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MEROMORPHIC FUNCTIONS AND THEIR kth DERIVATIVE SHARE ONLY ONE SMALL FUNCTION CM

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Abstract. It is shown that if a non-constant meromorphic function f and its derivative $f^{(k)}$ share one meromorphic small function $\beta \not\equiv 0, \infty$ CM (counting multiplicities), then either $T(r, f^{(k)}) = O(\bar{N}(r, \frac{1}{f^{(k)}}))$ or $f - \beta = (1 - \frac{p_{k-1}}{\beta})(f^{(k)} - \beta)$, where p_{k-1} is a polynomial of degree at most $k-1$ and $1 - \frac{p_{k-1}}{\beta} \not\equiv 0$. This result answers Brück's question and improves Al-Khaladi's result.

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1. Introduction and main results

In this paper, by meromorphic function we always mean a function which is meromorphic in the whole complex plane. We use the notation of the Nevanlinna theory of meromorphic functions (see [5], [8]). By $S(r, f)$ we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r with finite linear measure. Then the meromorphic function β is called a small function of f if $T(r, \beta) = S(r, f)$. We say that two non-constant meromorphic functions f and g share a small function β CM (counting multiplicities) if $f - \beta$ and $g - \beta$ have the same zeros with the same multiplicities. Let k be a positive integer, and let b be a small function of f or ∞, we denote by $N_{k)}(r, \frac{1}{f-b})$ the counting function of b-points of f with multiplicity $\leq k$ and by $N_{(k+1}(r, \frac{1}{f-b})$ the counting function of b-points of f with multiplicity $> k$. In the same way we define $\bar{N}_{k)}(r, \frac{1}{f-b})$ and $\bar{N}_{(k+1}(r, \frac{1}{f-b})$ where in counting the b-points of f we ignore the multiplicities (see [8]). Finally we denote by $N_0(r, \frac{1}{f^{(k+1)}})$ the counting function of the zeros of $f^{(k+1)}$ that are not zeros of $f^{(k)}$, where these zeros are counted according to their multiplicity while $\bar{N}_0(r, \frac{1}{f^{(k+1)}})$ counts these zeros only once.

Now we move to the problems of uniqueness of an entire function and its derivative that share some values. Rubel-Yang (see [6]) proved that if the entire function f and f' share two distinct finite values CM then $f \equiv f'$. In general, this result is false if f and f' share only one value. This may be seen in the examples given in [4]. Now the following question may be raised (see [4]). What conclusion can be made, if f and f' share only one value? In 1998, Zhang [9] answered this question and proved the following:

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Theorem A. *Let f be a non-constant meromorphic function. If f and f' share the value 1 CM, then either $f-1=c(f'-1)$, where c is a nonzero constant, or*

$$T(r,f') \le 2N\Big(r,\frac{1}{f'}\Big) + 2\bar{N}(r,f) + S(r,f).$$

In 2010 Al-Khaladi [2] improved and generalized this result and proved the following:

Theorem B. *Let f be a non-constant meromorphic function and let $\beta \not\equiv 0,\infty$ be a meromorphic small function of f. If f and $f^{(k)}$ share β CM, then either*

$$f-\beta = \Big(1-\frac{p_{k-1}}{\beta}\Big)(f^{(k)}-\beta), \tag{1.1}$$

where p_{k-1} is a polynomial of degree at most $k-1$ and $1-\frac{p_{k-1}}{\beta} \not\equiv 0$, or

$$T(r,f^{(k)}) \le (k+1)\bar{N}\Big(r,\frac{1}{f^{(k)}}\Big) + (k+1)\bar{N}(r,f) + S(r,f). \tag{1.2}$$

The aim of this paper is to show that the term involving $\bar{N}(r,f)$ in (1.2) can be dropped completely if one allows a large factor of $\bar{N}(r,1/f^{(k)})$. Note that in general situation one can omit $\bar{N}(r,f)$ be adding $\varepsilon T(r,f)$ ([7]). In fact, we shall prove the following theorem:

Theorem 1.1. *Let f be a non-constant meromorphic function and let $\beta \not\equiv 0,\infty$ be a meromorphic small function of f. If f and $f^{(k)}$ share β CM, then either (1.1) holds, or*

$$T(r,f^{(k)}) \le \begin{cases} \frac{(k+1)^2(k+2)}{k-1}\bar{N}(r,1/f^{(k)}) + S(r,f^{(k)}) & \text{if } k\ge 2 \\ 12\bar{N}(r,1/f') + S(r,f') & \text{if } k=1 \text{ and } \beta' \not\equiv 0 \\ 36\bar{N}(r,1/f') + S(r,f') & \text{if } k=1 \text{ and } \beta' \equiv 0 \end{cases} \tag{1.3}$$

From Theorem 1.1, we immediately deduce the following corollary:

Corollary 1.2. *Let f be a non-constant meromorphic function and let $\beta \not\equiv 0,\infty$ be a meromorphic small function of f. If f and $f^{(k)}$ share β CM, and if $\bar{N}(r,1/f^{(k)}) = S(r,f)$, then (1.1) holds.*

2. Some lemmas

For the proof of our theorem we need the following lemmas:

Lemma 2.1. *Let f' be a non-constant meromorphic function and let*

$$W = \Big(\frac{f''}{f'}\Big)^2 - 2\Big(\frac{f''}{f'}\Big)'. \tag{2.1}$$

Then

$$T(r,W) \le 2\bar{N}\Big(r,\frac{1}{f'}\Big) + 2\bar{N}_{(2}(r,f) + S(r,f'). \tag{2.2}$$

Proof. From Nevanlinna's fundamental estimate of the logarithmic derivative we obtain

$$m(r,W)\le 3m\Big(r,\frac{f''}{f'}\Big)+S\Big(r,\frac{f''}{f'}\Big)+O(1)=S(r,f')+S\Big(r,\frac{f''}{f'}\Big).$$

Since

$$\begin{aligned} T\Big(r,\frac{f''}{f'}\Big) &= N\Big(r,\frac{f''}{f'}\Big)+m\Big(r,\frac{f''}{f'}\Big)\le \bar N(r,f')+\bar N\Big(r,\frac{1}{f'}\Big)+S(r,f')\\ &\le 2T(r,f')+S(r,f'), \end{aligned}$$

this means that

$$m(r,W)=S(r,f'). \tag{2.3}$$

If z_∞ is a simple pole of f, then by an elementary calculation, we deduce from (2.1) that W is holomorphic at z_∞. We can also conclude from (2.1) that the poles of f with multiplicity $p\ge 2$ are poles of W with multiplicity 2 at most. And the zeros of f' with multiplicity $s\ge 1$ are poles of W with multiplicity 2. Thus

$$N(r,W)\le 2\bar N\Big(r,\frac{1}{f'}\Big)+2\bar N_{(2}(r,f).$$

Together with (2.3) we find (2.2). □

Our next lemma give extension of Lemma 2.5 in [1].

Lemma 2.2. *Let f' be a non-constant meromorphic function, and let β be a small function of f' such that $\beta\not\equiv 0,\infty$. Then*

$$T(r,f')\le 2\bar N\Big(r,\frac{1}{f'}\Big)+N_{1)}\Big(r,\frac{1}{f'-\beta}\Big)+2\bar N_{(2}\Big(r,\frac{1}{f'-\beta}\Big)+2\bar N_{(2}(r,f)+S(r,f'). \tag{2.4}$$

Proof. We set $F=f'/\beta$. Then it is clear that

$$\Big(\frac{f''}{f'}\Big)^2=\Big(\frac{F'}{F}\Big)^2+2\frac{\beta'}{\beta}\Big(\frac{F'}{F}\Big)+\Big(\frac{\beta'}{\beta}\Big)^2$$

and

$$\Big(\frac{f''}{f'}\Big)'=\Big(\frac{F'}{F}\Big)'+\Big(\frac{\beta'}{\beta}\Big)'=\frac{F''}{F}-\Big(\frac{F'}{F}\Big)^2+\Big(\frac{\beta'}{\beta}\Big)'$$

Substituting these two equations into (2.1) gives

$$W+2\Big(\frac{\beta'}{\beta}\Big)'-\Big(\frac{\beta'}{\beta}\Big)^2=\frac{F'}{F}\Big(3\frac{F'}{F}-2\frac{F''}{F'}+2\frac{\beta'}{\beta}\Big). \tag{2.5}$$

We distinguish the following two cases:

Case 1. If

$$W+2\Big(\frac{\beta'}{\beta}\Big)'-\Big(\frac{\beta'}{\beta}\Big)^2\equiv 0. \tag{2.6}$$

Then (2.5) becomes

$$3\frac{F'}{F} - 2\frac{F''}{F'} + 2\frac{\beta'}{\beta} \equiv 0$$

By integration once,

$$F^3\beta^2 = cF'^2, \tag{2.7}$$

where c is a nonzero constant. From (2.7) we obtain

$$\begin{aligned} 3m(r,F) &\leq m(r,F'^2) + S(r,f') \leq 2m(r,F') + S(r,f') \\ &\leq 2m\Big(r,\frac{F'}{F}\Big) + 2m(r,F) + S(r,f') = 2m(r,F) + S(r,f'), \end{aligned}$$

so that

$$m(r,f') = S(r,f'). \tag{2.8}$$

We can also conclude from (2.7) that

$$3N(r,F) = 2N(r,F') + S(r,f') = 2\Big(N(r,F) + \bar{N}(r,F)\Big) + S(r,f').$$

That is,

$$N(r,F) = 2\bar{N}(r,F) + S(r,f').$$

This implies that

$$N(r,f') = 2\bar{N}(r,f') + S(r,f').$$

Hence

$$N(r,f) + \bar{N}(r,f) = 2\bar{N}(r,f') + S(r,f'),$$

which, in view of $\bar{N}(r,f) = \bar{N}(r,f')$, leads to $N_{(2}(r,f) = S(r,f')$. Combining this with (2.8) we have

$$T(r,f') = 2N_{1)}(r,f) + S(r,f'). \tag{2.9}$$

Now suppose z_∞ is a simple pole of f and $\beta(z_\infty) \neq 0, \infty$ (otherwise the counting function of those simple poles of f which are the zeros or poles of β equal to $S(r,f')$). Therefore the Laurent expansion of f about z_∞ is

$$f(z) = a_{-1}(z - z_\infty)^{-1} + O(1), \tag{2.10}$$

where a_{-1} be the residue of f at z_∞. It is easy to see from (2.10) and (2.7) that

$$4c + a_{-1}\beta + 4c\frac{\beta'}{\beta}(z - z_\infty) + O(z - z_\infty)^2 \equiv 0,$$

from which it follows that $\beta'(z_\infty) = 0$. From this, if $\beta' \not\equiv 0$, then from (2.9) we find that

$$T(r,f') = 2N_{1)}(r,f) + S(r,f') \leq 2N(r,\frac{1}{\beta'}) + S(r,f') = S(r,f').$$

This is impossible. Therefore, we get $\beta' \equiv 0$. This and (2.6) imply that $W \equiv 0$. It follows from this and (2.1) that

$$2\Big(\frac{f''}{f'}\Big)^{-2}\Big(\frac{f''}{f'}\Big)' = 1.$$

By integrating twice we obtain $f'(z) = A/(z+B)^2$, where $A \neq 0$ and B are constants. From this we see that

$$\bar{N}\Big(r, \frac{1}{f'}\Big) + \bar{N}_{(2}\Big(r, \frac{1}{f'-\beta}\Big) + \bar{N}_{(2}(r, f) = 0$$

and

$$T(r, f') = N_{1)}\Big(r, \frac{1}{f'-\beta}\Big) + S(r, f').$$

In particular, (2.4) holds.

Case 2. If

$$W + 2\Big(\frac{\beta'}{\beta}\Big)' - \Big(\frac{\beta'}{\beta}\Big)^2 \not\equiv 0.$$

Then (2.5) may now be put in the form

$$\frac{1}{F-1} = \Big(\frac{F'}{F-1} - \frac{F'}{F}\Big)\Big(\frac{3\frac{F'}{F} - 2\frac{F''}{F'} + 2\frac{\beta'}{\beta}}{W + 2(\frac{\beta'}{\beta})' - (\frac{\beta'}{\beta})^2}\Big).$$

It follows from the fundamental estimate that

$$m\Big(r, \frac{1}{F-1}\Big) \leq m\Big(r, \frac{1}{W + 2(\frac{\beta'}{\beta})' - (\frac{\beta'}{\beta})^2}\Big) + S(r, f'). \tag{2.11}$$

If z_0 is a zero of $F-1$ with multiplicity $q \geq 3$ and $\beta(z_0) \in \mathbb{C} \cup \{\infty\}$, then z_0 is a simple pole of $\frac{\beta'}{\beta}$ at most, z_0 is a zero of $\frac{F'}{F}$ with multiplicity $q-1$ and z_0 is a zero of $\frac{F''}{F}$ with multiplicity $q-2$. Hence z_0 is a zero of $3(\frac{F'}{F})^2 - 2\frac{F''}{F} + 2\frac{\beta'}{\beta}(\frac{F'}{F})$ with multiplicity at least $q-2$. It is implied from (2.5) that z_0 is a zero of $W + 2(\frac{\beta'}{\beta})' - (\frac{\beta'}{\beta})^2$ with multiplicity $q-2$ at least. Thus,

$$N_{(3}\Big(r, \frac{1}{F-1}\Big) - 2\bar{N}_{(3}\Big(r, \frac{1}{F-1}\Big) \leq N\Big(r, \frac{1}{W + 2(\frac{\beta'}{\beta})' - (\frac{\beta'}{\beta})^2}\Big).$$

Combining this, (2.11) and Lemma 2.1 yields

$$\begin{aligned} & m\Big(r, \frac{1}{F-1}\Big) + N_{(3}\Big(r, \frac{1}{F-1}\Big) \\ \leq\ & T\Big(r, \frac{1}{W + 2(\frac{\beta'}{\beta})' - (\frac{\beta'}{\beta})^2}\Big) \\ +\ & 2\bar{N}_{(3}\Big(r, \frac{1}{F-1}\Big) + S(r, f') \\ \leq\ & T(r, W) + 2\bar{N}_{(3}\Big(r, \frac{1}{F-1}\Big) + S(r, f') \\ \leq\ & 2\bar{N}\Big(r, \frac{1}{f'}\Big) + 2\bar{N}_{(2}(r, f) \\ +\ & 2\bar{N}_{(3}\Big(r, \frac{1}{F-1}\Big) + S(r, f'). \end{aligned} \tag{2.12}$$

Since

$$\begin{aligned} T(r,F) &= m\Big(r,\frac{1}{F-1}\Big)+N\Big(r,\frac{1}{F-1}\Big)+O(1)\\ &= m\Big(r,\frac{1}{F-1}\Big)+N_{2)}\Big(r,\frac{1}{F-1}\Big)+N_{(3}\Big(r,\frac{1}{F-1}\Big)+O(1) \end{aligned}$$

and

$$\begin{aligned} & N_{2)}\Big(r,\frac{1}{F-1}\Big)+2\bar{N}_{(3}\Big(r,\frac{1}{F-1}\Big)\\ &= N_{1)}\Big(r,\frac{1}{F-1}\Big)+N_{=2}\Big(r,\frac{1}{F-1}\Big)\\ &+ 2\bar{N}_{(3}\Big(r,\frac{1}{F-1}\Big)\\ &= N_{1)}\Big(r,\frac{1}{F-1}\Big)+2\bar{N}_{=2}\Big(r,\frac{1}{F-1}\Big)\\ &+ 2\bar{N}_{(3}\Big(r,\frac{1}{F-1}\Big)\\ &= N_{1)}\Big(r,\frac{1}{F-1}\Big)+2\bar{N}_{(2}\Big(r,\frac{1}{F-1}\Big) \end{aligned}$$

where $N_{=2}(r,\frac{1}{F-1})$ ($\bar{N}_{=2}(r,\frac{1}{F-1})$) denotes the counting function (reduced counting function) of zeros of $F-1$ with multiplicity $=2$, it follows from (2.12) readily that

$$T(r,F)\le 2\bar{N}\Big(r,\frac{1}{f'}\Big)+N_{1)}\Big(r,\frac{1}{F-1}\Big)+2\bar{N}_{(2}\Big(r,\frac{1}{F-1}\Big)+2\bar{N}_{(2}(r,f)+S(r,f')$$

From this and $F=f'/\beta$ we arrive at the conclusion (2.4). □

Lemma 2.3 ([1]). *Let k be a positive integer, and let f be a non-constant meromorphic function. Then either*

$$f(z)=\frac{-(k+1)^{k+1}}{ck![z+c_1(k+1)]}+p_{k-1}(z), \tag{2.13}$$

where $c\neq 0$, c_1 are constants and p_{k-1} is a polynomial of degree at most $k-1$, or

$$kN_{1)}(r,f)\le \bar{N}_{(2}(r,f)+\bar{N}\Big(r,\frac{1}{f^{(k)}}\Big)+\bar{N}_0\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f). \tag{2.14}$$

Lemma 2.4. *Let $k\ge 2$ be a positive integer, and let f be a non-constant meromorphic function. Then either* (2.13) *holds, or*

$$N_{1)}(r,f)\le \frac{2}{k-1}\bar{N}_{(2}(r,f)+\frac{2}{k-1}\bar{N}\Big(r,\frac{1}{f^{(k)}}\Big)+S(r,f). \tag{2.15}$$

Proof. It is clear that

$$\begin{aligned}\bar{N}_0\Big(r,\frac{1}{f^{(k+1)}}\Big) &\le N\Big(r,\frac{1}{f^{(k+1)}/f^{(k)}}\Big)\le T\Big(r,\frac{f^{(k+1)}}{f^{(k)}}\Big)+O(1)\\ &= N\Big(r,\frac{f^{(k+1)}}{f^{(k)}}\Big)+m\Big(r,\frac{f^{(k+1)}}{f^{(k)}}\Big)+O(1)\\ &= N\Big(r,\frac{f^{(k+1)}}{f^{(k)}}\Big)+S(r,f)\\ &\le \bar{N}\Big(r,\frac{1}{f^{(k)}}\Big)+\bar{N}(r,f)+S(r,f)\\ &= \bar{N}\Big(r,\frac{1}{f^{(k)}}\Big)+N_{1)}(r,f)+\bar{N}_{(2}(r,f)+S(r,f)\end{aligned}$$

Combining this with (2.14) we obtain (2.15). □

Lemma 2.5. *Let $k\ge 2$ be a positive integer, let f be a non-constant meromorphic function and let $\beta\not\equiv 0,\infty$ be a meromorphic small function of f. If f and $f^{(k)}$ share β CM, then only* (2.15) *holds.*

Proof. Assume that (2.15) is not true. Then from Lemma 2.4 we have (2.13). It follows that

$$T(r,f^{(k)})=(k+1)T(r,z)+O(1)=(k+1)logr+O(1),$$

from which we deduce β is a constant. Thus from (2.13) we find $f-\beta$ has at most k of zeros, while $f^{(k)}-\beta$ has exactly $k+1$ of zeros. So f and $f^{(k)}$ can not share β CM which contradicts the condition of Lemma 2.5. □

Lemma 2.6. *Let k be a positive integer, let f be a non-constant meromorphic function and let $\beta\not\equiv 0,\infty$ be a meromorphic small function of f. If f and $f^{(k)}$ share β CM, then only* (2.14) *holds.*

Proof. By using the same methods as those in proof of Lemma 2.5, we obtain only (2.14). □

3. Proof of Theorem 1.1

Since f and f' share β CM, every $S(r,f)$ is also $S(r,f')$ and vice versa. Suppose that (1.1) is not true. In view of

$$T(r,f^{(k)})=m(r,f^{(k)})+N(r,f^{(k)})=m(r,f^{(k)})+N(r,f)+k\bar{N}(r,f),$$

from this and Theorem B we obtain that

$$m(r,f^{(k)})+N_{(2}(r,f)-\bar{N}_{(2}(r,f)\le(k+1)\bar{N}\Big(r,\frac{1}{f^{(k)}}\Big)+S(r,f).$$

Hence

$$\bar{N}_{(2}(r,f)\le(k+1)\bar{N}\Big(r,\frac{1}{f^{(k)}}\Big)+S(r,f). \tag{3.1}$$

We discuss the following three cases:

Case I. $k \geq 2$. Then from Lemma 2.5 and (3.1) we obtain

$$N_{1)}(r,f) \leq \frac{2(k+2)}{(k-1)} \bar{N}\Big(r, \frac{1}{f^{(k)}}\Big) + S(r,f).$$

It follows from this, (1.2) and (3.1) that

$$\begin{aligned} T(r,f^{(k)}) &\leq (k+1)\Big[\bar{N}\Big(r,\frac{1}{f^{(k)}}\Big) + N_{1)}(r,f) + \bar{N}_{(2}(r,f)\Big] + S(r,f) \\ &\leq \frac{(k+2)(k+1)^2}{(k-1)} \bar{N}\Big(r,\frac{1}{f^{(k)}}\Big) + S(r,f^{(k)}). \end{aligned}$$

This is (1.3) when $k \geq 2$.

Case II. $k = 1$ and $\beta' \not\equiv 0$. Suppose z_1 is a zero of $f - \beta$. Then

$$f(z) - \beta(z) = a_n(z - z_1)^n + \dots, \quad a_n \neq 0. \tag{3.2}$$

Since f and f' share β CM,

$$f'(z) - \beta(z) = b_n(z - z_1)^n + \dots, \quad b_n \neq 0. \tag{3.3}$$

Differentiating (3.2) once we obtain

$$f'(z) - \beta'(z) = na_n(z - z_1)^{n-1} + \dots. \tag{3.4}$$

Eliminating $f'(z)$ between (3.3) and (3.4) leads to

$$\beta'(z) - \beta(z) = -na_n(z - z_1)^{n-1} + \dots. \tag{3.5}$$

If $\beta' \equiv \beta$, then from (3.5) we find $a_n = 0$, a contradiction. Therefore we must have $\beta' \not\equiv \beta$. Further,

$$N_{(2}\Big(r, \frac{1}{f-\beta}\Big) = N_{(2}\Big(r, \frac{1}{f'-\beta}\Big) \leq 2N\Big(r, \frac{1}{\beta'-\beta}\Big) = S(r,f). \tag{3.6}$$

We set

$$H = \frac{(f'/\beta)'(f-\beta)}{f'(f'-\beta)} = \frac{f-\beta}{\beta^2}\Big[\frac{(f'/\beta)'}{(f'/\beta)-1} - \frac{(f'/\beta)'}{f'/\beta}\Big]. \tag{3.7}$$

Then it is clear that

$$m(r,H) \leq m(r,f) + S(r,f). \tag{3.8}$$

From (3.7) we see that if z_∞ is a pole of f with multiplicity $p \geq 1$ and $\beta(z_\infty) \neq 0, \infty$ (otherwise the counting function of those poles of f which are the zeros or poles of β is equal to $S(r,f)$),

$$H(z_\infty) = \frac{1}{\beta(z_\infty)}\Big(\frac{p+1}{p}\Big). \tag{3.9}$$

It follows from (3.2), (3.3) and (3.4) that if z_1 is a simple zero of $f-\beta$ and $\beta(z_1)\neq 0,\infty$ (otherwise the counting function of those simple zero of $f-\beta$ which are the zeros or poles of β is equal to $S(r,f)$), then

$$H(z_1)=\Big(\frac{\beta-\beta'}{\beta^2}\Big)(z_1). \tag{3.10}$$

Thus the pole of H can only occur at zeros of f'. However, the zeros of f' with multiplicity $s\geq 1$ are poles of H with multiplicity 1. Therefore from this, (3.8), (3.9), (3.10) and (3.6) we get

$$T(r,H)\leq m(r,f)+N(r,H)\leq \bar{N}\Big(r,\frac{1}{f'}\Big)+m(r,f)+S(r,f). \tag{3.11}$$

We consider the following two subcases:

Case II.1. $H\equiv\frac{\beta-\beta'}{\beta^2}$. From this and (3.9) (when $p=1$) we know that if z_∞ is a simple pole of f and $\beta(z_\infty)\neq 0,\infty$ (otherwise the counting function of those simple poles of f which are the zeros or poles of β equal to $S(r,f)$), then $(\beta+\beta')(z_\infty)=0$. If $\beta+\beta'\not\equiv 0$,

$$N_{1)}(r,f)\leq N\Big(r,\frac{1}{\beta+\beta'}\Big)+S(r,f)=S(r,f). \tag{3.12}$$

Therefore (1.2), (3.12) and (3.1) give that

$$T(r,f')\leq 6\bar{N}\Big(r,\frac{1}{f'}\Big)+S(r,f').$$

In particular, (1.3) holds when $k=1$ and $\beta'\not\equiv 0$.

If $\beta+\beta'\equiv 0$, from this, $H\equiv\frac{\beta-\beta'}{\beta^2}$ and (3.7) we find that

$$\frac{f''}{f'}+1=2\Big(\frac{f'-\beta}{f-\beta}\Big). \tag{3.13}$$

From this and $\beta+\beta'\equiv 0$ we see that

$$2\Big(\frac{f'-\beta'}{f-\beta}-\frac{f''-\beta'}{f'-\beta}\Big)+\frac{f''}{f'}=1.$$

Integrating both sides we obtain

$$\Big(\frac{f'-\beta}{f-\beta}\Big)^2=c\beta f', \tag{3.14}$$

where c is a nonzero constant. Eliminating $(f'-\beta)/(f-\beta)$ between (3.13) and (3.14) leads to

$$\Big(\frac{f''}{f'}+1\Big)^2=4c\beta f'. \tag{3.15}$$

Differentiating both sides of (3.15) with respect to z and then using $\beta+\beta'\equiv 0$ we obtain

$$2\Big(\frac{f''}{f'}+1\Big)\Big(\frac{f''}{f'}\Big)'=4c\beta'f'+4c\beta f''=-4c\beta f'+4c\beta f''=4c\beta f'\Big(\frac{f''}{f'}-1\Big),$$

and eliminating $4c\beta f'$ between this and (3.15) gives

$$2\Big(\frac{f''}{f'}+1\Big)\Big(\frac{f''}{f'}\Big)'=\Big(\frac{f''}{f'}+1\Big)^2\Big(\frac{f''}{f'}-1\Big).$$

Since, by (3.15), $\frac{f''}{f'}+1\not\equiv 0$, therefore

$$2\Big(\frac{f''}{f'}\Big)'=\Big(\frac{f''}{f'}+1\Big)\Big(\frac{f''}{f'}-1\Big)=\Big(\frac{f''}{f'}\Big)^2-1.$$

From this and (2.1) we find that

$$W=\Big(\frac{f''}{f'}\Big)^2-2\Big(\frac{f''}{f'}\Big)'\equiv 1.$$

Together with (2.1) we have $(f''/f')^2-2(f''/f')'-1=0$, which may also be written in the form

$$\frac{(f''/f')'}{(f''/f')-1}-\frac{(f''/f')'}{(f''/f')+1}=1.$$

By integrating once and then using $\beta+\beta'\equiv 0$ we conclude $T(r,f')=S(r,f')$ which is a contradiction.

Case II.2. $H\not\equiv\frac{\beta-\beta'}{\beta^2}$. Then by (3.10) and (3.11) we find that

$$\begin{aligned}N_{1)}\Big(r,\frac{1}{f-\beta}\Big)&\le N\Big(r,\frac{1}{H-\frac{\beta-\beta'}{\beta^2}}\Big)\le T(r,H)+S(r,f)\\&\le \bar N\Big(r,\frac{1}{f'}\Big)+m(r,f)+S(r,f).\end{aligned}$$

Combining this with (3.6) we obtain

$$N\Big(r,\frac{1}{f-\beta}\Big)\le \bar N\Big(r,\frac{1}{f'}\Big)+m(r,f)+S(r,f). \tag{3.16}$$

Since f and f' share β CM, every small function of f is small function of f' and vice versa. Now applying Lemma 2.2 to β' and then using the first fundamental theorem we deduce that

$$m\Big(r,\frac{1}{f'-\beta'}\Big)\le 2\bar N\Big(r,\frac{1}{f'}\Big)+2\bar N_{(2}(r,f)+S(r,f). \tag{3.17}$$

Since, by Milloux theory [5], we find that

$$m\Big(r,\frac{1}{f-\beta}\Big)\le m\Big(r,\frac{f'-\beta'}{f-\beta}\Big)+m\Big(r,\frac{1}{f'-\beta'}\Big)\le m\Big(r,\frac{1}{f'-\beta'}\Big)+S(r,f).$$

From this, (3.17), (3.16) and the first fundamental theorem we see that

$$T(r,f) \le m(r,f) + 3\bar{N}\Big(r, \frac{1}{f'}\Big) + 2\bar{N}_{(2}(r,f) + S(r,f).$$

This implies that

$$N_{1)}(r,f) \le 3\bar{N}\Big(r, \frac{1}{f'}\Big) + S(r,f).$$

Thus we find from this, (1.2) and (3.1) that

$$T(r,f') \le 12\bar{N}\Big(r, \frac{1}{f'}\Big) + S(r,f').$$

This is (1.3) when $k = 1$ and $\beta' \not\equiv 0$.

Case III. $k = 1$ and $\beta' \equiv 0$. Without loss of generality, we suppose $\beta = 1$, otherwise we make the transformation $(1/\beta)f$. As in [3] we set

$$F = 2\Big(\frac{f'}{f-1} - \frac{f''}{f'-1}\Big) + \frac{f''}{f'}. \tag{3.18}$$

From the fundamental estimate of the logarithmic derivative, $m(r,F) = S(r,f)$. If z_∞ is a simple pole of f, then from (3.18) we see that F can be continued holomorphically at z_∞. Since f and f' share 1 CM, we can also find from (3.18) that F can be continued holomorphically at the zeros of $f-1$ or $f'-1$. Thus,

$$T(r,F) = N(r,F) + m(r,F) \le \bar{N}\Big(r, \frac{1}{f'}\Big) + \bar{N}_{(2}(r,f) + S(r,f). \tag{3.19}$$

Let z_2 be a zero of f'' which is not a zero of f'. Since f and f' share 1 CM, $N_{(2}(r, \frac{1}{f-1}) = N_{(2}(r, \frac{1}{f'-1}) = 0$. Hence $f'(z_2) \ne 1$ and $f(z_2) \ne 1$. From (3.18) we see that

$$F(z_2) = \frac{2f'(z_2)}{f(z_2)-1}.$$

Differentiating (3.18) and then using $f''(z_2) = 0$, we arrive at

$$F'(z_2) = -2\Big(\Big(\frac{f'(z_2)}{f(z_2)-1}\Big)^2 + \frac{f'''(z_2)}{f'(z_2)-1}\Big) + \frac{f'''(z_2)}{f'(z_2)}.$$

Combining these two equations we obtain

$$-2F'(z_2) = F^2(z_2) + \frac{4f'''(z_2)}{f'(z_2)-1} - \frac{2f'''(z_2)}{f'(z_2)}.$$

On the other hand, by (2.1) we find that

$$W(z_2) = -\frac{2f'''(z_2)}{f'(z_2)}.$$

Substituting this into the last equation gives

$$f'(z_2)\Big(F^2(z_2) + 2F'(z_2) - W(z_2)\Big) = F^2(z_2) + 2F'(z_2) + W(z_2). \tag{3.20}$$

If $F^2(z_2)+2F'(z_2)-W(z_2)=0$, from (3.20) we get $W(z_2)=0$. If $W\equiv 0$, we may deduce from (2.1) that f and f' can not share β CM which contradicts the condition of Theorem 1.1. Therefore, we have $W\not\equiv 0$. Consequently, from (2.2) and (3.1),

$$\begin{aligned}\bar{N}_0\Big(r,\frac{1}{f''}\Big) &\le N\Big(r,\frac{1}{W}\Big)\le T(r,W)+O(1)\\ &\le 2\bar{N}\Big(r,\frac{1}{f'}\Big)+2\bar{N}_{(2}(r,f)+S(r,f)\\ &\le 6\bar{N}\Big(r,\frac{1}{f'}\Big)+S(r,f). \qquad (3.21)\end{aligned}$$

From Lemma 2.6, (3.1), (3.21) and (1.2) we obtain

$$T(r,f')\le 24\bar{N}\Big(r,\frac{1}{f'}\Big)+S(r,f')\le 36\bar{N}\Big(r,\frac{1}{f'}\Big)+S(r,f').$$

This is (1.3) when $k=1$ and $\beta'\equiv 0$.

In the following, we assume $F^2(z_2)+2F'(z_2)-W(z_2)\neq 0$. We write (3.20) as

$$f'(z_2)=\frac{F^2(z_2)+2F'(z_2)+W(z_2)}{F^2(z_2)+2F'(z_2)-W(z_2)}. \qquad (3.22)$$

Suppose z_1 is a zero of $f-1$. Since f and f' share 1 CM, the Taylor expansion of f about z_1 is

$$f(z)-1=(z-z_1)+a_2(z-z_1)^2+a_3(z-z_1)^3+\ldots,a_2\neq 0.$$

It follows from (3.18) and (2.1) that

$$F(z_1)=4a_2-\frac{3a_3}{a_2}\quad \textit{and}\quad W(z_1)=12(a_2^2-a_3).$$

That is,

$$2{f''}^2(z_1)-F(z_1)f''(z_1)-f'''(z_1)=0\quad \textit{and}\quad 3{f''}^2(z_1)-2f'''(z_1)-W(z_1)=0,$$

and eliminating ${f''}^2(z_1)$ from the last equations we obtain

$$f'''(z_1)-3F(z_1)f''(z_1)+2W(z_1)=0. \qquad (3.23)$$

Now we consider the following function

$$\Omega=\frac{f'''-3Ff''+2Wf'}{f'(f'-1)}. \qquad (3.24)$$

If we now eliminate f''' between (3.24) and (2.1) we obtain

$$2\Omega{f'}^2(f'-1)=3{f''}^2+3W{f'}^2-6Ff'f'', \qquad (3.25)$$

which, in view of (3.22), leads to

$$4\Omega(z_2) = 3\Big(F^2(z_2) + 2F'(z_2) - W(z_2)\Big).$$

If $4\Omega \not\equiv 3(F^2 + 2F' - W)$, then

$$\begin{aligned} \bar{N}_0\Big(r, \frac{1}{f''}\Big) &\le N\Big(r, \frac{1}{4\Omega - 3(F^2 + 2F' - W)}\Big) \\ &\le T(r, 4\Omega - 3(F^2 + 2F' - W)) + O(1) \\ &\le m(r, \Omega) + N\Big(r, 4\Omega - 3(F^2 + 2F' - W)\Big) + S(r, f) \\ &\le m\Big(r, \frac{1}{f' - 1}\Big) + 2\bar{N}\Big(r, \frac{1}{f'}\Big) + 2\bar{N}_{(2}(r, f) + S(r, f). \end{aligned} \tag{3.26}$$

By Lemma 2.2,

$$m\Big(r, \frac{1}{f' - 1}\Big) \le 2\bar{N}\Big(r, \frac{1}{f'}\Big) + 2\bar{N}_{(2}(r, f) + S(r, f).$$

Combining this with (3.26) we find that

$$\bar{N}_0\Big(r, \frac{1}{f''}\Big) \le 4\bar{N}\Big(r, \frac{1}{f'}\Big) + 4\bar{N}_{(2}(r, f) + S(r, f),$$

which together with Lemma 2.6, (3.1) and (1.2) implies that

$$T(r, f') \le 36\bar{N}\Big(r, \frac{1}{f'}\Big) + S(r, f').$$

This is the third part of (1.3).

If $4\Omega \equiv 3(F^2 + 2F' - W)$, then from (3.25) we have

$$(F^2 + 2F' - W)f'^2(f' - 1) = 2f''^2 + 2Wf'^2 - 4Ff'f''. \tag{3.27}$$

Differentiating this and then using $f''(z_2) = 0$, (2.1) and (3.22) we get

$$\Big(\frac{(F^2 + 2F' - W)'}{F^2 + 2F' - W}\Big)(z_2) = \Big(\frac{W'}{W} + F\Big)(z_2).$$

If $\frac{(F^2+2F'-W)'}{F^2+2F'-W} \not\equiv \frac{W'}{W} + F$, then from (2.1), (3.18), (2.2) and ((3.19)) we deduce that

$$\begin{aligned} \bar{N}_0\Big(r, \frac{1}{f''}\Big) &\le N\Big(r, \frac{1}{\frac{(F^2+2F'-W)'}{F^2+2F'-W} - \frac{W'}{W} - F}\Big) \\ &\le T\Big(r, \frac{(F^2 + 2F' - W)'}{F^2 + 2F' - W} - \frac{W'}{W} - F\Big) + O(1) \\ &= N\Big(r, \frac{(F^2 + 2F' - W)'}{F^2 + 2F' - W} - \frac{W'}{W} - F\Big) + S(r, f) \\ &\le \bar{N}\Big(r, \frac{1}{W}\Big) + \bar{N}(r, W) + \bar{N}\Big(r, \frac{1}{F^2 + 2F' - W}\Big) + S(r, f) \\ &\le 2T(r, W) + N(r, F^2 + 2F' - W) + S(r, f) \\ &\le 4\bar{N}\Big(r, \frac{1}{f'}\Big) + 4\bar{N}_{(2}(r, f) + S(r, f). \end{aligned}$$

Together with Lemma 2.6, (1.2) and (3.1) we get

$$T(r,f') \le 36\bar{N}\Big(r,\frac{1}{f'}\Big) + S(r,f').$$

This is (1.3) when $k=1$ and $\beta' \equiv 0$.

If $\frac{(F^2+2F'-W)'}{F^2+2F'-W} \equiv \frac{W'}{W} + F$, then by integrating once,

$$F^2 + 2F' - W = cf'W\Big(\frac{f-1}{f'-1}\Big)^2,$$

from which it follows that $\bar{N}(r,1/f') = 0$ and

$$m\Big(r,\frac{1}{f-1}\Big) \le m\Big(r,\frac{1}{f'-1}\Big) + \bar{N}_{(2}(r,f) + S(r,f).$$

From this and Lemma 2.2 we see that

$$m\Big(r,\frac{1}{f-1}\Big) \le 3\bar{N}_{(2}(r,f) + S(r,f).$$

On the other hand, it is clear that the formulas from (3.2) into (3.16) remain true if we replace β by 1. Thus, we have

$$N\Big(r,\frac{1}{f-1}\Big) \le m(r,f) + S(r,f).$$

Combining these two inequalities we obtain

$$T(r,f) \le m(r,f) + 3\bar{N}_{(2}(r,f) + S(r,f).$$

We may conclude that

$$N_{1)}(r,f) \le \bar{N}_{(2}(r,f) + S(r,f).$$

Together with (1.2) and (3.1) we see that

$$T(r,f') \le 10\bar{N}\Big(r,\frac{1}{f'}\Big) + S(r,f').$$

The proof is complete. □

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