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HOLOMORPH OF GENERALIZED BOL LOOPS

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Abstract. The notions of the holomorph of a generalized Bol loop and generalized flexible-Bol loop are characterized. With the aid of two self-mappings on the holomorph of a loop, it is shown that: the loop is a generalized Bol loop if and only if its holomorph is a generalized Bol loop; the loop is a generalized flexible-Bol loop if and only if its holomorph is a generalized flexible-Bol loop. Furthermore, elements of the Bryant Schneider group of a generalized Bol loop are characterized in terms of pseudo-automorphism, and the automorphisms gotten are used to build the holomorph of the generalized Bol loop.

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1. Introduction

The birth of Bol loops can be traced back to Gerrit Bol [9] in 1937 when he established the relationship between Bol loops and Moufang loops, the latter of which was discovered by Ruth Moufang [26]. Thereafter, a theory of Bol loops evolved through the Ph.D. thesis of Robinson [30] in 1964 where he studied the algebraic properties of Bol loops, Moufang loops and Bruck loops, isotopy of Bol loop and some other notions on Bol loops. Some later results on Bol loops and Bruck loops can be found in [4, 5], [8–11], [13], [33, 34] and [38]

In the 1980s, the study and construction of finite Bol loops caught the attention of many researchers among whom are Burn [13–15], Solarin and Sharma [39–41] and others like Chein and Goodaire [17–19], Foguel at. al. [22], Kinyon and Phillips [24, 25] in the present millennium. One of the most important results in the theory of Bol loops is the solution of the open problem on the existence of a simple Bol loop which was finally laid to rest by Nagy [27–29].

In 1978, Sharma and Sabinin [35, 36] introduced and studied the algebraic properties of the notion of half-Bol loops(left B-loops). Thereafter, Adeniran [2], Adeniran and Akinleye [4], Adeniran and Solarin [6] studied the algebraic properties of generalized Bol loops. Also, Ajmal [7] introduced and studied

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the algebraic properties of generalized Bol loops and their relationship with M-loops (cf. identity (2.9)).

Interestingly, the papers [3], [10], [12], [21], [23], [30, 31] are devoted to study the holomorphs of Bol loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops and Bruck loops.

The Bryant-Schneider group of a loop was introduced by Robinson [32], based on the motivation of [16]. Since the advent of the Bryant-Schneider group, some studies by Adeniran [1] and Chiboka [20] have been done on it relative to CC-loops and extra loops.

The objectives of this present work are to study the structure of the holomorph of a generalized Bol loop and generalized flexible Bol loop, and also to characterize elements of the Bryant-Schneider group of a generalized Bol loop (generalized flexible Bol loop) and use these elements to build the holomorph of a generalized Bol loop (generalized flexible Bol loop).

2. Preliminaries

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If for all $a, b \in L$, the equations:

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho \in L$ and $x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right and left inverse elements of x respectively. Here, $e^\rho \in L$ and $e^\lambda \in L$ satisfy the relations $xe^\rho = x$ and $e^\lambda x = x$ for all $x \in L$ and are respectively called the right and left identity elements. Now, if $e^\rho = e^\lambda = e \in L$, then e is called the identity element and $(L, \cdot)$ is called a loop. In case $x^\lambda = x^\rho$, then, we simply write $x^\lambda = x^\rho = x^{-1} = xJ$ and refer to x^{-1} as the inverse of x.

Let x be an arbitrarily fixed element in a loop $(G, \cdot)$. For any $y \in G$, the left and right translation maps of $x \in G$, L_x and R_x are respectively defined by

$$yL_x = x \cdot y \qquad \text{and} \qquad yR_x = y \cdot x$$

A loop $(L, \cdot)$ is called a (right) Bol loop if it satisfies the identity

$$(xy \cdot z)y = x(yz \cdot y) \tag{2.1}$$

A loop $(L, \cdot)$ is called a left Bol loop if it satisfies the identity

$$y(z \cdot yx) = (y \cdot zy)x \tag{2.2}$$

A loop $(L, \cdot)$ is called a Moufang loop if it satisfies the identity

$$(xy) \cdot (zx) = (x \cdot yz)x \tag{2.3}$$

A loop $(L, \cdot)$ is called a right inverse property loop (RIPL) if $(L, \cdot)$ satisfies right inverse property (RIP)

$$(yx)x^\rho = y \tag{2.4}$$

A loop $(L,\cdot)$ is called a left inverse property loop (LIPL) if $(L,\cdot)$ satisfies left inverse property (LIP)

$$x^{\lambda}(xy) = y \tag{2.5}$$

A loop $(L,\cdot)$ is called an automorphic inverse property loop (AIPL) if $(L,\cdot)$ satisfies automorphic inverse property (AIP)

$$(xy)^{-1} = x^{-1}y^{-1} \tag{2.6}$$

A loop $(L,\cdot)$ in which the mapping $x \mapsto x^2$ is a permutation, is called a Bruck loop if it is both a Bol loop and either AIPL or obeys the identity $xy^2 \cdot x = (yx)^2$. (Robinson [30])

Let $(L,\cdot)$ be a loop with a single valued self-map $\sigma : x \longrightarrow \sigma(x)$:

$(L,\cdot)$ is called a generalized (right) Bol loop or right B-loop if it satisfies the identity

$$(xy \cdot z)\sigma(y) = x(yz \cdot \sigma(y)) \tag{2.7}$$

$(L,\cdot)$ is called a generalized left Bol loop or left B-loop if it satisfies the identity

$$\sigma(y)(z \cdot yx) = (\sigma(y) \cdot zy)x \tag{2.8}$$

$(L,\cdot)$ is called an M-loop if it satisfies the identity

$$(xy) \cdot (z\sigma(x)) = (x \cdot yz)\sigma(x) \tag{2.9}$$

Let $(G,\cdot)$ be a groupoid (quasigroup, loop) and let A, B and C be three bijective mappings, that map G onto G. The triple $\alpha = (A, B, C)$ is called an autotopism of $(G,\cdot)$ if and only if

$$xA \cdot yB = (x \cdot y)C \ \forall\ x, y \in G.$$

Such triples form a group $AUT(G,\cdot)$ called the autotopism group of $(G,\cdot)$.

If $A = B = C$, then A is called an automorphism of the groupoid (quasigroup, loop) $(G,\cdot)$. Such bijections form a group $AUM(G,\cdot)$ called the automorphism group of $(G,\cdot)$.

The right nucleus of $(L,\cdot)$ is defined by $N_{\rho}(L,\cdot) = \{x \in L \mid zy \cdot x = z \cdot yx \ \forall\ y, z \in L\}$.

Definition 2.1. Let $(Q,\cdot)$ be a loop and $A(Q) \leq AUM(Q,\cdot)$ be a group of automorphisms of the loop $(Q,\cdot)$. Let $H = A(Q) \times Q$. Define $\circ$ on H as

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y) \text{ for all } (\alpha, x), (\beta, y) \in H.$$

$(H,\circ)$ is a loop and is called the A-holomorph of $(Q,\cdot)$.

The left and right translations maps of an element $(\alpha, x) \in H$ are respectively denoted by $\mathbb{L}_{(\alpha,x)}$ and $\mathbb{R}_{(\alpha,x)}$.

Remark 2.2. $(H, \circ)$ has a subloop $\{I\} \times Q$ that is isomorphic to $(Q, \cdot)$. As observed in Lemma 6.1 of Robinson [30], given a loop $(Q, \cdot)$ with an A-holomorph $(H, \circ)$, $(H, \circ)$ is a Bol loop if and only if $(Q, \cdot)$ is a θ-generalized Bol loop for all $\theta \in A(Q)$. Also in Theorem 6.1 of Robinson [30], it was shown that $(H, \circ)$ is a Bol loop if and only if $(Q, \cdot)$ is a Bol loop and $x^{-1} \cdot x\theta \in N_\rho(Q, \cdot)$ for all $\theta \in A(Q)$.

Definition 2.3. Let $(Q, \cdot)$ be a loop with a single valued self-map σ and let $(H, \circ)$ be the A-holomorph of $(Q, \cdot)$ with single valued self-map σ'. $(Q, \cdot)$ is called a σ-flexible loop (σ-flexible) if

$$xy \cdot \sigma(x\delta) = x \cdot y\sigma(x\delta) \text{ for all } x, y \in Q \text{ and some } \delta \in A(Q).$$

$(H, \circ)$ is called a σ'-flexible loop (σ'-flexible) if

$$(\alpha, x)(\beta, y) \circ \sigma'(\alpha, x) = (\alpha, x) \circ (\beta, y)\sigma'(\alpha, x) \text{ for all } (\alpha, x), (\beta, y) \in H.$$

If a loop is both a σ-generalized Bol loop and a σ-flexible loop, then it is called a σ-generalized flexible-Bol loop.

If in this triple $(A, B, C) \in AUT(G, \cdot)$, $B = C = AR_c$, then A is called a pseudo-automorphism of a quasigroup $(G, \cdot)$ with companion $c \in G$. Such bijections form a group $PS(G, \cdot)$ called the pseudo-automorphism group of $(G, \cdot)$.

Definition 2.4. [Robinson [32]]
Let $(G, \cdot)$ be a loop with symmetric group $SYM(G)$. A mapping $\theta \in SYM(G)$ is called a special map for G if there exist $f, g \in G$ so that
$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$.

Theorem 2.5. *[Robinson [32]]*
Let $(G, \cdot)$ be a loop with symmetric group $SYM(G)$. The set of all special maps in $(G, \cdot)$ i.e.

$$BS(G, \cdot) = \{\theta \in SYM(G, \cdot) \ : \ \exists\ f, g \in G\ (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)\}$$

is a subgroup of $SYM(G)$ and is called the Bryant-Schneider group of the loop $(G, \cdot)$.

Some existing results on generalized Bol loops and generalized Moufang loops are highlighted below.

Theorem 2.6. *[Adeniran and Akinleye [4]]*
If $(L, \cdot)$ is a generalized Bol loop, then:

1. *$(L, \cdot)$ is a RIPL.*
2. *$x^\lambda = x^\rho$ for all $x \in L$.*
3. *$R_{y \cdot \sigma(y)} = R_y R_{\sigma(y)}$ for all $y \in L$.*
4. *$[xy \cdot \sigma(x)]^{-1} = (\sigma(x))^{-1} y^{-1} \cdot x^{-1}$ for all $x, y \in L$.*

5. $(R_{y^{-1}}, L_y R_{\sigma(y)}, R_{\sigma(y)}), (R_y^{-1}, L_y R_{\sigma(y)}, R_{\sigma(y)}) \in AUT(L,\cdot)$ *for all* $y \in L$.

Theorem 2.7. *[Sharma and Sabinin [35]]*
If $(L,\cdot)$ *is a half Bol loop, then:*

1. $(L,\cdot)$ *is a LIPL.*
2. $x^{\lambda} = x^{\rho}$ *for all* $x \in L$.
3. $L_{(x)} L_{(\sigma(x))} = L_{(\sigma(x)x)}$ *for all* $x \in L$.
4. $(\sigma(x) \cdot yx)^{-1} = x^{-1} \cdot y^{-1}(\sigma(x))^{-1}$ *for all* $x, y \in L$.
5. $(R_{(x)} L_{(\sigma(x))}, L_{(x)^{-1}}, L_{(\sigma(x))}), (R_{(\sigma(x))} L_{(x)^{-1}}, L_{\sigma(x)}, L_{(x)^{-1}}) \in AUT(L,\cdot)$ *for all* $x \in L$.

Theorem 2.8. *[Ajmal [7]]*
Let $(L,\cdot)$ *be a loop. The following statements are equivalent:*

1. $(L,\cdot)$ *is a M-loop;*
2. $(L,\cdot)$ *is both a left B-loop and a right B-loop;*
3. $(L,\cdot)$ *is a right B-loop and satisfies the LIP;*
4. $(L,\cdot)$ *is a left B-loop and satisfies the RIP.*

Theorem 2.9. *[Ajmal [7]]*
Every isotope of a right B-loop with the LIP is a right B-loop.

Example 2.10. Let R be a ring of all 2×2 matrices taken over the field of three elements and let $G = R \times R$. For all $(u,f),(v,g) \in G$, define $(u,f)\cdot(v,g) = (u+v, f+g+uv^3)$. Then $(G,\cdot)$ is a loop which is not a right Bol loop but which is a σ-generalized Bol loop with $\sigma\ : x \mapsto x^2$.

We introduce the notions defined below for the first time.

Definition 2.11. [Twin Special Mappings]
Let $(G,\cdot)$ be a loop and let $\alpha,\beta \in SYM(G)$ such that $\alpha = \psi R_x, \beta = \psi R_y$, for some $x, y \in G$ and $\psi \in SYM(G)$. Then α and β are called twin special maps (twins). α (or β) is called a twin map (twin) of β (or α) or simply a twin map.

Let $(Q,\cdot)$ be a loop. Define

$$TBS_1(Q,\cdot) = \{\alpha \in SYM(Q) \mid \alpha \text{ is any twin map}\},$$
$$T_1(Q,\cdot) = T_1(Q) = \{\psi \in SYM(Q) \mid \alpha = \psi R_x \in TBS_1(Q,\cdot),\ x \in Q,\ \psi : e \mapsto e\},$$
$$TBS_2(Q,\cdot) = \{\alpha \in BS(Q,\cdot) \mid \alpha \in TBS_1(Q,\cdot)\},$$
$$T_2(Q,\cdot) = T_2(Q) = \{\psi \in SYM(Q) \mid \alpha = \psi R_x \in TBS_2(Q,\cdot),\ x \in Q,\ \psi : e \mapsto e\}$$
and
$$T_3(Q,\cdot) = T_3(Q) = \{\psi \in T_2(Q) \mid \alpha^{-1} \sim \beta^{-1} \text{ for any twin maps } \alpha,\beta \in SYM(Q)\}.$$

Define a relation $\sim$ on $SYM(Q)$ as $\alpha \sim \beta$ if there exists $x \in Q$ such that $\alpha^{-1} = R_x \beta^{-1}$.

The following results will be of judicious use to prove our main results.

Lemma 2.12. *[Bruck [10]]*
$(H,\circ)$ is a RIPL if and only if $(Q,\cdot)$ is a RIPL.

Lemma 2.13. *[Adeniran [2]]*
$(Q,\cdot)$ is a σ-generalized Bol loop if and only if $(R_x^{-1}, L_xR_{\sigma(x)}, R_{\sigma(x)}) \in AUT(Q,\cdot)$ for all $x\in Q$.

Lemma 2.14. *[Bruck [11]]*
Let $(Q,\cdot)$ be a RIPL. If $(U,V,W)\in AUT(Q,\cdot)$, then $(W, JVJ, U)\in AUT(Q,\cdot)$.

3. Main results

Theorem 3.1. *Let $(Q,\cdot)$ be a loop with a self-map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self-map σ' such that $\sigma' : (\alpha,x)\mapsto(\alpha,\sigma(x))$ for all $(\alpha,x)\in H$. The A-holomorph $(H,\circ)$ of $(Q,\cdot)$ is a σ'-generalized Bol loop if and only if $\Big(R_x^{-1}, L_xR_{[\sigma(x\gamma^{-1})]\alpha^{-1}}, R_{[\sigma(x\gamma^{-1})]\alpha^{-1}}\Big)\in AUT(Q,\cdot)$ for all $x\in Q$ and all $\alpha,\gamma\in A(Q)$.*

Proof. Define $\sigma' : H\to H$ as $\sigma'(\alpha,x)=(\alpha,\sigma(x))$. Let $(\alpha,x),(\beta,y),(\gamma,z)\in H$, then by Lemma 2.12 and Lemma 2.13, $(H,\circ)$ is a σ'-generalized Bol loop if and only if $(\mathbb{R}_{(\alpha,x)^{-1}}, \mathbb{L}_{(\alpha,x)}\mathbb{R}_{\sigma'(\alpha,x)}, \mathbb{R}_{\sigma'(\alpha,x)})\in AUT(H,\circ)$ for all $(\alpha,x)\in H$ if and only if $(\mathbb{R}_{(\alpha,x)^{-1}}, \mathbb{L}_{(\alpha,x)}\mathbb{R}_{(\alpha,\sigma(x))}, \mathbb{R}_{(\alpha,\sigma(x))})\in AUT(H,\circ) \Longleftrightarrow$

$$(\beta,y)\mathbb{R}_{(\alpha,x)^{-1}}\circ(\gamma,z)\mathbb{L}_{(\alpha,x)}\mathbb{R}_{(\alpha,\sigma(x))} = [(\beta,y)\circ(\gamma,z)]\mathbb{R}_{(\alpha,\sigma(x))} \tag{3.1}$$

(3.2)
$$\Leftrightarrow [(\beta,y)\circ(\alpha,x)^{-1}]\circ[((\alpha,x)\circ(\gamma,z))\circ(\alpha,\sigma(x))] = [(\beta,y)\circ(\gamma,z)]\circ(\alpha,\sigma(x))$$

Let $(\beta,y)\circ(\alpha,x)^{-1}=(\tau,t)$. Since $(\alpha,x)^{-1}=(\alpha^{-1},(x\alpha^{-1})^{-1})$, then

$$(\tau,t)=(\beta\alpha^{-1},(yx^{-1})\alpha^{-1}) \tag{3.3}$$

From (3.3),

$$(\tau,t)\circ[(\alpha\gamma, x\gamma\cdot z)\circ(\alpha,\sigma(x))] = (\beta\gamma, y\gamma\cdot z)\circ(\alpha,\sigma(x)) \tag{3.4}$$

$$\Leftrightarrow \Big(\tau\alpha\gamma\alpha, (t\alpha\gamma\alpha)\big((x\gamma\cdot z)\alpha\cdot\sigma(x)\big)\Big) = \big(\beta\gamma\alpha, (y\gamma\cdot z)\alpha\cdot\sigma(x)\big) \tag{3.5}$$

Putting (3.3) into (3.5), we have

(3.6)
$$\Big(\beta\alpha^{-1}\alpha\gamma\alpha, (yx^{-1})\alpha^{-1}(\alpha\gamma\alpha)\big((x\gamma\cdot z)\alpha\cdot\sigma(x)\big)\Big) = \big(\beta\gamma\alpha, (y\gamma\cdot z)\alpha\cdot\sigma(x)\big)$$

$$\Leftrightarrow \Big(\beta\gamma\alpha, (yx^{-1})\gamma\alpha\big[(x\gamma\cdot z)\alpha\cdot\sigma(x)\big]\Big) = \big(\beta\gamma\alpha, (y\gamma\cdot z)\alpha\cdot\sigma(x)\big) \tag{3.7}$$

$$\Leftrightarrow (yx^{-1})\gamma\alpha\cdot[(x\gamma\cdot z)\alpha\cdot\sigma(x)] = (y\gamma\cdot z)\alpha\cdot\sigma(x) \tag{3.8}$$

$$\Leftrightarrow \big[(yx^{-1})\gamma\cdot[(x\gamma\cdot z)\cdot(\sigma(x)\alpha^{-1})]\big]\alpha = [(y\gamma\cdot z)\cdot(\sigma(x)\alpha^{-1})]\alpha \tag{3.9}$$

$$\Leftrightarrow (y\gamma x^{-1}\gamma)[(x\gamma\cdot z)\cdot(\sigma(x)\alpha^{-1})] = (y\gamma\cdot z)(\sigma(x)\alpha^{-1}) \tag{3.10}$$

Let $\bar{y} = y\gamma$, then (3.10) becomes

$$(\bar{y} \cdot x^{-1}\gamma)[(x\gamma \cdot z)(\sigma(x)\alpha^{-1})] = (\bar{y} \cdot z)(\sigma(x)\alpha^{-1}) \tag{3.11}$$

$$\Leftrightarrow \left(R_{x\gamma}^{-1}, L_{x\gamma}R_{[\sigma(x)\alpha^{-1}]}, R_{[\sigma(x)\alpha^{-1}]}\right) \in AUT(Q, \cdot) \tag{3.12}$$

(3.13)
and replacing $x\gamma$ by x, $\left(R_x^{-1}, L_xR_{[\sigma(x\gamma^{-1})]\alpha^{-1}}, R_{[\sigma(x\gamma^{-1})]\alpha^{-1}}\right) \in AUT(Q, \cdot)$.

□

Corollary 3.2. *Let $(Q, \cdot)$ be a loop with a self-map σ and let $(H, \circ)$ be the A-holomorph of $(Q, \cdot)$ with a self-map σ' such that $\sigma' : (\alpha, x) \mapsto (\alpha, \sigma(x))$ for all $(\alpha, x) \in H$. $(H, \circ)$ is a σ'-generalized Bol loop if and only if $(Q, \cdot)$ is a $\alpha^{-1}\sigma\gamma^{-1}$-generalized Bol loop for any $\alpha, \gamma \in A(Q)$.*

Proof. From Theorem 3.1, $(H, \circ)$ is a σ'-generalized Bol loop if and only if

$$\left(R_x^{-1}, L_xR_{[\sigma(x\gamma^{-1})]\alpha^{-1}}, R_{[\sigma(x\gamma^{-1})]\alpha^{-1}}\right) \in AUT(Q, \cdot) \Leftrightarrow$$
$$\left(R_x^{-1}, L_xR_{\sigma''(x)}, R_{\sigma''(x)}\right) \in AUT(Q, \cdot)$$

where $\sigma'' = \alpha^{-1}\sigma\gamma^{-1}$, for all $x \in Q$ and all $\alpha, \gamma \in A(Q)$. It is equivalent to the fact that $(Q, \cdot)$ is a σ''-generalized Bol loop. □

Theorem 3.3. *Let $(Q, \cdot)$ be a loop with a self-map σ and let $(H, \circ)$ be the holomorph of $(Q, \cdot)$ with a self-map σ' such that $\sigma' : (\alpha, x) \mapsto (\alpha, \alpha\sigma\gamma(x))$ for all $(\alpha, x) \in H$. Then $(Q, \cdot)$ is a σ-generalized Bol loop if and only if $(H, \circ)$ is a σ'-generalized Bol loop.*

Proof. The proof of this follows from the proof of Theorem 3.1. □

Theorem 3.4. *Let $(Q, \cdot)$ be a loop with a self-map σ and let $(H, \circ)$ be the holomorph of $(Q, \cdot)$ with a self-map σ' such that $\sigma' : (\alpha, x) \mapsto (\alpha, \sigma(x))$ for all $(\alpha, x) \in H$. Then for any $\gamma \in A(Q)$, $(Q, \cdot)$ is a $\sigma\alpha\gamma^{-1}$-generalized flexible-Bol loop if and only if $(H, \circ)$ is a σ'-generalized flexible-Bol loop.*

Proof.

$$(R_x^{-1}, L_xR_{\sigma(x)}, R_{\sigma(x)})^{-1} = (R_x, L_x^{-1}R_{\sigma(x)}^{-1}, R_{\sigma(x)}^{-1}) \tag{3.14}$$

$$\Leftrightarrow L_x^{-1}R_{\sigma(x)}^{-1} = (L_xR_{\sigma(x)})^{-1} \Leftrightarrow R_{\sigma(x)}L_x = L_xR_{\sigma(x)} \tag{3.15}$$

$$\Leftrightarrow xy \cdot \sigma(x) = x \cdot y\sigma(x) \tag{3.16}$$

Let $(\alpha, x), (\beta, y), (\gamma, z) \in H$, then by Lemma 2.12 and Lemma 2.13, $(H, \circ)$ is a σ'-generalized Bol loop if and only if $(\mathbb{R}_{(\alpha,x)^{-1}}, \mathbb{L}_{(\alpha,x)}\mathbb{R}_{\sigma'(\alpha,x)}, \mathbb{R}_{\sigma'(\alpha,x)}) \in$

$AUT(H,\circ)$ for all $(\alpha,x)\in H$. Thus, following (3.14) to (3.16),

$$(\mathbb{R}^{-1}_{(\alpha,x)},\mathbb{L}_{(\alpha,x)}\mathbb{R}_{\sigma'(\alpha,x)},\mathbb{R}_{\sigma'(\alpha,x)})^{-1}=(\mathbb{R}_{(\alpha,x)},\mathbb{L}^{-1}_{(\alpha,x)}\mathbb{R}^{-1}_{\sigma'(\alpha,x)},\mathbb{R}^{-1}_{\sigma'(\alpha,x)}) \tag{3.17}$$

$$\Leftrightarrow \mathbb{L}_{(\alpha,x)}\mathbb{R}_{\sigma'(\alpha,x)}=\mathbb{R}_{\sigma'(\alpha,x)}\mathbb{L}_{(\alpha,x)} \tag{3.18}$$

$$\Leftrightarrow (\alpha,x)(\beta,y)\circ\sigma'(\alpha,x)=(\alpha,x)\circ(\beta,y)\sigma'(\alpha,x) \tag{3.19}$$

$$\Leftrightarrow (\alpha,x)(\beta,y)\circ(\alpha,\sigma x)=(\alpha,x)\circ(\beta,y)(\alpha,\sigma x) \tag{3.20}$$

$$\Leftrightarrow \big(\alpha\beta\alpha,(x\beta\cdot y)\alpha\cdot\sigma(x)\big)=\big(\alpha\beta\alpha,x\beta\alpha\cdot(y\alpha\cdot\sigma(x))\big) \tag{3.21}$$

$$\Leftrightarrow (x\beta\alpha\cdot y\alpha)\sigma(x)=x\beta\alpha\cdot(y\alpha\cdot\sigma(x)) \tag{3.22}$$

$$\Leftrightarrow (x\gamma^{-1}\alpha\beta\alpha\cdot y)\sigma\big(x\gamma^{-1}\alpha\big)=x\gamma^{-1}\alpha\beta\alpha\cdot\big(y\cdot\sigma\big(x\gamma^{-1}\alpha\big)\big) \tag{3.23}$$

$$\Leftrightarrow (x\gamma^{-1}\alpha\beta\alpha\cdot y)\sigma\alpha\gamma^{-1}(x)=x\gamma^{-1}\alpha\beta\alpha\cdot(y\cdot\sigma\alpha\gamma^{-1}(x)) \tag{3.24}$$

$$\Leftrightarrow (xy)\sigma\alpha\gamma^{-1}\big((x(\alpha\beta\alpha)^{-1}\gamma\big)=x\cdot(y\cdot\sigma\alpha\gamma^{-1}\big((x(\alpha\beta\alpha)^{-1}\gamma\big) \tag{3.25}$$

$$\Leftrightarrow (xy)\sigma\alpha\gamma^{-1}(x\delta)=x\cdot(y\cdot\sigma\alpha\gamma^{-1}(x\delta) \tag{3.26}$$

where $\delta=(\alpha\beta\alpha)^{-1}\gamma\in A(Q)$. So, by (3.19) to (3.26), $(H,\circ)$ is σ'-flexible if and only if $(Q,\cdot)$ is $\sigma\alpha\gamma^{-1}$-flexible.

Now, following (3.17), $(H,\circ)$ is a σ'-generalized Bol loop if and only if

$$(\mathbb{R}_{(\alpha,x)},\mathbb{L}^{-1}_{(\alpha,x)}\mathbb{R}^{-1}_{\sigma'(\alpha,x)},\mathbb{R}^{-1}_{\sigma'(\alpha,x)})\in AUT(Q,\cdot) \tag{3.27}$$

$$\Leftrightarrow (\beta,y)\mathbb{R}_{(\alpha,x)}\circ(\gamma,z)\mathbb{L}^{-1}_{(\alpha,x)}\mathbb{R}^{-1}_{\sigma'(\alpha,x)}=[(\beta,y)\circ(\gamma,z)]\mathbb{R}^{-1}_{\sigma'(\alpha,x)} \tag{3.28}$$

Let $(\gamma,z)\mathbb{L}^{-1}_{(\alpha,x)}\mathbb{R}^{-1}_{\sigma'(\alpha,x)}=(\mu,u)$ in (3.28), then $(\gamma,z)=\big(\alpha\mu\alpha,x\mu\alpha(u\alpha\cdot\sigma(x))\big)\Rightarrow$ $\gamma=\alpha\mu\alpha$ and $z=x\mu\alpha(u\alpha\cdot\sigma(x))$. Consequently,

$$\mu=\alpha^{-1}\gamma\alpha^{-1}\text{ and }u=zL^{-1}_{(x\alpha^{-1}\gamma)}R_{(\sigma(x))^{-1}}\alpha^{-1}=\Big[\big((x\alpha^{-1}\gamma)\backslash z\big)(\sigma(x))^{-1}\Big]\alpha^{-1} \tag{3.29}$$

Also, if $[(\beta,y)\circ(\gamma,z)]\mathbb{R}^{-1}_{\sigma'(\alpha,x)}=(\beta\gamma,y\gamma\cdot z)\mathbb{R}_{(\alpha,\sigma(x))^{-1}}=(\tau,v)$ in (3.28), then

$$(\tau,v)=\Big(\beta\gamma\alpha^{-1},(y\gamma\cdot z)\alpha^{-1}\cdot\big((\sigma(x))^{-1}\big)\alpha^{-1}\Big) \tag{3.30}$$

Substituting (3.29) and (3.30) into (3.28), we get

$$[(\beta,y)\circ(\alpha,x)]\circ(\mu,u)=(\tau,v)\Leftrightarrow(\beta\alpha\mu,(y\alpha\cdot x)\mu\cdot u)=(\tau,v) \tag{3.31}$$

$$\Leftrightarrow\Big(\beta\gamma\alpha^{-1},(y\alpha\cdot x)\alpha^{-1}\gamma\alpha^{-1}\cdot\Big([(x\alpha^{-1}\gamma)\backslash z](\sigma(x))^{-1}\Big)\alpha^{-1}\Big) \tag{3.32}$$

$$=\Big(\beta\gamma\alpha^{-1},(y\gamma\cdot z)\alpha^{-1}\cdot\alpha^{-1}\big((\sigma(x))^{-1}\big)\Big) \tag{3.33}$$

$$\Leftrightarrow\Big\{(y\alpha\cdot x)\alpha^{-1}\gamma\cdot\Big([(x\alpha^{-1}\gamma)\backslash z](\sigma(x))^{-1}\Big)\Big\}\alpha^{-1}=\Big[(y\gamma\cdot z)(\sigma(x))^{-1}\Big]\alpha^{-1} \tag{3.34}$$

$$\Leftrightarrow (y\alpha\cdot x)\alpha^{-1}\gamma\cdot\Big([(x\alpha^{-1}\gamma)\backslash z](\sigma(x))^{-1}\Big)=(y\gamma\cdot z)(\sigma(x))^{-1} \tag{3.35}$$

$$\Leftrightarrow (y\gamma\cdot x\alpha^{-1}\gamma)\cdot\Big([(x\alpha^{-1}\gamma)\backslash z](\sigma(x))^{-1}\Big)=(y\gamma\cdot z)(\sigma(x))^{-1} \tag{3.36}$$

$$\Leftrightarrow \bar{y}R_{\bar{x}}\cdot zL^{-1}_{\bar{x}}R^{-1}_{[\sigma(\bar{x}\gamma^{-1}\alpha)]}=(\bar{y}z)R^{-1}_{[\sigma(\bar{x}\gamma^{-1}\alpha)]} \tag{3.37}$$

$$\Leftrightarrow\Big(R_{\bar{x}},L^{-1}_{\bar{x}}R^{-1}_{[\sigma(\bar{x}\gamma^{-1}\alpha)]},R^{-1}_{[\sigma(\bar{x}\gamma^{-1}\alpha)]}\Big)\in AUT(Q,\cdot) \tag{3.38}$$

where $\bar{y} = y\gamma$ and $\bar{x} = x\alpha^{-1}\gamma$. Based on (3.26) and the reverse of the procedure from (3.14) to (3.16), (3.38) is true if and only if $(Q, \cdot)$ is a $\sigma\alpha\gamma^{-1}$-generalized Bol loop.

$\therefore$ $(Q, \cdot)$ is a $\sigma\alpha\gamma^{-1}$-generalized flexible-Bol loop if and only if $(H, \circ)$ is a σ'-generalized flexible-Bol loop. □

Theorem 3.5. *Let $(Q, \cdot)$ be a loop with a self-map σ and let $(H, \circ)$ be the holomorph of $(Q, \cdot)$ with a self-map σ' such that $\sigma' : (\alpha, x) \mapsto (\alpha, \sigma\gamma\alpha^{-1}(x))$ for all $(\alpha, x) \in H$. Then for any $\gamma \in A(Q)$, $(Q, \cdot)$ is a σ-generalized flexible-Bol loop if and only if $(H, \circ)$ is a σ'-generalized flexible-Bol loop.*

Proof. The proof of this follows in the sense of Theorem 3.4. □

Theorem 3.6. *Let $(Q, \cdot)$ be a generalized Bol loop. If a mapping $\alpha \in BS(Q, \cdot)$ such that $\alpha = \psi R_x$, where $\psi : e \mapsto e$, then ψ is a unique pseudo-automorphism with companion $xg^{-1} \cdot \sigma(x)$ for some $g \in Q$ and for all $x \in Q$.*

Proof. If $\alpha \in BS(Q, \cdot)$, then $(\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \in AUT(Q, \cdot)$ for some $f, g \in Q$. So, applying Lemma 2.14, $(\alpha, J\alpha L_f^{-1}J, \alpha R_{g^{-1}}) \in AUT(Q, \cdot)$ for some $f, g \in Q$. Since, $(R_{x^{-1}}, L_x R_{\sigma(x)}, R_{\sigma(x)}) \in AUT(Q, \cdot)$ for all $x \in Q$, then

$$(\alpha, J\alpha L_f^{-1}J, \alpha R_{g^{-1}})(R_x^{-1}, L_x R_{\sigma(x)}, R_{\sigma(x)}) = \tag{3.39}$$

$$(\alpha R_{x^{-1}}, J\alpha L_f^{-1}JL_x R_{\sigma(x)}, \alpha R_{g^{-1}} R_{\sigma(x)}) \in AUT(Q, \cdot) \tag{3.40}$$

Let $\theta = J\alpha L_f^{-1}JL_x R_{\sigma(x)}$. Then (3.40) becomes

$$u\alpha R_{x^{-1}} \cdot v\theta = (u \cdot v)\alpha R_{g^{-1}} R_{\sigma(x)} \tag{3.41}$$

for all $u, v \in Q$. If $\alpha = \psi R_x$, then $\alpha R_x^{-1} = \psi$. Thus, $\theta = J\psi R_x L_f^{-1} JL_x R_{\sigma(x)}$ and (3.41) becomes

$$u\psi \cdot v\theta = (u \cdot v)\psi R_x R_{g^{-1}} R_{\sigma(x)} \tag{3.42}$$

Let $u = e$ in (3.41), then we have $e\psi \cdot v\theta = (e \cdot v)\psi R_x R_{g^{-1}} R_{\sigma(x)} \Longrightarrow$

$$\theta = \psi R_x R_{g^{-1}} R_{\sigma(x)} \tag{3.43}$$

So by (3.42) and (3.41), (3.40) becomes

$$(\psi, \theta, \psi R_x R_{g^{-1}} R_{\sigma(x)}) = \langle \psi, \psi R_x R_{g^{-1}} R_{\sigma(x)}, \psi R_x R_{g^{-1}} R_{\sigma(x)}) \in AUT(Q, \cdot)$$

for all $x \in Q$ and some $g \in Q$. Since $(Q, \cdot)$ is a generalized Bol loop,
$R_x R_{g^{-1}} R_{\sigma(x)} = R_{xg^{-1} \cdot \sigma(x)}$. Hence,

$$(\psi, \psi R_{xg^{-1} \cdot \sigma(x)}, \psi R_{xg^{-1} \cdot \sigma(x)}) \in AUT(Q, \cdot). \tag{3.44}$$

for all $x \in Q$ and some $g \in Q$. Thus, ψ is a pseudo-automorphism with a companion $xg^{-1}\sigma(x)$.

Let $\psi_1 R_{x_1} = \psi_2 R_{x_2}$ where $\psi_1, \psi_2 : e \mapsto e$ and $x_1, x_2 \in Q$. Then $R_{x_1} R_{x_2}^{-1} = \psi_1^{-1}\psi_2$. So, $eR_{x_1} R_{x_2}^{-1} = e\psi_1^{-1}\psi_2$, thus, $x_1 x_2^{-1} = e$. Hence $x_1 = x_2$, so $\psi_1 = \psi_2$. And this implies that for all $x \in Q$, there exists a unique ψ such that $\alpha = \psi R_x$. Therefore, $\alpha = \psi R_x$ if and only if $\psi \in PS(Q, \cdot)$ with companion $xg^{-1} \cdot \sigma(x)$ for some $g \in Q$ and all $x \in Q$. □

Corollary 3.7. *Let $(Q,\cdot)$ be a σ-generalized Bol loop with $\sigma : x \mapsto (xg^{-1})^{-1}$ for all $x \in Q$ and some $g \in Q$. If a mapping $\alpha \in BS(Q,\cdot)$ is such that $\alpha = \psi R_x$, where $\psi : e \mapsto e$, then $\psi \in AUM(Q,\cdot)$ is unique.*

Proof. Using (3.44),
$(\psi, \psi R_{xg^{-1}\cdot\sigma(x)}, \psi R_{xg^{-1}\cdot\sigma(x)}) = (\psi, \psi R_{xg^{-1}\cdot(xg^{-1})^{-1}}, \psi R_{xg^{-1}\cdot(xg^{-1})^{-1}}) =$
$(\psi,\psi,\psi) \in AUT(Q,\cdot)$. Thus, ψ is an automorphism of Q. □

Theorem 3.8. *Let $(Q,\cdot)$ be a σ-generalized Bol loop in which $\sigma\big(x^{-1}\big) = (\sigma(x))^{-1}$ and $xy\cdot\sigma(x) = x\cdot y\sigma(x)$ for all $x, y \in Q$. If a mapping $\alpha \in BS(Q,\cdot)$ is such that $\alpha = \psi R_x^{-1}$, where $\psi : e \mapsto e$, then ψ is a unique pseudo-automorphism with companion $x^{-1}g^{-1}\cdot(\sigma(x))^{-1}$ for some $g \in Q$ and for all $x \in Q$.*

Proof. By Lemma 2.12 and Lemma 2.13, $(Q,\cdot)$ is a generalized Bol loop if and only if $(R_{x^{-1}}, L_x R_{\sigma(x)}, R_{\sigma(x)}) \in AUT(Q,\cdot)$ for all $x \in Q$. Since $xy\cdot\sigma(x) = x\cdot y\sigma(x)$, then $(R_{x^{-1}}, L_x R_{\sigma(x)}, R_{\sigma(x)})^{-1} = (R_x, L_x^{-1}R_{(\sigma(x))^{-1}}, R_{(\sigma(x))^{-1}}) \in AUT(Q,\cdot)$. $\alpha \in BS(Q,\cdot) \Longleftrightarrow (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \in AUT(Q,\cdot) \Longrightarrow (\alpha, J\alpha L_{f^{-1}}J, \alpha R_g^{-1}) \in AUT(Q,\cdot)$ for some $g, f \in Q$ by Lemma 2.14. Now, the product

$$(\alpha, J\alpha L_f^{-1}J, \alpha R_g^{-1})(R_x, L_x^{-1}R_{(\sigma(x))^{-1}}, R_{(\sigma(x))^{-1}}) = \tag{3.45}$$

$$(\alpha R_x, J\alpha L_f^{-1}JL_x^{-1}R_{(\sigma(x))^{-1}}, \alpha R_g^{-1}R_{(\sigma(x))^{-1}}) \in AUT(Q,\cdot) \tag{3.46}$$

for all $x \in Q$ and some $g, f \in Q$. Substituting $\alpha = \psi R_x^{-1}$ into (3.46), we have

$$(\psi, J\psi R_x^{-1}L_f^{-1}JL_x^{-1}R_{(\sigma(x))^{-1}}, \psi R_x^{-1}R_g^{-1}R_{(\sigma(x))^{-1}}) \in AUT(Q,\cdot) \tag{3.47}$$

for all $x \in Q$ and some $g \in Q$. Now, for all $y, z \in Q$

$$y\psi \cdot zJ\psi R_x^{-1}L_f^{-1}JL_x^{-1}R_{(\sigma(x))^{-1}} = (yz)\psi R_x^{-1}R_g^{-1}R_{(\sigma(x))^{-1}} \tag{3.48}$$

Putting $y = e$ in (3.48), we have

$$J\psi R_x^{-1}L_f^{-1}JL_x^{-1}R_{(\sigma(x))^{-1}} = \psi R_x^{-1}R_g^{-1}R_{(\sigma(x))^{-1}} \tag{3.49}$$

for all $x \in Q$ and some $g \in Q$. Thus, using (3.49) in (3.48),

$$(\psi, \psi R_x^{-1}R_g^{-1}R_{(\sigma(x))^{-1}}, \psi R_x^{-1}R_g^{-1}R_{(\sigma(x))^{-1}}) \in AUT(Q,\cdot) \tag{3.50}$$

for all $x \in Q$ and some $g \in Q$.

Since $(Q,\cdot)$ is a generalized Bol loop,

$$R_{x^{-1}}R_{g^{-1}}R_{(\sigma(x))^{-1}} = R_{x^{-1}g^{-1}\cdot(\sigma(x))^{-1}}.$$

Hence,

$$\big(\psi, \psi R_{x^{-1}g^{-1}\cdot(\sigma(x))^{-1}}, \psi R_{x^{-1}g^{-1}\cdot(\sigma(x))^{-1}}\big) \in AUT(Q,\cdot). \tag{3.51}$$

The proof the uniqueness of ψ is similar to that in Theorem 3.6. Therefore, ψ is a unique pseudo-automorphism of $(Q,\cdot)$ with companion $x^{-1}g^{-1}\cdot(\sigma(x))^{-1}$. □

Corollary 3.9. *Let $(Q,\cdot)$ be a σ-generalized Bol loop and an AIPL in which $\sigma\big(x^{-1}\big)=(\sigma(x))^{-1}$ and $xy\cdot\sigma(x)=x\cdot y\sigma(x)$ for all $x,y\in Q$ where $\sigma\ :\ x\mapsto(xg)^{-1}$ for all $x\in Q$ and some $g\in Q$. If a mapping $\alpha\in BS(Q,\cdot)$ is such that $\alpha=\psi R_x^{-1}$, where $\psi:e\mapsto e$, then $\psi\in AUM(Q,\cdot)$ is unique.*

Proof. Using (3.51),

$$\big(\psi,\psi R_{x^{-1}g^{-1}\cdot(\sigma(x))^{-1}},\psi R_{x^{-1}g^{-1}\cdot(\sigma(x))^{-1}}\big)=$$
$$\big(\psi,\psi R_{x^{-1}g^{-1}\cdot((xg)^{-1})^{-1}},\psi R_{x^{-1}g^{-1}\cdot((xg)^{-1})^{-1}}\big)=(\psi,\psi,\psi)\in AUT(Q,\cdot).$$

Thus, ψ is an automorphism of Q. □

Lemma 3.10. *Let $(Q,\cdot)$ be a σ-generalized Bol loop. Then*

1. *$\sim$ is an equivalence relation over $SYM(Q)$.*
2. *For any $\alpha,\beta\in SYM(Q)$, $\alpha\sim\beta$ if and only if $\alpha,\beta\in TBS_1(Q,\cdot)$.*
3. *$TBS_1(Q,\cdot)=\displaystyle\bigcup_{[\alpha]\in SYM(Q)/\sim}[\alpha]$.*

Proof.

1. Let $\alpha,\beta,\gamma\in SYM(Q)$. With $x=e$, $\alpha^{-1}=R_e\alpha^{-1}$ and so $\alpha\sim\alpha$. Thus, $\sim$ is reflexive. Let $\alpha\sim\beta$, then there exists $x\in Q$ such that $\alpha^{-1}=R_x\beta^{-1}\Longrightarrow\beta^{-1}=R_{x^{-1}}\alpha^{-1}\Longrightarrow\beta\sim\alpha$. Thus, $\sim$ is symmetric. Let $\alpha\sim\beta$ and $\beta\sim\gamma$, then there exist $x,y\in Q$ such that $\alpha^{-1}=R_x\beta^{-1}$ and $\beta^{-1}=R_y\gamma^{-1}\Longrightarrow\alpha^{-1}=R_xR_y\gamma^{-1}$. Choose $y=\sigma(x)$, so that $\alpha^{-1}=R_xR_{\sigma(x)}\gamma^{-1}=R_{x\sigma(x)}\gamma^{-1}\Longrightarrow\alpha\sim\gamma$. $\therefore$ $\sim$ is an equivalence relation over $SYM(Q)$.

2. Let $\alpha,\beta\in SYM(Q)$. Let $\alpha\sim\beta$, then there exists $y\in Q$ such that $\alpha^{-1}=R_y\beta^{-1}$. Take $y=x\sigma(x)$, then $\alpha^{-1}=R_{x\sigma(x)}\beta^{-1}=R_xR_{\sigma(x)}\beta^{-1}\Rightarrow\alpha R_x=\beta R_{\sigma(x)^{-1}}$. Say, $\alpha R_x=\beta R_{\sigma(x)^{-1}}=\psi$, then $\alpha=\psi R_{x^{-1}}$ and $\beta=\psi R_{\sigma(x)}$. So, $\alpha,\beta\in TBS_1(Q,\cdot)$.

   Let $\alpha,\beta\in TBS_1(Q,\cdot)$. Then there exist $x,y\in Q$, $\psi\in SYM(Q)$ such that $\alpha=\psi R_x$ and $\beta=\psi R_y$. This implies $\psi=\alpha R_x^{-1}=\beta R_y^{-1}\Rightarrow\alpha^{-1}=R_{x^{-1}}R_y\beta^{-1}$. Take $y=\sigma(x^{-1})$, then $\alpha^{-1}=R_{x^{-1}}R_{\sigma(x^{-1})}\beta^{-1}=R_{x^{-1}\sigma(x^{-1})}\beta^{-1}\Rightarrow\alpha\sim\beta$.

3. Use 1. and 2.

□

Lemma 3.11. *Let $(Q,\cdot)$ be a loop. Then*

1. *$TBS_1(Q,\cdot)\le SYM(Q)$ if and only if $\alpha^{-1}\sim\beta^{-1}$ for any twin maps $\alpha,\beta\in SYM(Q)$. Hence, $T_1(Q,\cdot)\le SYM(Q)$.*
2. *$TBS_2(Q,\cdot)\le BS(Q,\cdot)$ if and only if $\alpha^{-1}\sim\beta^{-1}$ for any twin maps $\alpha,\beta\in SYM(Q)$. Hence, $T_2(Q,\cdot)\le PS(Q,\cdot)$.*

Proof.

1. $TBS_1(Q,\cdot) \neq \emptyset$ because $I = IR_e$ and $I^{-1} = I^{-1}R_e$ and so, $I, I^{-1} \in TBS_1(Q,\cdot)$. Let $\alpha_1, \alpha_2 \in TBS_1(Q,\cdot)$ and let $\psi_1, \psi_2 \in SYM(Q)$. Then there exist $x_1, y_1, x_2, y_2 \in Q$, $\psi_1, \psi_2 \in SYM(Q)$ and $\beta_1, \beta_2 \in SYM(Q)$ such that $\alpha_1 = \psi_1 R_{x_1}$, $\beta_1 = \psi_1 R_{y_1}$ and $\alpha_2 = \psi_2 R_{x_2}$, $\beta_2 = \psi_2 R_{y_2}$. So, $\alpha_1\alpha_2^{-1} = \psi_1 R_{x_1} R_{x_2}^{-1}\psi_2^{-1}$. Now, $\alpha_1\alpha_2^{-1} \in TBS_1(Q,\cdot) \Leftrightarrow \alpha_1\alpha_2^{-1} = \psi R_x$ and $\beta_1\beta_2^{-1} = \psi R_y$ for some $x, y \in Q$ and $\psi \in SYM(Q)$. Taking $\psi = \psi_1\psi_2^{-1}$ and $x = x_2$, then $\alpha_1\alpha_2^{-1} = \psi_1\psi_2^{-1}R_x \Leftrightarrow \psi_1 R_{x_1} R_{x_2}^{-1}\psi_2^{-1}$

   $= \psi_1\psi_2^{-1}R_x \Leftrightarrow \psi_2 R_{x_1} = R_x\psi_2 R_{x_2} \Leftrightarrow \psi_2 R_{x_1} = R_x\alpha_2 \Leftrightarrow \psi_2 R_{y_2} = R_x\alpha_2$ with $x_1 = y_2 \Leftrightarrow \beta_2 = R_x\alpha_2 \Leftrightarrow \alpha_2^{-1} \sim \beta_2^{-1}$. Thus, $TBS_1(Q,\cdot) \le SYM(Q)$ if and only if $\alpha_2^{-1} \sim \beta_2^{-1}$.

   Assuming that $TBS_1(Q,\cdot) \le SYM(Q)$, then $T_1(Q,\cdot) \neq \emptyset$ because $I \in T_1(Q,\cdot)$. As earlier shown, $\alpha_1\alpha_2^{-1} = \psi_1\psi_2^{-1}R_x$ for any $\psi_1, \psi_2 \in T_1(Q,\cdot)$ and $\alpha_1, \alpha_2 \in TBS_1(Q,\cdot)$. So, $T_1(Q,\cdot) \le SYM(Q)$.

2. $TBS_2(Q,\cdot) \neq \emptyset$ because $TBS_1(Q,\cdot) \neq \emptyset$ and $BS(Q,\cdot) \neq \emptyset$. For any $\alpha_1, \alpha_2 \in TBS_2(Q,\cdot)$, $\alpha_1\alpha_2^{-1} \in BS(Q,\cdot)$. So, $\alpha_1\alpha_2^{-1} \in TBS_2(Q,\cdot) \Leftrightarrow \alpha_1\alpha_2^{-1} \in TBS_1(Q,\cdot) \Leftrightarrow \alpha_2^{-1} \sim \beta_2^{-1}$. $\therefore$ $TBS_2(Q,\cdot) \le BS(Q,\cdot) \Leftrightarrow \alpha_2^{-1} \sim \beta_2^{-1}$.

   Assuming that $TBS_2(Q,\cdot) \le BS(Q,\cdot)$, then $T_2(Q,\cdot) \neq \emptyset$ because $I \in T_2(Q,\cdot)$. Let $\psi \in T_2(Q,\cdot)$, then there exists $\alpha \in BS(Q,\cdot)$, and $\alpha = \psi R_x \in TBS_2(Q,\cdot)$ for some $x \in Q$. Recall that $\alpha \in BS(Q,\cdot)$ implies there exist $f, g \in Q$ such that $(\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \in AUT(Q,\cdot)$. Taking $g = x$ and $f = e$, $(\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) = (\psi R_x R_x^{-1}, \psi R_x L_e^{-1}, \psi R_x) = (\psi, \psi R_x, \psi R_x) \in AUT(Q,\cdot) \Rightarrow \alpha \in PS(Q,\cdot)$. Thus, $T_2(Q,\cdot) \subseteq PS(Q,\cdot)$.

   Let $\psi_1, \psi_2 \in T_2(Q,\cdot)$, then there exist $\alpha_1, \alpha_2 \in TBS_2(Q,\cdot)$ such that $\alpha_1 = \psi_1 R_{x_1}$ and $\alpha_2 = \psi_2 R_{x_2}$. In fact, $\alpha_1, \alpha_2 \in TBS_1(Q,\cdot)$ and so, following 1., $\alpha_1\alpha_2^{-1} = \psi_1\psi_2^{-1}R_y \in TBS_1(Q,\cdot)$ for some $y \in Q$. This implies that $\alpha_1\alpha_2^{-1} = \psi_1\psi_2^{-1}R_y \in TBS_2(Q,\cdot)$ for some $y \in Q$ and so $\psi_1\psi_2^{-1} \in T_2(Q,\cdot)$. Thus, $T_2(Q,\cdot) \le PS(Q,\cdot)$.

□

In what follows, in a loop $(Q,\cdot)$ with A-holomorph $(H,\circ)$ where $H = A(Q) \times Q$, we shall replace $A(Q)$ by $T_3(Q)$ whenever $T_3(Q) \le AUM(Q,\cdot)$ and then call $(H,\circ)$ a T_3-holomorph of $(Q,\cdot)$.

Corollary 3.12. *Let $(Q,\cdot)$ be a loop with a self-map $\sigma : x \mapsto (xg^{-1})^{-1}$ for all $x \in Q$ and some $g \in Q$ and let $(H,\circ)$ be the T_3-holomorph of $(Q,\cdot)$ with a self-map σ' such that $\sigma' : (\alpha, x) \mapsto \big(\alpha, (xg^{-1})^{-1}\big)$ for all $(\alpha, x) \in H$. Then $(H,\circ)$ is a σ'-generalized Bol loop if $(Q,\cdot)$ is a $\alpha^{-1}\sigma\gamma^{-1}$-generalized Bol loop for any $\alpha, \gamma \in T_3$.*

Proof. This is proved using Lemma 3.11, Corollary 3.2 and Corollary 3.7. □

Corollary 3.13. *Let $(Q,\cdot)$ be a loop with a self-map $\sigma : x \mapsto (xg^{-1})^{-1}$ for all $x \in Q$ and some $g \in Q$ and let $(H,\circ)$ be the T_3-holomorph of $(Q,\cdot)$ with a self-map σ' such that $\sigma' : (\alpha, x) \mapsto \Big(\alpha, \big[\alpha\gamma(x)(\alpha(g))^{-1}\big]^{-1}\Big)$ for all $(\alpha, x) \in H$ and* *any $\gamma \in T_3$. If $(Q, \cdot)$ is a σ-generalized Bol loop, then $(H, \circ)$ is a σ'-generalized Bol loop.*

Proof. This is proved using Lemma 3.11, Theorem 3.3 and Corollary 3.7. $\square$

Corollary 3.14. *Let $(Q, \cdot)$ be a loop with a self-map $\sigma : x \mapsto (xg^{-1})^{-1}$ for all $x \in Q$ and some $g \in Q$ and let $(H, \circ)$ be the T_3-holomorph of $(Q, \cdot)$ with a self-map σ' such that $\sigma' : (\alpha, x) \mapsto \big(\alpha, (xg^{-1})^{-1}\big)$ for all $(\alpha, x) \in H$. If for any $\gamma \in T_3$, $(Q, \cdot)$ is a $\sigma\alpha\gamma^{-1}$-generalized flexible-Bol loop, then $(H, \circ)$ is a σ'-generalized flexible-Bol loop.*

Proof. This is proved using Lemma 3.11, Theorem 3.4 and Corollary 3.7. $\square$

Corollary 3.15. *Let $(Q, \cdot)$ be a loop with a self-map $\sigma : x \mapsto (xg^{-1})^{-1}$ for all $x \in Q$ and some $g \in Q$ and let $(H, \circ)$ be the T_3-holomorph of $(Q, \cdot)$ with a self-map σ' such that $\sigma' : (\alpha, x) \mapsto \Big(\alpha, \big[\big(\gamma\alpha^{-1}(x)\big)g^{-1}\big]^{-1}\Big)$ for all $(\alpha, x) \in H$ and any $\gamma \in T_3$. If $(Q, \cdot)$ is a σ-generalized flexible-Bol loop, then $(H, \circ)$ is a σ'-generalized flexible-Bol loop.*

Proof. This is proved using Lemma 3.11, Theorem 3.5 and Corollary 3.7. $\square$

Remark 3.16. In Corollary 3.12, 3.13, 3.14, 3.15, the holomorph of a loop is built on the group of automorphisms gotten via the group of twin mappings.

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