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JOIN-CLOSED SUBSETS OF AN ORDERED SET. UNICITY OF THE JOIN-BASIS

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Abstract. We define the notions of cluster points and isolated points of a subset of an arbitrary closure space. We recall the notion of free subset and the notion of basis.

We apply all that to the closure space made of the join-closed subsets of an arbitrary ordered set E. We establish that a join-closed subset has at most one basis. The set $I(E)$ of the isolated points of E is exactly the set of the completely join-irreducible elements of E. When $I(E)$ generates E, $I(E)$ is the unique basis of E (we give examples). When $I(E)$ does not generate E, E has no basis.

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1. Basic notions

Throughout this paper, $(E, \mathcal{F})$ is a *closure space*, i.e. E is a set and $\mathcal{F}$ is a Moore system on E ($\mathcal{F}$ is a subset of 2^E such that for any $\mathcal{G} \subset \mathcal{F}$, $\bigcap \mathcal{G} \in \mathcal{F}$, whence $E \in \mathcal{F}$). Let $f : 2^E \mapsto 2^E$ be the associated closure (when $X \subset E$, $f(X) = \bigcap \{Y \in \mathcal{F} : X \subset Y\}$). Let us call *open subsets* the complements of the closed subsets. Let us call *neighborhood* of a point $x \in E$ each subset containing an open subset containing x. Let us denote by $C : 2^E \mapsto 2^E$ the mapping defined by $C(X) = E \setminus X$. Let us say that a point x is *adherent* to X when it belongs to $f(X)$.

A point x is adherent to X if and only if each neighborhood of x meets X.

Indeed, if we denote by Ω the complete lattice of the open sets, we get :

$$x \in f(X) \Leftrightarrow (X \subset Y \in \mathcal{F} \Rightarrow x \in Y) \Leftrightarrow \forall Y \in \mathcal{F}, (X \cap C(Y) = \emptyset \Rightarrow x \in Y)$$

$$\Leftrightarrow (x \in Z \in \Omega \Rightarrow X \cap Z \neq \emptyset).$$

Let us call *cluster point of* X each point x of E such that $x \in f(X \setminus \{x\})$. Equivalently, x is a cluster point of X if and only if each neighborhood of x meets X in (at least) a point different from x.

Let us write $\overline{X}$ for $f(X)$. Let us denote by $A(X)$ the set of the cluster points of X. Clearly, $\overline{X} = X \cup A(X)$. We deduce that X is closed if and only if $A(X) \subset X$.

Let us define the *induced* closure space on X by the Moore system $\{X \cap F : F \in \mathcal{F}\}$. Let us say that a point x of X is *X-isolated* when there exists a

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neighborhood V of x such that $V \cap X = \{x\}$, i.e. when $x \notin A(X)$. It comes to say that there exists $F \in \mathcal{F}$ such that $X \setminus F = \{x\}$. The set of the X-isolated points is $I(X) = X \setminus A(X)$.

The E-*isolated* (or, more simply : *isolated*) points are the x such that $\{x\}$ is open. Observe the inclusion $X \cap I(E) \subset I(X)$, which can be strict (taking $E = \mathbb{R}$ with the usual closed subsets and $X = \mathbb{N}$), or not (taking $E = \emptyset$). For more about this please check [5].

Lemma 1.1. *Whenever $x \in X \subset E$, the following properties are equivalent :*
(1) $x \in A(X)$
(2) $\exists Y \subset X \setminus \{x\}$ such that $x \in \overline{Y}$
(3) $f(X \setminus \{x\}) = f(X)$

Proof. Relations (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious.
To prove (2) $\Rightarrow$ (3) let us put $H = X \setminus \{x\}$.
We get $\overline{X} = f(H \cup \{x\}) = f(\overline{H} \cup \{x\}) = \overline{H}$. □

Let us put $S(X) = X \cap A(X)$. Since the mapping $A : 2^E \mapsto 2^E$ is increasing, S is also increasing. Let us call X-*superfluous* the points of $S(X)$.

A subset X of E is said to be *free* when it is minimal among the subsets Y of E such that $f(Y) = f(X)$.

Let us say that a subset Z of E *generates* the closed subset F when $f(Z) = F$. This subset Z is said to be a *basis* for F when it is a free generating subset of F (i.e. when Z is a minimal generating subset for F).

Lemma 1.2. *A subset X is free if and only if it is devoid of X-superfluous points.*

Proof. Suppose X is not free. It can be found a proper subset Y of X such that $f(Y) = f(X)$. Choose $x \in X \setminus Y$. We get $x \in f(X) = f(Y)$ hence, by Lemma 1.1, x is superfluous, and $S(X) \neq \emptyset$.

Conversely, suppose $S(X) \neq \emptyset$. Let $x \in S(X)$ and put $Y = X \setminus \{x\}$. Since Y is a proper subset of X and generates $f(X)$, X is not free. □

It follows that X is free if and only if $X = I(X)$.

Lemma 1.3. *The set $I(X)$ of the X-isolated points of X is free.*

Proof. Since $I(X) \subset X$, $S(I(X)) \subset S(X)$, therefore $S(I(X)) \subset S(X) \cap I(X) = \emptyset$. □

Let us return to the the induced space X. The associated closure $\varphi_X : 2^X \mapsto 2^X$ is defined by $\varphi_X(Y) = X \cap \overline{Y}$. It is easily seen that, when $Y \subset X$, Y is free if and only if it is φ_X-free. When F is closed, we get for any $Y \subset F$ (since $\overline{Y} \subset F$), $\varphi_F(Y) = \overline{Y}$. It follows that for any subset Y of F, $\varphi_F(Y) = F$ if and only if $f(Y) = F$, and that the bases of F are the same as the φ_F-bases.

Let us say that a point x of the closed subspace F is *extremal* when $F \setminus \{x\} \in \mathcal{F}$, which comes to say that $x \in I(F)$.

Lemma 1.4. *Let F be a closed subset of E.*

1. If the subset G generates F, $I(F) \subset G$.

2. In the case where $I(F)$ generates F, the subsets X generating F are those verifying $I(F) \subset X \subset F$; moreover, $I(F)$ is the smallest basis of F.

Proof. 1. Suppose $z \in I(F) \setminus G$. Since $G \subset F \setminus \{z\}$, it follows $F = \overline{G} \subset f(F \setminus \{z\}) \subset F$, hence $f(F \setminus \{z\}) = F$. Then $F \setminus \{z\}$ is not closed, i.e. $z \notin I(F)$, a contradiction.

2. Suppose $I(F) \subset X \subset F$. We deduce $f(I(F)) \subset f(X) \subset f(F)$, whence $f(X) = F$.

Since $I(F)$ is free, it is a basis of F. If X is any basis of F, X generates F, whence $I(F) \subset X$. □

Lemma 1.5. *For any subset X of E, the following statements are equivalent :*

(1) $A(X) = A(\overline{X})$

(2) $S(\overline{X}) = A(X)$

(3) $I(X) = I(\overline{X})$.

Proof. (1) ⇒ (2). $S(\overline{X}) = A(\overline{X}) \cap \overline{X} = A(X) \cap \overline{X} = A(X)$.

(2) ⇒ (1). Since $\overline{X}$ is closed, $A(\overline{X}) \subset \overline{X}$, so $A(\overline{X}) \subset A(\overline{X}) \cap \overline{X} = S(\overline{X}) = A(X)$.

(2) ⇒ (3). $I(\overline{X}) = \overline{X} \setminus S(\overline{X}) = \overline{X} \setminus A(X) = I(X)$.

(3) ⇒ (2). $S(\overline{X}) = \overline{X} \setminus I(\overline{X}) = \overline{X} \setminus I(X) = A(X)$. □

Proposition 1.6. *Suppose that the closure space $(E, \mathcal{F})$ is such that, for every free subset X of E, $A(X) = A(\overline{X})$.*

Then, a closed subset F has at most one basis. When $I(F)$ generates F, the unique basis of F is $I(F)$, otherwise F has no basis.

Proof. Let B be a basis of F. Since B is free, $B = I(B)$. We deduce, by Lemma 1.5, $B = I(B) = I(\overline{B}) = I(F)$. □

2. Results

In the sequel, E is an *ordered set*. Let us call $\mathcal{R}$ the set of all subsets of E admitting a supremum. We put, for $x \in E$, $T_x = \{t \in E : t < x\}$ and $\downarrow x = \{t \in E : t \leq x\}$. We will use the axiom of choice.

Definition 2.1. *([1, p.26], [2, p.53], [4]) An element x of E is said to be completely join-irreducible when one of the two following equivalent properties holds true :*

(1) For any $Y \in \mathcal{R}$, $x = \bigvee Y$ implies $x \in Y$.

(2) x is not the supremum of T_x.

Proof. (of the equivalence)

(1) → (2). If x was the supremum of T_x, we could deduce $x \in T_x$.

(2) → (1). Let y be an upper bound of T_x such that $y \ngeq x$. Suppose $x = \bigvee Y$. If $x \notin Y$, $Y \subset T_x$ hence y is an upper bound of Y, and $y \geq \bigvee Y = x$, a contradiction. We conclude that $x \in Y$. □

Let us say that a subset X of E is $\vee$-*closed* (or more simply *closed*) when it contains all existing supremums of subsets of X. We can define on E a closure space by considering the Moore system $\mathcal{F}$ of all $\vee$-closed subsets.

Lemma 2.2. *The closure of an arbitrary subset X of E is*

$$Y = \{\vee H : H \subset X \text{ and } H \in \mathcal{R}\}.$$

Proof. 1. Let us first prove that Y is closed. Let Z be a subset of Y possessing a supremum. To each $z \in Z$ we associate a subset M_z of X such that $z = \bigvee M_z$. The sets Z and $N = \bigcup\{M_z : z \in Z\}$ have the same upper bounds. So, $N \in \mathcal{R}$ and $\bigvee N = \bigvee Z$. We have simultaneously $N \subset X$, $N \in \mathcal{R}$ and $\bigvee N = \bigvee Z$. By the definition of Y it follows $\bigvee Z \in Y$.

2. So, we have $X \subset Y \in \mathcal{F}$.

3. Suppose $X \subset P \in \mathcal{F}$. Let $x \in Y$. It is possible to find $H \subset X$ such that $x = \bigvee H$. Since P is closed, $x \in P$.

4. We conclude that Y is the smallest closed subset containing X. □

Let us now localize, for a given subset X, the X-superfluous and the X-isolated elements of X. First, by the definition of the cluster points, we get:

$$A(X) = \{x \in E : \exists H \subset X \setminus \{x\} \text{ such that } x = \bigvee H\}$$

i.e.

$$A(X) = \{x \in E : \exists H \subset X \text{ such that } x = \bigvee H \text{ and } x \notin H\}.$$

We deduce

$$S(X) = \{x \in X : \exists H \subset X \text{ such that } x = \bigvee H \text{ and } x \notin H\}$$

and

$$I(X) = \{x \in X : \forall H \subset X,\ (x = \bigvee H \Rightarrow x \in H)\}.$$

We observe that $I(E)$ *is the set of all completely $\vee$-irreducible elements.*

Proposition 2.3. *Whenever X is an arbitrary subset of E, $A(X) = A(\overline{X})$.*

Proof. Since $A(X) \subset A(\overline{X})$, we need only prove that $x \notin A(X) \Rightarrow x \notin A(\overline{X})$.

We must estabish that, whenever H is a subset of $\overline{X}$ such that $x = \bigvee H$, we can deduce that $x \in H$.

Since $H \subset \overline{X}$, we can associate to each $y \in H$ a subset $M_y \subset X$ such that $y = \bigvee M_y$. We easily observe that, if we define $P = \bigcup\{M_y : y \in H\}$, then $\bigvee P = x$. Since $x \notin A(X)$ and $P \subset X$, we deduce $x \in P$. It is then possible to find $z \in H$ such that $x \in M_z$. We get $x \leq \bigvee M_z = z \leq \bigvee H = x$, therefore $x = z \in H$. □

By Lemma 1.5, we deduce that the X-isolated points are exactly the extremal points of the closed subset generated by X.

Proposition 2.4. *Let F be a $\bigvee$-closed subset of the ordered set E. If $I(F)$ generates F, then $I(F)$ is the unique basis of F, otherwise F admits no basis.*

Proof. It is an obvious consequence of Proposition 1.6. □

In particular, if $I(E)$ generates E, $I(E)$ is the unique basis of E, otherwise E has no basis. When $I(E)$ generates E, the subsets G generating E are exactly those verifying $I(E) \subset G$ (Lemma 1.4).

Notice that the closure f satisfies the *anti-exchange property* :

If $x \in (f(X \cup \{y\})) \setminus (f(X))$ and $y \in f(X \cup \{x\})$, then $x = y$.

Indeed, there exists $Z \subset X \cup \{y\}$ such that $x = \bigvee Z$. The condition $y \notin Z$ would induce $Z \subset X$, whence $x \in f(X)$; therefore $y \in Z$, so $x \geq y$. The relation $y \in f(X)$ would imply $x \in f(X)$. So, by symmetry, $y \geq x$.

Let us say that a subset of E is *join-dense* when it generates E.

3. Examples

A. E is well-founded

Let us say that the ordered set E is *well-founded* when each non-empty subset of E has a minimal element. For instance, any finite ordered set, and the set $\mathbb{N}$ of natural numbers ordered by divisibility, are well-founded.

Lemma 3.1. *If x, y are two elements of a well-founded ordered set E verifying $x \not\leq y$, there exists an isolated point i such that $i \leq x$ and $i \not\leq y$.*

Proof. The set $Z = \{z \in E : z \leq x \text{ and } z \not\leq y\}$ is non-empty, since $x \in Z$. Let i be a minimal element of Z. For each $a \in T_i$, we get, by the minimality of i, $a \notin Z$, hence $a \leq y$. Since $i \not\leq y$, and since y is an upper bound for T_i, it follows (cf. Definition 2.1) that $i \in I(E)$. □

Proposition 3.2. *Let E be a well-founded ordered set. For each $x \in E$:*

$$x = \bigvee (I(E) \cap (\downarrow x))$$

Proof. Put $H = I(E) \cap (\downarrow x))$. Let us show that $x \leq y$ whenever y is an upper bound of H. Suppose $x \not\leq y$. Let i be an isolated point such that $i \leq x$ and $i \not\leq y$ (c.f. Lemma 3.1). Since $i \in H$, it follows $i \leq y$, a contradiction. □

Proposition 3.3. *Let E be a well-founded ordered set. Then :*

A subset X of E is join-dense if and only if $I(E) \subset X$.

The set $I(E)$ is the unique basis of E.

Proof. These are easy consequences of Lemma 1.4, Propositions 1.6 and 3.2. □

For instance, the basis of the complete lattice $\mathbb{N}$ of natural numbers ordered by divisibility, is made of the powers other than 1 of prime integers.

B. E is a complete lattice

Let H be an arbitrary subset of a complete lattice E. Let E^* be the dual of E. Let us define the two mappings $a : 2^H \mapsto E$ and $b : E \mapsto 2^H$ by $a(X) = \vee X$ and $b(t) = H \cap (\downarrow t)$.

Since $a(X) \leq t \Leftrightarrow X \subset b(t)$, if we consider a and b as mappings from 2^H to E^* and from E^* to 2^H, respectively, then the pair (a, b) is a Galois connexion. It follows that $c = b \circ a$ is a closure on 2^H and $\omega = a \circ b$ is an aperture on E. The set of all fixed points of c is $\mathcal{P} = \{H \cap (\downarrow t) : t \in E\}$. The set of all fixed points of ω is the join-closed subset $\overline{H}$ generated by H. The mappings $i : \mathcal{P} \mapsto \overline{H}$ and $j : \overline{H} \mapsto \mathcal{P}$ defined by $i(X) = X$ and $j(t) = b(t)$ are reciprocal isomorphisms.

When $H = I(E)$ and $\overline{H} = E$, we know (Proposition 1.6) that $I(E)$ is the unique basis of E, and we observe that E is isomorphic to $\mathcal{P}$.

Suppose now that H denotes the set set of the isolated points of E. Denote by $\mathcal{M}$ the system of all initial subsets of $\overline{H}$. Let us define the mapping $m : \mathcal{M} \to E$ by $m(X) = \bigvee X$. It can be easily verified that m is surjective if and only if H is the (unique) basis of L.

C. Alexandroff system

Let us name *Alexandroff system* on a set S each join-closed Moore system $\mathcal{A}$. Clearly, if we denote by a the closure associated to $\mathcal{A}$, we can write, for $X \in \mathcal{A}$,

$$X = \bigcup\{a(\{x\}) : x \in X\} \quad (1)$$

Proposition 3.4. *Let $\mathcal{A}$ be an Alexandroff system on S.*

1. The set of isolated elements of $\mathcal{A}$ is

$$\mathcal{B} = \{a(\{x\}) : x \in S\}.$$

2. $\mathcal{B}$ is the unique basis of $\mathcal{A}$.

Proof. 1. Let X be an isolated element, (i.e. an element completely join-irreducible). By (1), there exists $t \in S$ such that $X = a(\{t\})$.

Conversely, suppose $X = a(\{z\})$. If $X = \bigcup \mathcal{M}$ ($\mathcal{M} \subset \mathcal{A}$), $z \in M \in \mathcal{M}$. Whence $X \subset M \subset \bigcup \mathcal{M} = X$. So $X = M \in \mathcal{M}$.

2. By (1), $\mathcal{B}$ is join-dense. □

Since the set Δ of all Alexandroff systems on S is meet-closed, $(2^{2^S}, \Delta)$ is a closure space.

Proposition 3.5. *Let $(S, \mathcal{F})$ be a closure space.*

Let us denote by $\overline{x}$ the closed subset generated by $x \in S$.

1. The Alexandroff system generated by $\mathcal{F}$ is $\mathcal{A} = \{X \subset S : \forall x \in X, \overline{x} \subset X\}$.

2. Whenever $x \in S$, $a(x) = \overline{x}$.

3. The unique basis of $\mathcal{A}$ is $\{\overline{x} : x \in S\}$.

Proof. 1. It is easily seen that $\mathcal{F} \subset \mathcal{A} \in \Delta$. Suppose $\mathcal{F} \subset \mathcal{N} \in \Delta$: it is easy to see that $\mathcal{A} \subset \mathcal{N}$.

2. Clearly, if $X \subset S$, $a(X) = \bigcup\{\overline{x} : x \in X\}$. Therefore, $a(x) = \overline{x}$. □

It seems not to be known in what case $\mathcal{F}$ has a basis.

4. Annex

1. Edelman and Jamison proved (in [3]) that each anti-exchange finite closure space has a unique join-basis. Whence we deduce that, if in the closure space on a poset E associated to the Moore system of all the join-closed subsets of E there is no basis, E is infinite. Example of that situation: E is the real interval $[0, 1]$; since $I(E)$ is empty, the closure of $I(E)$ is $\{0\}$; hence E has no base.

2. We present some other examples, based on the fact that, when a complete lattice is atomistic (i.e., when each element is a supremum of atoms), the atoms are exactly the completely meet-irreducible elements.

2a.(Szpilrajn, [6]) Let S be an arbitrary set. Let Ω be the set of orders on S. Let $\Omega' = \Omega \cup \{S^2\}$. The obtained complete lattice has for the unique meet-basis the set of total orders on S.

2b. Let L be an arbitrary lattice (with 0 and 1). Let F be the Moore system of the filters on L. Let U be the set of the ultra-filters. The complete lattice F possesses the unique join-basis the set U.

References

[1] Caspard, N., Leclerc, B., Monjardet, B., Les ensembles ordonnés finis : concepts, résultats et usages. Springer, 2007.

[2] Davey, B.A., Priestley, H.A., Introduction to lattices and order. Cambridge, 2003.

[3] Edelmanand, P.H., Jamison, E., The thoery of convex geometries. Geometrae dedicata 19 (1989), 247-270.

[4] Erné, M., Šešelja, B., Tepavčević, A., Posets generated by irreducible elements. Order 20 (2003), 79-89.

[5] Ore, O., Some studies on closure relations. Duke J. of Math. 10 (1943), 761-785.

[6] Szpilrajn, E., Sur l'extension de l'ordre partiel. Fundamenta mathemeticae 16 (1930), 386-389.

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