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A REMARK ON COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED G-METRIC SPACES

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Abstract. In this paper we present some coupled fixed point theorems for mixed monotone mappings in partially ordered G-metric spaces.

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1. Introduction

In a recent paper Bhaskar and Lakshmikantham [6] introduced mixed monotone operator and established coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces. After their work, many authors studied about coupled fixed point [2, 4, 5, 7, 14, 15]. Some authors generalized the concept of metric spaces. Mustafa and Sims [12] introduced the notion of G-metric. Some authors studied some fixed point theorems in partially ordered G-metric space [1, 3, 7, 13].

In this paper, the letters $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all nonnegative real numbers and the set of all natural numbers, respectively. Mustafa and Simis [12] introduced following definition and obtained following results.

Definition 1. [12] Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$.

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$.

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables).

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

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Definition 2. [12] Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to $x \in X$ if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.

Proposition 1. *[12] Let (X,G) be a G-metric space. The following are equivalent:*

(1) (x_n) is G-convergent to x.
(2) $G(x_n,x_n,x) \to 0$ as $n \to +\infty$.
(3) $G(x_n,x,x) \to 0$ as $n \to +\infty$.
(4) $G(x_n,x_m,x) \to 0$ as $n,m \to +\infty$.

Definition 3. [12] Let (X,G) be a G-metric space. A sequence (x_n) is called a G-Cauchy sequence if, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$ for all $m,n,l \geq N$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 2. *[12] Let (X,G) be a G-metric space. Then the following are equivalent*

(1) the sequence (x_n) is G-Cauchy
(2) for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \epsilon$, for all $m,n \geq N$.

Proposition 3. *[12] Let (X,G) be a G-metric space. A mapping $f : X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Proposition 4. *[12] Let (X,G) be a G-metric space. Then, the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Proposition 5. *[12] Let (X,G) be a G-metric space, then for any $x,y,z,a \in X$ it follows*

(1) if $G(x,y,z) = 0$ then $x = y = z$,
(2) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,
(3) $G(x,y,y) \leq 2G(y,x,x)$,
(4) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$.

Proposition 6. *[12] A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).*

Definition 4. [7] Let (X,G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences (x_n) and (y_n) converging to x and y respectively, $\{F(x_n,y_n)\}$ is G-convergent to $F(x,y)$.

Bhaskar and Lakshmikantham in [6] introduced the concept of a mixed monotone property and following definitions.

Definition 5. [6] Let $(X,\leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is , for any $x,y \in X$,

$$x_1,x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1,y) \leq F(x_2,y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 6. [6] An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping F if

$$F(x, y) = x \quad and \quad F(y, x) = y.$$

In this paper we generalize the result of Berinde [4] into the context of partially ordered G-metric space.

2. Main result

Theorem 1. *Let $(X, \leq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that $F : X \times X \to X$ is a mapping having the mixed monotone property on X and there exists a constant $k \in [0, 1)$ such that*

$$G(F(x, y), F(u, v), F(z, t)) + G(F(y, x), F(v, u), F(t, z)) \leq k[G(x, u, z) + G(y, v, t))]$$

for all $x, y, u, v, z, t \in X$ with $x \geq u \geq z$ and $y \leq v \leq t$.

If there exist $x_0, y_0 \in X$ with

$$x_0 \leq F(x_0, y_0) \quad and \quad y_0 \geq F(y_0, x_0)$$

then F has a coupled fixed point.

Proof. First we define the functional $G_2 : X^2 \times X^2 \times X^2 \to \mathbb{R}_+$ by

$$G_2(X, U, Z) = \frac{1}{2}[G(x, u, z) + G(y, v, t)],$$

for all $X = (x, y), U = (u, v), Z = (z, t) \in X \times X$.

It is easily to seen that G_2 is a G-metric on X^2 and, if (X, G) is complete, then (X^2, G_2) is a complete G-metric space, too. If we define the operator $T : X^2 \to X^2$ by

$$T(X) = (F(x, y), F(y, x)), \quad \forall X = (x, y) \in X^2,$$

and we choose $X = (x, y)$, $U = (u, v)$ and $Z = (z, t) \in X^2$, by the definition of G_2, we have

$$G_2(T(X), T(U), T(Z)) = \frac{1}{2}[G(F(x, y), F(u, v), F(z, t)) + G(F(y, x), F(v, u), F(t, z))],$$

and

$$G_2(X, U, Z) = \frac{1}{2}[G(x, u, z) + G(y, v, t)].$$

Therefore, using the contractive condition, we obtain

$$G_2(T(X), T(U), T(Z)) \leq kG_2(X, U, Z). \tag{2.1}$$

Since $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, we denote $W_0 = (x_0, y_0)$ and we define the sequence (W_n) by

$$W_{n+1} = T(W_n), \tag{2.2}$$

with $W_n = (x_n, y_n)$. We show that $W_n \leq W_{n+1}$ for all $n \geq 0$.

For $n = 0$ since F has the mixed monotone property we have

$$W_0 = (x_0, y_0) \leq (F(x_0, y_0), F(y_0, x_0)) = (x_1, y_1) = W_1,$$

suppose that for some n it holds, then we have

$$W_n = (x_n, y_n) \leq (F(x_n, y_n), F(y_n, x_n)) = (x_{n+1}, y_{n+1}) = W_{n+1},$$

which implies that the mapping T is monotone and the sequence (W_n) is non-decreasing. Now if we take $X = U = W_n$ and $Z = W_{n-1}$ in (2.1), we obtain

$$\begin{aligned} & G_2(W_{n+1}, W_{n+1}, W_n) \\ & \quad = \quad G_2(T(W_n), T(W_n), T(W_{n-1})) \\ & \quad \leq \quad kG_2(W_n, W_n, W_{n-1}), \quad n \geq 1. \end{aligned}$$

By induction, we obtain

$$\begin{aligned} & G_2(W_{n+1}, W_{n+1}, W_n) \\ & \quad = \quad G_2(T(W_n), T(W_n), T(W_{n-1})) \\ & \quad \leq \quad k^n G_2(W_1, W_1, W_0), \quad n \geq 1. \end{aligned}$$

This implies that (W_n) is a G-Cauchy sequence in the G-metric space (X, G_2). Indeed, let $m > n$, then

$$\begin{aligned} & G_2(W_m, W_m, W_n) \\ & \quad \leq \quad \sum_{i=n+1} G_2(W_i, W_i, W_{i-1}) \\ & \quad \leq \quad (k^n + k^{n+1} + \cdots + k^{m-n-1}) G_2(W_1, W_1, W_0) \\ & \quad \leq \quad k^n \frac{1 - k^{m-n-1}}{1 - k} G_2(W_1, W_1, W_0). \end{aligned}$$

So, (W_n) is a G-Cauchy sequence in the complete G-metric space (X, G_2) and hence there exists a $W \in X \times X$ such that

$$\lim_{n \to \infty} W_n = W.$$

Since, from contractive condition it follows that T is continuous in (X^2, G_2) and using (2.2) it follows that W is a fixed point of T, that is

$$T(W) = W.$$

Suppose that $W = (x, y)$. From the definition of T, we get

$$x = F(x, y) \quad y = F(y, x),$$

and the proof is finished. □

Remark 1. Notice that, since the contractivity condition in Theorem 1 is valid only for comparable elements, therefore Theorem 1 cannot guarantee the uniqueness of coupled fixed point.

Now we prove the existence and uniqueness theorem of coupled fixed point. Notice that if $(X, \leq)$ is a partially ordered set, we endow the product space $X \times X$ with the partial order relation given by

$$(u, v) \leq (x, y) \Leftrightarrow x \geq u \quad \text{and} \quad y \leq v$$

Theorem 2. *In addition to the hypothesis of Theorem 1, suppose that for all $X = (x, y)$, $X^* = (x^*, y^*) \in X \times X$, there exists $U = (u, v) \in X \times X$ such that $U \in X \times X$ is comparable to X and X^*. Then F has a unique coupled fixed point.*

Proof. Suppose that $X = (x, y)$ and $X^* = (x^*, y^*)$ are coupled fixed point of F. We distinguish two cases.

Case 1. IF X is comparable to X^*. Then, from the definition of G_2 and using contractive condition we obtain

$$G_2(T(X), T(X), T(X^*)) = G_2(X, X, X^*) \leq kG_2(X, X, X^*),$$

since $0 \leq k < 1$, this implies that $G_2(X, X, X^*) \leq 0$. That is $X = X^*$.

Case 2. If X is not comparable to X^*. Then there exists a $U \in X \times X$ comparable to X and X^*. From monotonicity of T it follows that $T^n(U)$ is comparable to $T^n(X) = X$ and to $T^n(X^*) = X^*$.

Again, from rectangle inequality, the definition of G_2 and using contractive condition we obtain

$$\begin{aligned} G_2(X, X, X^*) &= G_2(T^n(X), T^n(X), T^n(X^*)) \\ &\leq G_2(T^n(X), T^n(U), T^n(U)) + G_2(T^n(U), T^n(X^*), T^n(X^*)) \\ &\leq k^n[G_2(X, U, U) + G_2(U, X^*, X^*)]. \end{aligned}$$

Taking $n \to \infty$, it follows that $G_2(X, X, X^*) \leq 0$. That is $X = X^*$. □

Theorem 3. *Under the hypotheses of Theorem 1, suppose that x_0 and y_0 are comparable then the coupled fixed point $(x, y) \in X \times X$ satisfies $x = y$.*

Proof. Following the proof of Theorem 1, we only have to show that $x = F(x, x)$. Assume $y_0 \leq x_0$ (similar argument for $x_0 \leq y_0$). Then we get

$$y_0 \leq y_n \leq \cdots \leq y_1 \leq y_0 \leq x_0 \leq x_1 \leq \cdots \leq x_n \leq x.$$

Thus, we have $y \leq x$. Using contractive condition, we obtain

$$\begin{aligned} & G(x, x, y) + G(y, y, x) \\ &= G(F(x, y), F(x, y), F(y, x)) + G(F(y, x), F(y, x), F(x, y)) \\ &\leq k[G(x, x, y) + G(y, y, x)]. \end{aligned}$$

Since $0 \leq k < 1$, it follows that $G(x,x,y)+G(y,y,x) = 0$, it means $G(x,x,y) = G(y,y,x) = 0$.

Thus $x = y$. □

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