$. Hence $\mathbf{A}^{\mathbf{B}}$ is antisymmetric with regard to $\mathcal{K}$. □

Lemma 3.5. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 3$ and let $\mathcal{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of the set H. Let $\mathbf{A}^{\mathbf{B}} = (Hom(\mathbf{B}, \mathbf{A}), r)$. If $\mathbf{B}$ is reflexive and $(f_1, \dots, f_n) \in {}^1r_{\mathcal{K}}$, then $(f_1(x), \dots, f_n(x)) \in {}^1R_{\mathcal{K}}$ for each $x \in B$.*

Proof. Assume $|K_1| = |K_2| = |K_3| = l, |K_4| = m$. Let $\varphi \in {}^1r_{\mathcal{K}}$. Then there exists a mapping $\psi \in r$ such that $\varphi(K_1) = \psi(K_2), \varphi(K_2) = \psi(K_3), \varphi(K_3) = \psi(K_1)$ and $\varphi(K_4) = \psi(K_4)$.
Let $\varphi = (f_1, \dots, f_n), \psi = (g_1, \dots, g_n) \in (Hom(\mathbf{B}, \mathbf{A}))^H$. Then there exist the permutations α, γ and β of $\{1, \dots, l\}$ such that

$$f_i = g_{l+\alpha(i)},\ f_{l+i} = g_{2l+\gamma(i)} \text{ and } f_{2l+i} = g_{\beta(i)} \text{ for each } i = 1, \dots, l, \tag{3.11}$$

and there exists a permutation η of $\{1, \dots, m\}$ such that

$$f_{3l+j} = g_{3l+\eta(j)} \text{ for each } j = 1, \dots, m. \tag{3.12}$$

Let $f_x = (f_1(x), \dots, f_n(x))$, $g_x = (g_1(x), \dots, g_n(x))$ for each $x \in B$. Since $\psi \in r$ and $\mathbf{B}$ is reflexive, $g_x \in R$. By (3.11) and (3.12), we get $f_x(K_1) = g_x(K_2), f_x(K_2) = g_x(K_3), f_x(K_3) = g_x(K_1)$ and $f_x(K_4) = g_x(K_4)$. So $f_x \in {}^1R_{\mathcal{K}}$. □

Theorem 3.6. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 3$ and let $\mathcal{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of the set H. If $\mathbf{B}$ is reflexive, then the following statements hold.*

(i) If $\mathbf{A}$ is both diagonal and cyclic with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is cyclic with regard to $\mathcal{K}$.

(ii) If $\mathbf{A}$ is acyclic with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is acyclic with regard to $\mathcal{K}$.

(iii) If $\mathbf{A}$ is anticyclic with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is anticyclic with regard to $\mathcal{K}$.

Proof. Since $\mathbf{B}$ is reflexive, $\mathbf{A}$ is diagonal and cyclic with regard to $\mathcal{K}$, and by using Lemma 3.5, we can prove, similarly to Theorem 3.3(i), that $\mathbf{A}^{\mathbf{B}}$ is cyclic with regard to $\mathcal{K}$. The proof of (ii) is the same as of Theorem 3.3(ii), by using Lemma 3.5, we conclude (ii).

(iii) Assume $|K_1| = |K_2| = |K_3| = l, |K_4| = m$. Let $\varphi = (f_1, \ldots, f_n) \in (Hom(\mathbf{B}, \mathbf{A}))^H$ and $\varphi \in r \cap {}^1r_{\mathcal{K}}$. Because the reflexivity of $\mathbf{B}$ and Lemma 3.5(i), we have $(f_1(x), \ldots, f_n(x)) \in R \cap {}^1R_{\mathcal{K}}$ for each $x \in B$. Since $\mathbf{A}$ is anticyclic with regard to $\mathcal{K}$, we get for each $x \in B$, there exist the permutations μ_x, ν_x of $\{1, \ldots, l\}$ such that

$$f_i(x) = f_{l+\mu_x(i)}(x) = f_{2l+\nu_x(i)}(x) \text{ for } i = 1, \ldots, l. \tag{3.13}$$

We will show that $\varphi(K_1) = \varphi(K_2) = \varphi(K_3)$. Suppose $\varphi(K_1) \neq \varphi(K_2)$ or $\varphi(K_1) \neq \varphi(K_3)$ or $\varphi(K_2) \neq \varphi(K_3)$. If $\varphi(K_1) \neq \varphi(K_2)$, then for any permutation λ of $\{1, \ldots, l\}$ there exists $i \in \{1, \ldots, l\}$ such that $f_i \neq f_{l+\lambda(i)}$. Therefore, $f_i(x) \neq f_{l+\lambda(i)}(x)$ for some $x \in B$. This contradicts with (3.13). We can prove the cases $\varphi(K_1) \neq \varphi(K_3)$ and $\varphi(K_2) \neq \varphi(K_3)$ in the same manner. So $\varphi(K_1) = \varphi(K_2) = \varphi(K_3)$. Hence $\mathbf{A}^{\mathbf{B}}$ is anticyclic with regard to $\mathcal{K}$. $\square$

Lemma 3.7. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems with card $H = n, n \geq 3$ and let $\mathcal{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of the set H. Let $\mathbf{A}^{\mathbf{B}} = (Hom(\mathbf{B}, \mathbf{A}), r)$ and α be a permutation of the set $\{1, 2, 3\}$ and a mapping $\varphi \in (Hom(\mathbf{B}, \mathbf{A}))^H$ for which there exists a mapping $\psi \in r$ such that $\varphi(K_i) = \psi(K_{\alpha(i)})$ for $i = 1, 2, 3$ and $\varphi(K_4) = \psi(K_4)$. If $\mathbf{B}$ is reflexive, $\varphi = (f_1, \ldots, f_n)$ and $\psi = (g_1, \ldots, g_n)$, then $f_x(K_i) = g_x(K_{\alpha(i)})$ for $i = 1, 2, 3$ and $f_x(K_4) = g_x(K_4)$, where $f_x = (f_1(x), \ldots, f_n(x))$ and $g_x = (g_1(x), \ldots, g_n(x))$ for each $x \in B$.*

Proof. Assume $|K_1| = |K_2| = |K_3| = l, |K_4| = m$. Let α be a permutation of the set $\{1, 2, 3\}$ and a mapping $\varphi \in (Hom(\mathbf{B}, \mathbf{A}))^H$ for which there exists a mapping $\psi \in r$ such that $\varphi(K_i) = \psi(K_{\alpha(i)})$ for $i = 1, 2, 3$ and $\varphi(K_4) = \psi(K_4)$. Suppose $\varphi = (f_1, \ldots, f_n)$ and $\psi = (g_1, \ldots, g_n)$. Then, there exists a permutation β_j of $\{1, \ldots, l\}$ for $j = 1, 2, 3$ such that

$$f_{(j-1)l+i} = g_{(\alpha(j)-1)l+\beta_j(i)} \text{ for each } i = 1, \ldots, l, \tag{3.14}$$

and there exists a permutation η of $\{1, \ldots, m\}$ such that

$$f_{3l+k} = g_{3l+\eta(k)} \text{ for each } k = 1, \ldots, m. \tag{3.15}$$

Let $x \in B$, $f_x = (f_1(x), \ldots, f_n(x))$ and $g_x = (g_1(x), \ldots, g_n(x))$. Since $\psi \in r$ and $\mathbf{B}$ is reflexive, $(g_1(x), \ldots, g_n(x)) \in R$. By (3.14) and (3.15), we have $f_x(K_i) = g_x(K_{\alpha(i)})$ for $i = 1, 2, 3$ and $f_x(K_4) = g_x(K_4)$. $\square$

Theorem 3.8. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems with card $H = n, n \geq 3$ and let $\mathcal{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of the set H. If $\mathbf{B}$ is reflexive then the following statements hold.*

(i) If $\mathbf{A}$ is both diagonal and strongly symmetric with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is strongly symmetric with regard to $\mathcal{K}$.

(ii) *If* $\mathbf{A}$ *is strongly asymmetric with regard to* $\mathcal{K}$*, then* $\mathbf{A}^{\mathbf{B}}$ *is strongly asymmetric with regard to* $\mathcal{K}$*.*

(iii) *If* $\mathbf{A}$ *is strongly antisymmetric with regard to* $\mathcal{K}$*, then* $\mathbf{A}^{\mathbf{B}}$ *is strongly antisymmetric with regard to* $\mathcal{K}$*.*

Proof. Since $\mathbf{B}$ is reflexive, $\mathbf{A}$ is diagonal and strongly symmetric with regard to $\mathcal{K}$ by using Lemma 3.7. Similarly to Theorem 3.3(i) we can prove that $\mathbf{A}^{\mathbf{B}}$ is strongly symmetric with regard to $\mathcal{K}$.

(ii) Assume $|K_1| = |K_2| = |K_3| = l, |K_4| = m$ and $\mathbf{A}^{\mathbf{B}} = (Hom(\mathbf{B}, \mathbf{A}), r)$. Let α be an odd permutation of the set $\{1, 2, 3\}$ and a mapping $\varphi \in (Hom(\mathbf{B}, \mathbf{A}))^H$ for which there exists a mapping $\psi \in r$ such that $\varphi(K_i) = \psi(K_{\alpha(i)})$ for $i = 1, 2, 3$ and $\varphi(K_4) = \psi(K_4)$. Let $\varphi = (f_1, \ldots, f_n)$ and $\psi = (g_1, \ldots, g_n)$. We will show that $\varphi \notin r$. Suppose $\varphi \in r$ and $f_y = (f_1(y), \ldots, f_n(y))$, $g_y = (g_1(y), \ldots, g_n(y))$ for each $y \in R$. Since $\mathbf{B}$ is reflexive and $\psi \in r$, $g_y \in R$. From Lemma 3.7, we have $f_y(K_i) = g_y(K_{\alpha(i)})$ for $i = 1, 2, 3$ and $f_y(K_4) = g_y(K_4)$. As $\varphi \in r$ and $\mathbf{B}$ is reflexive, $f_y \in R$. This is a contradiction because $\mathbf{A}$ is strongly asymmetric with regard to $\mathcal{K}$. Hence $\mathbf{A}^{\mathbf{B}}$ is strongly asymmetric with regard to $\mathcal{K}$.

We can prove similarly to Theorem 3.6(iii) because $\mathbf{B}$ is reflexive, $\mathbf{A}$ is strongly antisymmetric with regard to $\mathcal{K}$ by using Lemma 3.7, we can get $\mathbf{A}^{\mathbf{B}}$ is strongly antisymmetric with regard to $\mathcal{K}$. □

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