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SOLVABILITY OF CERTAIN SEQUENCE SPACES EQUATIONS WITH OPERATORS

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Abstract. In this paper we deal with special *sequence space equations (SSE) with operators,* which are determined by an identity whose each term is a *sum or a sum of products of sets of the form* $\chi_a(T)$ *and* $\chi_{f(x)}(T)$ where f map U^+ to itself and χ is any of the symbols s, s^0, or $s^{(c)}$. Among other things under some conditions we solve (SSE) with operators $\chi_a(C(\lambda)D_\tau)+\chi_x(C(\mu)D_\tau)=\chi_b$, and $\chi_a(C(\lambda)C(\mu))+\chi_x(C(\lambda\sigma)C(\mu))=\chi_b$ where $\chi\in\{s,s^0\}$, and $\chi_a(C(\lambda)D_\tau)+s_x^0(C(\mu)D_\tau)=\chi_b$ where χ is either of the symbols s, or $s^{(c)}$ and $C(\nu)D_\tau$ is a factorable matrix.

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1. Introduction

In [21] Wilansky introduced sets of the form $a^{-1}*\chi$, where $a=(a_n)_{n\geq1}$ is a sequence satisfying $a_n\neq0$ for all n, and χ is any set of sequences. Recall that $x=(x_n)_{n\geq1}$ belongs to $a^{-1}*\chi$ if $(a_nx_n)_{n\geq1}$ belongs to χ. In this way, for any strictly positive sequence a, are defined the sets s_a^0, $s_a^{(c)}$ and s_a by $a^{-1}*\chi$ where χ is either of the sets c_0, c, and ℓ_∞ respectively. In [5, 8] the sum s_a+s_b and the product s_a*s_b of the sets s_a and s_b were defined, and characterizations of matrix transformations mapping in the sets $s_a+s_b^0(\Delta^q)$ and $s_a+s_b^{(c)}(\Delta^q)$ were given, where Δ is the operator of the first difference. In [16] de Malafosse and Malkowsky gave among other things properties of the matrix of weighted means considered as operator in the set s_a. Characterizations of matrix transformations mapping in $s_\alpha^0\left((\Delta-\lambda I)^h\right)+s_\beta^{(c)}\left((\Delta-\mu I)^l\right)$ with λ, μ, h, $l\in\mathbb{C}$ can be found in [9]. There are many other results using the sets s_a^0, $s_a^{(c)}$ and s_a, let us cite for instance applications to the following topics, $\sigma-$*core*, [7], *solvability of infinite tridiagonal systems*, [6], *measure of noncompactness*, [18], *Hardy theorem*, [20] and *statistical convergence*, [19].

In this paper our aim is to solve special *sequence spaces equations (SSE),* which are determined by an identity whose each term is a *sum or a sum of products of sets of the form* $\chi_a(T)$ *and* $\chi_{f(x)}(T)$ where f maps U^+ to itself, and χ is any of the symbols s, s^0, or $s^{(c)}$, the sequence x is the unknown and T is

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a given triangle. The resolution of such (SSE) consists in determining the set of all sequences x satisfying the identity, see for instance [13, 11, 15, 12, 17, 14, 10].

This paper is organized as follows. In Section 2 we recall some results on the sum and the product of sets of the form χ_a, where χ is either of the symbols s, or s^0. In Section 3 we solve the sequence spaces equations $\chi_a\left(C(\lambda)D_\tau\right)+\chi_x\left(C(\mu)D_\tau\right)=\chi_b$ and $\chi_a\left(\overline{N}_q\right)+\chi_x\left(\overline{N}_pD_{q/p}\right)=\chi_b$ for $\chi\in\left\{s,s^0\right\}$, where $\overline{N}_q$ is the operator of weighted means in some cases and we solve another type of (SSE) defined by $\chi_a\left(C(\lambda)D_\tau\right)+s_x^0\left(C(\mu)D_\tau\right)=s_b^0$ where χ is either of the symbols s, or $s^{(c)}$. In Section 4 we deal with (SSE) with operators represented by products of triangles of the form $\chi_a\left(C(\lambda)C(\mu)\right)+\chi_x\left(C(\lambda\sigma)C(\mu)\right)=\chi_b$ where $C(\nu)D_\tau$ is a factorable matrix for $\chi\in\left\{s,s^0\right\}$.

2. Sum and product of sequence spaces of the form χ_a, where χ is either of the symbols s, s^0

2.1. The sets χ_a, where χ is either of the symbols s, s^0, or $s^{(c)}$ for $a\in U^+$

We write s, ℓ_∞, c and c_0 for the sets of all complex, bounded, convergent and convergent to naught sequences, respectively. For a given infinite matrix $\Lambda=(\lambda_{nm})_{n,m\geq1}$ we define the operators Λ_n, for any integer $n\geq1$, by $\Lambda_n(\xi)=\sum_{m=1}^{\infty}\lambda_{nm}\xi_m$, where $\xi=(\xi_m)_{m\geq1}$, and the series are assumed convergent for all n. So we are led to the study of the operator Λ defined by $\Lambda\xi=\left(\Lambda_n(\xi)\right)_{n\geq1}$ mapping a sequence space into another sequence space.

A Banach space E of complex sequences with the norm $\|\|_E$ is a *BK space if each projection* $P_n{:}E\to\mathbb{C}$ defined by $P_n\xi=\xi_n$ *is continuous.* A BK space E is *said to have AK if every sequence* $\xi\in E$ has a *unique representation* $\xi=\sum_{n=1}^{\infty}\xi_ne^{(n)}$ where $e^{(n)}$ is the sequence with 1 in the n-th position, and 0 otherwise.

Let a be a nonzero sequence. Using Wilansky's notations we write $1/a*E$ for the set of all $\xi=(\xi_n)_{n\geq1}$ such that $(a_n\xi_n)_{n\geq1}\in E$. Let U^+ be the set of all real sequences ξ with $\xi_n>0$ for all n. If $\xi\in s$ we define D_ξ as the diagonal matrix defined by $[D_\xi]_{nn}=\xi_n$ for all n, we have $D_a*E=(1/a)^{-1}*E$ and it can be easily shown $\Lambda\in(D_a*E,D_b*F)$ if and only if $D_{1/b}\Lambda D_a\in(E,F)$ where E, $F\subset s$. Recall that for $a\in U^+$ we have $s_a=D_a*\ell_\infty$, $s_a^0=D_a*c_0$ and $s_a^{(c)}=D_a*c$. Each of the previous sets is a BK space *normed by* $\|\xi\|_{s_a}$, where $\|\xi\|_{s_a}=\sup_n\left(|\xi_n|/a_n\right)<\infty$. So we can define s_a as the set of all sequences ξ such that $(\xi_n/a_n)_n\in\ell_\infty$, s_a^0 as the set of all sequences ξ such that $\xi_n/a_n\to0$ $(n\to\infty)$ and $s_a^{(c)}$ as the set of all sequences ξ such that $\xi_n/a_n\to l$ $(n\to\infty)$ for some $l\in\mathbb{C}$, (cf. [3, 4]). If $a=(r^n)_{n\geq1}$, we write $\chi_a=\chi_r$ where χ is any of the symbols s, s^0, or $s^{(c)}$ to simplify. When $r=1$, we obtain $s_1=\ell_\infty$, $s_1^0=c_0$ and $s_1^{(c)}=c$. If we let $e=(1,1,...)$, then we have $s_e=s_1=\ell_\infty$, $s_e^0=s_1^0=c_0$ and $s_e^{(c)}=s_1^{(c)}=c$. When Λ maps E into F we write $\Lambda\in(E,F)$, see [2]. So we have $\Lambda\xi\in F$ for all $\xi\in E$, ($\Lambda\xi\in F$ means that for each $n\geq1$ the series defined by $\Lambda_n(\xi)=\sum_{m=1}^{\infty}\lambda_{nm}\xi_m$ is convergent and $\left(\Lambda_n(\xi)\right)_{n\geq1}\in F$). The set S_a of all infinite matrices $\Lambda=(\lambda_{nm})_{n,m\geq1}$ such that

$\|\Lambda\|_{S_a} = \sup_{n\geq 1}\left(a_n^{-1}\sum_{m=1}^{\infty}|\lambda_{nm}|\,a_m\right) < \infty$ is a Banach algebra with identity normed by $\|\Lambda\|_{S_a}$. Recall that if $\Lambda \in (s_a, s_a)$, then $\|\Lambda\xi\|_{s_a} \leq \|\Lambda\|_{S_a}\|\xi\|_{s_a}$ for all $\xi \in s_a$. It is well-known that $S_a = \left(s_a^0, s_a\right) = \left(s_a^{(c)}, s_a\right) = (s_a, s_a)$.

2.2. Sum of sets of the form χ_a where χ is either of the symbols s^0, or s.

In this subsection we recall some properties of the *sum* $E + F$ of sets of the form s_a^0, or s_a.

Let E, $F \subset s$ be two linear vector spaces. We write $E + F$ for the set of all sequences $\xi = \zeta + \zeta'$ where $\zeta \in E$ and $\zeta' \in F$. In the next result we use the notation $[\max(a,b)]_n = \max(a_n, b_n)$. We prove the following results.

Proposition 1. *Let a, $b \in U^+$ and assume χ is either of the symbols s^0, or s. Then we have*

(i) $\chi_a \subset \chi_b$ if and only if there is $K > 0$ such that $a_n \leq Kb_n$ for all n.

(ii) $\chi_a = \chi_b$ if and only if $s_a = s_b$, that is, there are K_1, $K_2 > 0$ such that

$$K_1 \leq \frac{b_n}{a_n} \leq K_2 \text{ for all } n.$$

(iii) $\chi_a + \chi_b = \chi_{a+b} = \chi_{\max(a,b)}$.

(iv) $\chi_a + \chi_b = \chi_a$ if and only if $b/a \in \ell_\infty$.

Proof. The case $\chi = s$ was shown in [5, Proposition 1, p. 244], and [8, Theorem 4, p. 293]. The case $\chi = s^0$ can be shown similarly, since we have $s_a = s_b$ if and only if $s_a^0 = s_b^0$. □

Notice that $\chi_a \subset \chi_b$ is equivalent to $a \in s_b$.

2.3. Solvability of the equation $\chi_a + \chi_x = \chi_b$ where χ is either of the symbols s^0, or s.

In the following we determine the set of all sequences $x = (x_n)_{n\geq 1} \in U^+$ such that $y_n = b_n O(1)$ $(n \to \infty)$ if and only if there are u, $v \in s$ such that $y = u + v$ and $u_n = a_n O(1)$ and $v_n = x_n O(1)$ $(n \to \infty)$ for all $y \in s$. Similarly we determine the sequences $x \in U^+$ such that $y_n = b_n o(1)$ if and only if there are u, $v \in s$ such that $y = u + v$ and $u_n = a_n o(1)$ and $v_n = x_n o(1)$ $(n \to \infty)$.

Theorem 2. *Let a, $b \in U^+$, and consider the equation*

$$\chi_a + \chi_x = \chi_b \tag{1}$$

where χ is either of the symbols s^0, or s and $x = (x_n)_{n\geq 1} \in U^+$ is the unknown. Then

(i) if $a/b \in c_0$, then equation (1) holds if and only if there are K_1, $K_2 > 0$ depending on x, such that $K_1 b_n \leq x_n \leq K_2 b_n$ for all n, that is $s_x = s_b$.

(ii) If a/b, $b/a \in \ell_\infty$, then equation (1) holds if and only if there is $K > 0$ depending on x such that $0 < x_n \leq Kb_n$ for all n; that is, $x \in s_b$.

(iii) If $a/b \notin \ell_\infty$, then equation (1) has no solution in U^+.

Proof. The case of equation (1) where $\chi = s$ was shown in [1]. For equation (1) with $\chi = s^0$ it is enough to note that $s_a + s_x = s_b$ can be written in the form $s_{a+x} = s_b$ which is turn in $s^0_{a+x} = s^0_b$ and $s^0_a + s^0_x = s^0_b$. This concludes the proof. □

In the following corollary we write $cl\,(u)$, $u > 0$, for the set of all sequences ξ such that $Ku^n \leq \xi_n \leq K'u^n$ for all n and for some K, $K' > 0$. This set is an equivalence class for the relation $\xi \mathcal{R} \xi'$ if $s_\xi = s_{\xi'}$ with $\xi' = (u^n)_n$. The following (SSE) is completely solved.

Corollary 3. *Let r, $u > 0$. The set Λ_χ of all $x \in U^+$ that satisfy the equation*

$$\chi_r + \chi_x = \chi_u \text{ where } \chi \in \left\{s^0, s\right\} \tag{2}$$

is defined by

$$\Lambda_\chi = \begin{cases} cl\,(u) & \text{for } r < u, \\ s_u \bigcap U^+ & \text{for } r = u, \\ \varnothing & \text{for } r > u. \end{cases}$$

2.4. Product of sequence spaces of the form χ_a for $\chi \in \left\{s^0, s\right\}$.

In this subsection we will deal with some properties of the *product* $E * F$ of particular subsets E and F of s. For any sequences $\xi \in E$ and $\eta \in F$ we put $\xi\xi' = \left(\xi_n \xi'_n\right)_{n \geq 1}$. Most of the following results were shown in [5].

For any sets of sequences E and F, we write $E*F$ for the set of all sequences $\xi\xi'$ such that $\xi \in E$ and $\xi' \in F$. We immediately have the following results where $\mathcal{S}_\chi$, $\chi \in \left\{s^0, s\right\}$, is constituted of all the sets of the form χ_a with $a \in U^+$.

Proposition 4. *The set $\mathcal{S}_\chi$, where $\chi \in \left\{s^0, s\right\}$ with multiplication $*$ is a commutative group with χ_1 as the unit element.*

Proof. First it can easily be seen that $\chi_a * \chi_b = \chi_{ab}$. We deduce the map $\psi : U^+ \mapsto \mathcal{S}_\chi$ defined by $\psi\,(a) = \chi_a$ is a surjective homomorphism and since U^+ with the multiplication of sequences is a group it is the same for $\mathcal{S}_\chi$. Then the unit element of $\mathcal{S}_\chi$ is $\psi\,(e) = \chi_1$. □

Remark 5. Note that the inverse of χ_a is $\chi_{1/a}$ with $\chi \in \left\{s^0, s\right\}$.

As a direct consequence of Proposition 4 we deduce the following corollary.

Corollary 6. *Let a, b, $c \in U^+$ and let $\chi \in \left\{s^0, s\right\}$. Then*

(i) $\chi_a * \chi_b = \chi_{ab}$.

(ii) $\chi_a * \chi_b = \chi_a * \chi_c$ *if and only if* $\chi_b = \chi_c$.

(iii) The sequence $x = (x_n)_{n \geq 1} \in U^+$ satisfies the equation $\chi_a * \chi_x = \chi_b$ if and only if $K_1 b_n / a_n \leq x_n \leq K_2 b_n / a_n$ for all n and for some K_1, $K_2 > 0$ depending on x.

Throughout this paper the unknown of each sequence spaces equation is a sequence $x \in U^+$.

3. The (SSE) with operators represented by factorable matrices

In this section we deal with the resolution of (SSE) of the form $\chi_a(C(\lambda)D_\tau) + \chi_x(C(\mu)D_\tau) = \chi_b$ and $\chi_a(\overline{N}_q) + \chi_x(\overline{N}_p D_{q/p}) = \chi_b$ for $\chi \in \{s, s^0\}$ where $\overline{N}_q$ is the operator of weighted means in some cases. Then we solve the (SSE) $\chi_a(C(\lambda)D_\tau) + s_x^0(C(\mu)D_\tau) = s_b^0$, where χ is either of the symbols s, or $s^{(c)}$.

3.1. The operators $C(\eta)$, $\Delta(\eta)$ and the sets $\widehat{\Gamma}$, Γ and $\widehat{C_1}$

The infinite matrix $T = (t_{nm})_{n,m\geq 1}$ is said to be a triangle if $t_{nm} = 0$ for $m > n$ and $t_{nn} \neq 0$ for all n. Now let U be the set of all sequences $(u_n)_{n\geq 1} \in s$, with $u_n \neq 0$ for all n. The infinite matrix $C(\eta) = (c_{nm})_{n,m\geq 1}$, for $\eta = (\eta_n)_{n\geq 1} \in U$, is defined by

$$c_{nm} = \begin{cases} \dfrac{1}{\eta_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that the matrix $\Delta(\eta) = (d_{nm})_{n,m\geq 1}$ with

$$d_{nm} = \begin{cases} \eta_n & \text{if } m = n, \\ -\eta_{n-1} & \text{if } m = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

is the inverse of $C(\eta)$, that is $C(\eta)(\Delta(\eta)\xi) = \Delta(\eta)(C(\eta)\xi)$ for all $\xi \in s$. If $\eta = e$ we get the well known operator of the first difference represented by $\Delta(e) = \Delta$. We then have $\Delta\xi_n = \xi_n - \xi_{n-1}$ for all $n \geq 1$, with the convention $\xi_0 = 0$. It is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and $\Delta, \Sigma \in S_R$ for any $R > 1$.

Consider the sets

$$\widehat{C_1} = \left\{\xi \in U^+ : \quad [C(\xi)\xi]_n = \frac{1}{\xi_n}\sum_{m=1}^{n}\xi_m = O(1)\right\},$$

$$\widehat{\Gamma} = \left\{\xi \in U^+ : \lim_{n\to\infty}\left(\frac{\xi_{n-1}}{\xi_n}\right) < 1\right\}, \Gamma = \left\{\xi \in U^+ : \limsup_{n\to\infty}\left(\frac{\xi_{n-1}}{\xi_n}\right) < 1\right\}$$

and

$$G_1 = \left\{\xi \in U^+ : \text{there exist } C > 0 \text{ and } \gamma > 1 \text{ such that } \xi_n \geq C\gamma^n \text{ for all } n\right\}.$$

By [4, Proposition 2.1, p. 1786] and [16, Proposition 2.2 p. 88], we obtain the next lemma.

Lemma 7. $\widehat{\Gamma} \subset \Gamma \subset \widehat{C_1} \subset G_1$.

We also need the following results.

Lemma 8. *[8, Proposition 9, p. 300] Let $a, b \in U^+$. Then*

(i) the following statements are equivalent

(α) $\chi_a(\Delta) = \chi_b$ where χ is any of the symbols s, or s^0,

(β) $a \in \widehat{C_1}$ and $s_a = s_b$.

(ii) $a \in \widehat{\Gamma}$ if and only if $s_a^{(c)}(\Delta) = s_a^{(c)}$.

rom the preceding results we deduce the following:

3.2. Application to the equation $\chi_a(C(\lambda)D_\tau)+\chi_x(C(\mu)D_\tau)=\chi_b$ where x is the unknown

Let a, b, λ, μ, $\tau\in U^+$ and consider the equation

$$\chi_a(C(\lambda)D_\tau)+\chi_x(C(\mu)D_\tau)=\chi_b,\text{ where }\chi=s^0,\text{ or }s \tag{3}$$

and $x\in U^+$ is the unknown. The operator represented by $C(\lambda)D_\tau=D_{1/\lambda}\Sigma D_\tau$ is called a factorable matrix. For $\chi=s^0$ solving the (SSE) (3) consists of determining all sequences $x\in U^+$ such that the condition $y_n/b_n\to 0$ $(n\to\infty)$ holds if and only if there are u, $v\in s$ such that $y=u+v$ and

$$\frac{\tau_1u_1+...+\tau_nu_n}{\lambda_na_n}\to 0\text{ and }\frac{\tau_1v_1+...+\tau_nv_n}{\mu_nx_n}\to 0\ \ (n\to\infty)\ \text{ for all }y\in s.$$

We then have the following result.

Theorem 9. *Let a, b, λ, μ, $\tau\in U^+$. Then*

(i) If $b\tau\notin\widehat{C_1}$, then equation (3) where x is the unknown has no solutions.

(ii) If $b\tau\in\widehat{C_1}$ we then have

(a) if $a\lambda/b\tau\in c_0$, then equation (3) is equivalent to $s_x=s_{b\tau/\mu}$, that is $K_1b_n\tau_n/\mu_n\le x_n\le K_2b_n\tau_n/\mu_n$ for all n and for some K_1, $K_2>0$.

(b) if $a\lambda/b\tau$, $b\tau/a\lambda\in\ell_\infty$, then the solutions of (3) are all sequences that satisfy $x\in s_{b\tau/\mu}$, that is, $x_n\le Kb_n\tau_n/\mu_n$ for all n and for some $K>0$.

(c) If $a\lambda/b\tau\notin\ell_\infty$, then (3) has no solution.

Proof. We have $[C(\lambda)D_\tau]^{-1}=D_{1/\tau}\Delta(\lambda)$ and $[C(\mu)D_\tau]^{-1}=D_{1/\tau}\Delta(\mu)$ then

$$\chi_a(C(\lambda)D_\tau)=[C(\lambda)D_\tau]_a^{-1}\chi_a=D_{1/\tau}\Delta(\lambda)\chi_a$$

and $\chi_x(C(\mu)D_\tau)=D_{1/\tau}\Delta(\mu)\chi_x$ and equation (3) is equivalent to

$$D_{1/\tau}\Delta(\lambda)\chi_a+D_{1/\tau}\Delta(\mu)\chi_x=\chi_b,$$

that is $D_{1/\tau}\Delta\left(\chi_{a\lambda}+\chi_{\mu x}\right)=\chi_b$. Since $\Delta(\lambda)=\Delta D_\lambda$ and $\Delta(\mu)=\Delta D_\mu$ we deduce

$$\chi_{a\lambda}+\chi_{\mu x}=\chi_b\left(D_{1/\tau}\Delta\right)=\chi_{b\tau}(\Delta). \tag{4}$$

Then (4) is equivalent to $\chi_{a\lambda+\mu x}=\chi_{b\tau}(\Delta)$ itself equivalent to $\chi_{a\lambda+\mu x}=\chi_{b\tau}$ and $b\tau\in\widehat{C_1}$ by Lemma 8. So if $b\tau\notin\widehat{C_1}$ equation (3) has no solution and if $b\tau\in\widehat{C_1}$ it is enough to apply Theorem 1 to the equation $\chi_{a\lambda}+\chi_{\mu x}=\chi_{b\tau}$. $\square$

We can state the following corollaries.

Corollary 10. *Consider the equation*

$$\chi_1(C(\lambda)D_\tau)+\chi_x(C(\mu)D_\tau)=\chi_1\text{ with }\chi=s^0,\text{ or }s. \tag{5}$$

(i) If $\tau\notin\widehat{C_1}$, then (5) has no solutions.

(ii) If $\tau\in\widehat{C_1}$, then

(a) if $\lambda\in s_\tau^0$, then (5) is equivalent to $s_x=s_{\tau/\mu}$;

(b) if $\lambda\in s_\tau$, $\tau\in s_\lambda$, then the solutions of (SSE) (5) are all sequences that satisfy $x\in s_{\tau/\mu}$;

(c) if $\lambda\notin s_\tau$, then (5) has no solution.

Proof. It is enough to take $a = b = e$ in Theorem 9. □

In the following remark where $C(\lambda) = C((n)_n)$ is the Cesàro operator denoted by C_1, the (SSE) is completely solved.

Remark 11. Consider the (SSE)

$$\chi_1(C_1 D_\tau) + \chi_x(C_1 D_\tau) = \chi_1 \text{ with } \chi = s^0, \text{ or } s. \tag{6}$$

If $\tau \notin \widehat{C_1}$ then (6) has no solution. If $\tau \in \widehat{C_1}$ the solutions of (6) are all the sequences that satisfy $s_x = s_{(\tau_n/n)_n}$. This means that there are $K_1, K_2 > 0$ such that $K_1\tau_n/n \leq x_n \leq K_2\tau_n/n$ for all n. Indeed, we have $\lambda_n = n$ and since $\tau \in \widehat{C_1}$ implies that there is $\gamma > 1$ such that $\tau_n \geq K\gamma^n$ for all n and for some $K > 0$, we deduce that $n/\tau_n \to 0$ $(n \to \infty)$. So it is enough to apply Corollary 10 (ii).

To state the next result, consider the equation

$$\chi_a(C(\lambda)) + \chi_x(C(\mu)) = \chi_b \text{ with } \chi = s^0, \text{ or } s. \tag{7}$$

Corollary 12. *Let $a, b, \lambda, \mu \in U^+$. Then*

(i) If $b \notin \widehat{C_1}$, then equation (7) has no solution.

(ii) If $b \in \widehat{C_1}$, then 3 cases are possible,

(a) if $a\lambda/b \in c_0$ then the solutions $x \in U^+$ of equation (7) are all sequences that satisfy $s_x = s_{b/\mu}$;

(b) if there are $k_1, k_2 > 0$ such that $k_1 \leq a_n\lambda_n/b_n \leq k_2$ for all n, then equation (7) is equivalent to $x \in s_{b/\mu}$;

(c) if $a\lambda/b \notin \ell_\infty$, then equation (7) has no solution.

Proof. This result follows from Theorem 9 with $\tau = e$. □

When $a = e$ we obtain the next corollary where the (SSE) is totally solved.

Corollary 13. *The equation $\chi_1(C_1) + \chi_x(C_1) = \chi_b$ with $\chi = s^0$, or s, has no solution if $b \notin \widehat{C_1}$ and if $b \in \widehat{C_1}$ the solutions are determined by $K_1 b_n/n \leq x_n \leq K_2 b_n/n$ for all n and for some $K_1, K_2 > 0$.*

Proof. This result follows from Corollary 12 with $a = e$, $\lambda_n = \mu_n = n$ for all n. Indeed, the condition $b \in \widehat{C_1}$ implies that there is $\gamma > 1$ such that $b_n \geq K\gamma^n$ for all n. Then we have $a_n\lambda_n/b_n \leq Kn\gamma^{-n} = o(1)$ $(n \to \infty)$. □

Now state the next result where we put $\lambda_0 = (n)_{n\geq 1}$. Here the (SSE) is also totally solved.

Corollary 14. *Let $r_1, r_2 > 0$ and consider the equation*

$$\chi_{r_1}(C_1 D_{\lambda_0}) + \chi_x(C_1 D_{\lambda_0}) = \chi_{r_2} \text{ with } \chi = s^0, \text{ or } s. \tag{8}$$

(i) If $r_2 \leq 1$, then equation (8) has no solution.

(ii) If $r_2 > 1$, then

(a) if $r_1 < r_2$, then equation (8) is equivalent to $s_x = s_{r_2}$;

(b) if $r_1 = r_2$, then equation (8) is equivalent to $x \in s_{r_2}$;

(c) if $r_1 > r_2$, then equation (8) has no solution.

Proof. i) If $r_2 \leq 1$, then we have $(r_2^n)_{n\geq 1} \notin \widehat{C_1}$, since by Lemma 7 we have $\widehat{C_1} \subset G_1$ and by Corollary 12 equation (8) has no solutions. ii) Case when $r_2 > 1$. (a) First we have

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\frac{r_2^{n-1}}{r_2^n}\right) = \frac{1}{r_2} < 1$$

and since $\widehat{\Gamma} \subset \widehat{C_1}$ we deduce $(nr_2^n)_{n\geq 1} \in \widehat{C_1}$. So by Theorem 9 we have

$$\frac{a_n\lambda_n}{b_n\tau_n} = \frac{nr_1^n}{nr_2^n} = \left(\frac{r_1}{r_2}\right)^n = o(1) \quad (n\to\infty)$$

and $s_x = s_{r_2}$. The cases (b) and (c) can be shown similarly. □

3.3. The (SSE) with operators of the weighted means

In this subsection we use the operator of weighted means $\overline{N}_q$ defined by the triangle $\left[\overline{N}_q\right]_{nm} = q_m/Q_n$ for $m \leq n$, where $Q_n = \sum_{m=1}^{n} q_m$, for all n, with $q \in U^+$.

Consider now the equation

$$\chi_a\left(\overline{N}_q\right) + \chi_x\left(\overline{N}_p D_{q/p}\right) = \chi_b \text{ where } \chi = s^0, \text{ or } s \tag{9}$$

for p, $q \in U^+$. The question in the case when $\chi = s$ is: what are the sequences $x \in U^+$ such that $y_n = O(b_n)$ $(n\to\infty)$ if and only if there are u, $v \in s$ such that $y = u + v$ and

$$\frac{q_1u_1 + ... + q_nu_n}{Q_n} = O(a_n) \text{ and } \frac{q_1v_1 + ... + q_nv_n}{P_n} = O(x_n) \quad (n\to\infty) \text{ for all } y?$$

Since we have $\overline{N}_q = D_{1/Q}\Sigma D_q = C(Q)D_q$, it is enough to take $Q = \lambda$, $\mu = P$ and $\tau = q$ in Theorem 9. We then have

Corollary 15. *Let a, b, p, $q \in U^+$. Then*

(i) If $bq \notin \widehat{C_1}$, then (9) has no solution;

(ii) if $bq \in \widehat{C_1}$, then

(a) $aQ/bq \in c_0$ implies that (SSE) (9) is equivalent to $s_x = s_{bq/P}$.

(b) If there are k_1, $k_2 > 0$ such that $k_1 \leq a_nQ_n/b_nq_n \leq k_2$ for all n, the solutions of (9) are all the sequences that satisfy $x \in s_{bq/P}$ (that is, $x_n \leq Kb_nq_n/P_n$ for all n).

(c) If $aQ/bq \notin \ell_\infty$, then (9) has no solution.

This result leads to the following application.

Example 16. Let $R > 0$ and let S be the set of all sequences $x \in U^+$ that satisfy the statement: $y_n/R^n \to 0$ $(n\to\infty)$ if and only if there are u, v such that $y = u + v$ and

$$\frac{1}{2^n-1}\sum_{m=1}^{n} 2^m u_m \to 0 \text{ and } \frac{1}{nx_n}\sum_{m=1}^{n} 2^m v_m \to 0 \quad (n\to\infty) \text{ for all } y \in s.$$

It can be shown that the set S is empty if $R < 1$; if $R = 1$, it is equal to $s_{(1/n)_n}$ and if $R > 1$ it is determined by $K_1(2R)^n/n \leq x_n \leq K_2(2R)^n/n$ for all n.

To end this section consider a new type of (SSE) using the sets $s_a^{(c)}$.

3.4. On the (SSE) $\chi_a (C(\lambda) D_\tau) + s_x^0 (C(\mu) D_\tau) = s_b^0$ where χ is either s, or $s^{(c)}$

Consider now another type of (SSE) with factorable matrices using the set $s_a^{(c)}$ and that are totally solved. Here we determine the set of all the sequences $x \in U^+$ such that the condition $y_n/b_n \to 0$ $(n \to \infty)$ holds if and only if there are $u, v \in s$ such that $y = u + v$ and

$$\frac{\tau_1 u_1 + ... + \tau_n u_n}{\lambda_n a_n} \to l \text{ and } \frac{\tau_1 v_1 + ... + \tau_n v_n}{\mu_n x_n} \to 0 \ \ (n \to \infty)$$

for all $y \in s$ and for some scalar l. We state the next lemma, which is a direct consequence of [17, Theorem 4.4, p. 7].

Lemma 17. *Let a, $b \in U^+$ and consider the (SSE)*

$$\chi_a + s_x^0 = s_b^0, \text{ where } \chi \text{ is either } s, \text{ or } s^{(c)}. \tag{10}$$

(i) if $a/b \in c_0$, then the solutions of (10) are all the sequences that satisfy $s_x = s_b$.

(ii) if $a/b \notin c_0$, then (10) has no solution.

From Lemma 17 and Theorem 9 we deduce the resolution of the (SSE)

$$\chi_a (C(\lambda) D_\tau) + s_x^0 (C(\mu) D_\tau) = s_b^0 \text{ where } \chi \text{ is either } s, \text{ or } s^{(c)}. \tag{11}$$

Theorem 18. *Let a, b, λ, μ, $\tau \in U^+$. Then*

(i) if $b\tau \notin \widehat{C_1}$, then (SSE) (11) has no solution.

(ii) If $b\tau \in \widehat{C_1}$, then two cases are possible,

(a) if $a\lambda/b\tau \in c_0$, then the solutions of (11) are all the sequences that satisfy $s_x = s_{b\tau/\mu}$;

(b) if $a\lambda/b\tau \notin c_0$, then (11) has no solution.

Proof. Let χ be any of the symbols s, or $s^{(c)}$. Show that if x satisfies (11), then $\chi_{a\lambda} + s_{\mu x}^0 = s_{b\tau}^0$ and $b\tau \in \widehat{C_1}$. Reasoning as in the proof of Theorem 9, we have that (11) is equivalent to

$$\chi_{a\lambda} + s_{\mu x}^0 = s_b^0 \left(D_{1/\tau} \Delta\right) = s_{b\tau}^0 (\Delta), \tag{12}$$

and since we have $s_{a\lambda}^0 \subset \chi_{a\lambda} \subset s_{a\lambda}$ and $s_{\mu x}^0 \subset s_{\mu x}$, we deduce

$$s_{a\lambda+\mu x}^0 = s_{a\lambda}^0 + s_{\mu x}^0 \subset \chi_{a\lambda} + s_{\mu x}^0 \subset s_{a\lambda} + s_{\mu x} = s_{a\lambda+\mu x}.$$

Then

$$s_{a\lambda+\mu x}^0 \subset s_{b\tau}^0 (\Delta) \subset s_{a\lambda+\mu x}.$$

The first inclusion is equivalent to $I \in \left(s_{a\lambda+\mu x}^0, s_{b\tau}^0 (\Delta)\right)$ and to $D_{1/b\tau} \Delta D_{a\lambda+\mu x} \in (c_0, c_0)$. Since $(c_0, c_0) \subset (c_0, s_1) = S_1$, we deduce

$$\frac{a_n \lambda_n + \mu_n x_n}{b_n \tau_n} \leq K \text{ for all } n.$$

The second inclusion yields $\Delta^{-1}=\Sigma\in\left(s^0_{b\tau},s_{a\lambda+\mu x}\right)$, that is $D_{1/(a\lambda+\mu x)}\Sigma D_{b\tau}\in(c_0,\ell_\infty)=S_1$ and

$$\frac{b_1\tau_1+...+b_n\tau_n}{a_n\lambda_n+\mu_n x_n}\leq K' \text{ for all } n.$$

We deduce

$$\frac{b_1\tau_1+...+b_n\tau_n}{b_n\tau_n}=\frac{b_1\tau_1+...+b_n\tau_n}{a_n\lambda_n+\mu_n x_n}\frac{a_n\lambda_n+\mu_n x_n}{b_n\tau_n}\leq KK' \text{ for all } n.$$

We conclude $b\tau\in\widehat{C_1}$ and by (12) and Lemma 8 we have $\chi_{a\lambda}+s^0_{\mu x}=s^0_{b\tau}$.

Conversely if $\chi_{a\lambda}+s^0_{\mu x}=s^0_{b\tau}$ and $b\tau\in\widehat{C_1}$, then (12) and (11) hold. We conclude the proof using Lemma 17. □

Remark 19. Note that the (SSE) in (11) has solutions if and only if $b\tau\in\widehat{C_1}$ and $a\lambda/b\tau\in c_0$.

4. On the equation $\chi_a\left(C(\lambda)C(\mu)\right)+\chi_x\left(C(\lambda\sigma)C(\mu)\right)=\chi_b$

In this section for a, b, λ, μ, $\sigma\in U^+$ we consider an equation that generalizes (SSE) (3) and defined for $b\in\widehat{C_1}$ by

$$\chi_a\left(C(\lambda)C(\mu)\right)+\chi_x\left(C(\lambda\sigma)C(\mu)\right)=\chi_b, \tag{13}$$

where χ is any of the symbols s, or s^0. For $\chi=s^0$ the resolution of equation (13) consists in determining the set of all $x\in U^+$ such that for every $y\in s$ the condition $y_n/b_n\to 0$ $(n\to\infty)$ holds if and only if there are u, $v\in s$ such that $y=u+v$ and

$$\frac{1}{\lambda_n a_n}\sum_{m=1}^{n}\left(\frac{1}{\mu_m}\sum_{k=1}^{m}u_k\right)\to 0 \text{ and } \frac{1}{\lambda_n\sigma_n x_n}\sum_{m=1}^{n}\left(\frac{1}{\mu_m}\sum_{k=1}^{m}v_k\right)\to 0\ \ (n\to\infty). \tag{14}$$

To solve equation (13) we state the following proposition.

Proposition 20. *Assume that $b\in\widehat{C_1}$. Then*

(i) if $b/\mu\notin\widehat{C_1}$, then equation (13) has no solution.

(ii) Let $b/\mu\in\widehat{C_1}$. Then

(a) if $a\lambda\mu/b\in c_0$, then equation (13) holds if and only if $s_x=s_{b/\lambda\sigma\mu}$;

(b) if $a\lambda\mu/b$, $b/a\lambda\mu\in\ell_\infty$, then equation (13) holds if and only if $x\in s_{b/\lambda\sigma\mu}$;

(c) if $a\lambda\mu/b\notin\ell_\infty$, then equation (13) has no solution.

Proof. Equation (13) is equivalent to $\Delta(\mu)\left(\Delta(\lambda)\chi_a+\Delta(\lambda\sigma)\chi_x\right)=\chi_b$, that is

$$\Delta(\lambda)\chi_a+\Delta(\lambda\sigma)\chi_x=\chi_b\left(\Delta(\mu)\right)=D_{1/\mu}\chi_b(\Delta) \tag{15}$$

and since $b\in\widehat{C_1}$, we have $D_{1/\mu}\chi_b(\Delta)=D_{1/\mu}\chi_b=\chi_{b/\mu}$. So equation (15) is equivalent to $\chi_{a\lambda}+\chi_{\lambda\sigma x}=\chi_{b/\mu}(\Delta)$. Then by Lemma 8 equation (15) is equivalent to $b/\mu\in\widehat{C_1}$ and $\chi_{a\lambda}+\chi_{\lambda\sigma x}=\chi_{b/\mu}$. We conclude by Theorem 1 and Corollary 6 that if $a\lambda\mu/b\in c_0$ equation, $\chi_{a\lambda}+\chi_{\lambda\sigma x}=\chi_{b/\mu}$ is equivalent to $s_x=s_{b/\lambda\sigma\mu}$. The cases (b) and (c) follow immediately from Theorem 1.

Example 21. The set of all $x \in U^+$ such that $y_n/2^n = O(1)$ $(n \to \infty)$ holds if and only if there are u, $v \in s$ such that $y = u + v$ and

(16)
$$\frac{1}{n}\sum_{m=1}^{n}\left(\frac{1}{m}\sum_{k=1}^{m}u_k\right) = O(1) \text{ and } \frac{1}{x_n}\sum_{m=1}^{n}\left(\frac{1}{m}\sum_{k=1}^{m}v_k\right) = \frac{1}{n}O(1) \quad (n \to \infty)$$

for all y is given by

$$K_1 2^n \le x_n \le K_2 2^n \text{ for all } n. \tag{17}$$

Indeed, the previous statement is equivalent to the equation

$$\ell_\infty\left(C_1^2\right) + s_x\left(C\left(\left(1/n\right)_n\right)C_1\right) = s_2. \tag{18}$$

We have $b = (2^n)_{n\ge 1} \in \widehat{C_1}$, $b/\mu = (2^n/n)_{n\ge 1} \in \widehat{C_1}$ and $a_n\lambda_n\mu_n/b_n = n^2 2^{-n} \to 0$ $(n \to \infty)$. So we obtain (17). Furthermore for each x satisfying (17), we have

$$\left(\ell_\infty\left(C_1^2\right) + s_x\left(C\left(\left(1/n\right)_n\right)C_1\right), s_\alpha\right) = \left(s_2, s_\alpha\right) \text{ for } \alpha \in U^+.$$

So $A \in \left(\ell_\infty\left(C_1^2\right) + s_x\left(C\left(\left(1/n\right)_n\right)C_1\right), s_\alpha\right)$ if and only if

$$\sup_n\left(\alpha_n^{-1}\sum_{m=1}^{\infty}\left|a_{nm}\right| 2^m\right) < \infty.$$

□

References

[1] Farés A., de Malafosse, B., Sequence spaces equations and application to matrix transformations Int. Math. Forum 3. 19 (2008), 911-927.

[2] Maddox, I.J., Infinite matrices of operators. Berlin, Heidelberg and New York: Springer-Verlag, 1980.

[3] de Malafosse, B., Contribution à l'étude des systèmes infinis. Thèse de Doctorat de 3è cycle, Univ. Paul Sabatier, Toulouse III, 1980.

[4] de Malafosse, B., On some BK space. Int. J. Math. Math. Sci. 28 (2003), 1783-1801.

[5] de Malafosse, B., Sum and product of certain BK spaces and matrix transformations between these spaces. Acta Math. Hung. 104 (3), (2004), 241-263.

[6] de Malafosse, B., The Banach algebra $B(X)$, where X is a BK space and applications. Mat. Vesnik, 57 (2005), 41-60.

[7] de Malafosse, B., Space $S_{\alpha,\beta}$ and $\sigma-$core. Studia Math. 172 (2006), 229-241.

[8] de Malafosse, B., Sum of sequence spaces and matrix transformations. Acta Math. Hung. 113 (3) (2006), 289-313.

[9] de Malafosse, B., Sum of sequence spaces and matrix transformations mapping in $s_\alpha^0\left((\Delta - \lambda I)^h\right) + s_\beta^{(c)}\left((\Delta - \mu I)^l\right)$. Acta math. Hung., 122 (2008), 217-230.

[10] de Malafosse, B., Application of the infinite matrix theory to the solvability of certain sequence spaces equations with operators. Mat. Vesnik 54, 1 (2012), 39-52.

[11] de Malafosse, B., Applications of the summability theory to the solvability of certain sequence spaces equations with operators of the form $B(r,s)$. Commun. Math. Anal. 13, 1 (2012), 35-53.

[12] de Malafosse B., Solvability of certain sequence spaces inclusion equations with operators, Demonstratio Math. 46, 2 (2013), 299-314.

[13] de Malafosse, B., Solvability of sequence spaces equations using entire and analytic sequences and applications. J. Ind. Math. Soc. 81 N°1-2, (2014), 97-114.

[14] de Malafosse, B., Malkowsky, E., On the solvability of certain (SSIE) with operators of the form $B(r,s)$. Math. J. Okayama. Univ. 56 (2014), 179-198.

[15] de Malafosse, B., Malkowsky, E., On sequence spaces equations using spaces of strongly bounded and summable sequences by the Cesàro method. Antartica J. Math. 10 (6) (2013), 589-609.

[16] de Malafosse, B., Malkowsky, E., Matrix transformations in the sets $\chi\left(\overline{N}_p\overline{N}_q\right)$ where χ is in the form s_ξ, or $\overset{\circ}{s}_\xi$, or $s_\xi^{(c)}$. Filomat 17 (2003), 85-106.

[17] de Malafosse, B., Rakočević V., Matrix transformations and sequence spaces equations. Banach J. Math. Anal. 7 (2) (2013), 1-14.

[18] de Malafosse, B., Rakočević, V., Applications of measure of noncompactness in operators on the spaces s_α, s_α^0, $s_\alpha^{(c)}$ and l_α^p. J. Math. Anal. Appl. 323 (2006), 131-145.

[19] de Malafosse, B., Rakočević, V., Matrix Transformations and Statistical convergence. Linear Algebra Appl. 420 (2007), 377-387.

[20] de Malafosse, B., Rakočević V., A generalization of a Hardy theorem. Linear Algebra Appl. 421 (2007), 306-314.

[21] Wilansky, A., Summability through Functional Analysis. North-Holland Mathematics Studies 85, 1984.

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