Novi Sad J. Math.
Vol. 44, No. 1, 2014, 75-90

CONVERGENCE OF THE EXPLICIT ITERATION METHOD FOR STRICTLY ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS IN THE INTERMEDIATE SENSE

G. S. Saluja[1]

Abstract. In this paper, we establish a weak convergence theorem and some strong convergence theorems of an explicit iteration process for a finite family of strictly asymptotically pseudo-contractive mappings in the intermediate sense and also establish a strong convergence theorem by a new hybrid method for above said iteration scheme and mappings in the setting of Hilbert spaces.

AMS Mathematics Subject Classification (2010): 47H09, 47H10.

Key words and phrases: Strictly asymptotically pseudo-contractive mapping in the intermediate sense, common fixed point, explicit iteration process, strong convergence, weak convergence, Hilbert space.

1. Introduction and Preliminaries

Throughout this paper, let H be a real Hilbert space with the scalar product and norm denoted by the symbols $\langle .,. \rangle$ and $\| . \|$ respectively. Let C be a closed convex subset of H, we denote by $P_C(.)$ the metric projection from H onto C. It is known that $z = P_C(x)$ is equivalent to $\langle z - y, x - z \rangle \geq 0$ for every $y \in C$. A point $x \in C$ is a fixed point of T provided that $Tx = x$. Denote by $F(T)$ the set of fixed point of T, that is, $F(T) = \{x \in C : Tx = x\}$. It is known that $F(T)$ is closed and convex. Let T be a (possibly) nonlinear mapping from C into C. We now consider the following classes:

T is contractive, i.e., there exists a constant $k < 1$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \tag{1.1}$$

for all $x, y \in C$.

T is nonexpansive, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \tag{1.2}$$

for all $x, y \in C$.

T is uniformly L-Lipschitzian, i.e., if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \tag{1.3}$$

[1]Department of Mathematics, Govt. Nagarjuna P.G. College of Science, Raipur (C.G.), India, e-mail: saluja_1963@rediffmail.com

for all $x, y \in C$ and $n \in \mathbb{N}$.

T is pseudo-contractive, i.e.,

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2, \tag{1.4}$$

for all $x, y \in C$.

T is asymptotically nonexpansive [6], i.e., if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x-y\|, \tag{1.5}$$

for all $x, y \in C$ and $n \geq 1$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] as a generalization of the class of nonexpansive mappings. T is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \big(\|T^n x - T^n y\| - \|x-y\|\big) \leq 0. \tag{1.6}$$

Observe that if we define

$$G_n = \max\Big\{0, \sup_{x,y\in C} \big(\|T^n x - T^n y\| - \|x-y\|\big)\Big\}, \tag{1.7}$$

then $G_n \to 0$ as $n \to \infty$. It follows that (1.7) is reduced to

$$\|T^n x - T^n y\| \leq \|x-y\| + G_n, \tag{1.8}$$

for all $x, y \in C$ and $n \geq 1$.

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [3]. It is known [8] that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Recall that T is said to be a k-strictly pseudocontraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \tag{1.9}$$

for all $x, y \in C$.

T is said to an asymptotically k-strictly pseudocontraction with sequence $\{r_n\}$ if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$\begin{aligned} \|T^n x - T^n y\|^2 &\leq (1+r_n)\|x-y\|^2 \\ &\quad + k\|(x - T^n x) - (y - T^n y)\|^2, \end{aligned} \tag{1.10}$$

for some $k \in [0, 1)$ for all $x, y \in C$ and $n \geq 1$.

Remark 1.1. (see [13]) If T is k-strictly asymptotically pseudo-contractive mapping, then it is uniformly L-Lipschitzian with $L = \sup_{n\geq 1}\{(a_n+\sqrt{k})/(1+\sqrt{k}) : n \in N\}$ where $\{a_n\}$ is a sequence in $[1,\infty)$ with $a_n \to 1$ as $n \to \infty$, but the converse does not hold.

The class of asymptotically k-strictly pseudocontraction was introduced by Qihou [9] in 1996. Kim and Xu [7] studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically k-strictly pseudocontraction with sequence $\{r_n\}$ is a uniformly L-Lipschitzian mapping with $L = \sup_{n\geq 1}\{(k+\sqrt{1+(1-k)r_n})/(1+k) : n \in N\}$.

Recently, Sahu et al. [19] introduced a class of new mappings: asymptotically k-strictly pseudocontractive mappings in the intermediate sense. Recall that T is said to be an asymptotically k-strictly pseudocontraction in the intermediate sense with sequence $\{r_n\}$ if there exists a sequence $\{r_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ and a constant $k \in [0,1)$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \Big(\|T^n x - T^n y\|^2 - (1+r_n)\|x-y\|^2 - k\|(I-T^n)x - (I-T^n)y\|^2\Big) \leq 0. \tag{1.11}$$

Throughout this paper, we assume that

$$s_n = \max\Big\{0, \sup_{x,y\in C}\Big(\|T^n x - T^n y\|^2 - (1+r_n)\|x-y\|^2 - k\|(I-T^n)x - (I-T^n)y\|^2\Big)\Big\}. \tag{1.12}$$

It follows that $s_n \to 0$ as $n \to \infty$ and (1.11) is reduced to the relation

$$\|T^n x - T^n y\|^2 \leq (1+r_n)\|x-y\|^2 + k\|(I-T^n)x - (I-T^n)y\|^2 + s_n, \tag{1.13}$$

for all $x, y \in C$ and $n \geq 1$.

Remark 1.2. (see [19]) (1) T is not necessarily uniformly L-Lipschitzian (see Lemma 2.6 of [19]).

(2) When $s_n = 0$ for all $n \in \mathbb{N}$ in (1.13) then T is an asymptotically k-strictly pseudocontractive mapping with sequence $\{r_n\}$.

Remark 1.3. When $s_n = 0$ for all $n \in \mathbb{N}$ and $k = 0$ in (1.13), then T is an asymptotically nonexpansive mapping with sequence $\{r_n\} \subset [0,\infty)$ such that $\lim_{n\to\infty} r_n = 0$, a concept introduced by Goebel and Kirk [6] in 1972.

They obtained a weak convergence theorem of modified Mann iterative processes for the class of mappings which is not necessarily Lipschitzian. Moreover, a strong convergence theorem was also established in a real Hilbert space by hybrid projection method; see [19] for more details.

In 2001, Xu and Ori [22] have introduced the following implicit iteration process for common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in Hilbert spaces:

$$x_n = t_n x_{n-1} + (1-t_n)T_n x_n, \quad n \geq 1 \tag{1.14}$$

where $T_n = T_{n \text{ mod } N}$. (Here the mod N function takes values in $\{1, 2, \ldots, N\}$). And they proved the weak convergence of the process (1.14).

In 2003, Sun [20] modified the implicit iteration process of Xu and Ori [22] and applied the modified averaging iteration process for the approximation of fixed points of asymptotically quasi-nonexpansive mappings. Sun introduced the following implicit iteration process for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings $\{T_i\}_{i=1}^N$ in Banach spaces:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \geq 1 \tag{1.15}$$

where $n = (k - 1)N + i$, $i \in I = \{1, 2, \ldots, N\}$.

Assuming that the implicit iteration process is defined in C where C is a nonempty closed convex subset of a Banach space E, Sun proved the strong convergence theorem for said class of mappings in uniformly convex Banach spaces.

We note that it is the same as Mann's iterations [10] that have only weak convergence theorems with implicit iteration scheme (1.14) and (1.15) (also, see [1, 4, 5]). In this paper, we introduce the following explicit iteration scheme and modify it by hybrid method, so strong convergence theorems are obtained:

Let C be a closed convex subset of a Hilbert space H and let $\{T_i\}_{i=1}^N$ be N asymptotically k-strictly pseudocontraction in the intermediate sense on C such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$. The explicit iteration scheme generates a sequence $\{x_n\}_{n=1}^\infty$ in the following way:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_i^k x_n, \tag{1.16}$$

where $n = (k - 1)N + i$, $i \in I = \{1, 2, \ldots, N\}$.

The goal of this paper is to establish a weak convergence theorem and some strong convergence theorems of an explicit iteration scheme (1.16) to approximating a common fixed point for a finite family of strictly asymptotically pseudo-contractive mappings in the intermediate sense in Hilbert spaces. The results presented in the paper extend and improve some recent results of [2, 7, 9, 12, 14, 15, 17, 18, 22].

In order to prove our main results, we need the following lemma:

Lemma 1.4. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and let $T_i \colon C \to C$ be asymptotically k_i-strictly pseudocontractive mappings in the intermediate sense for $i = 1, 2, \ldots, N$ with a sequence $\{r_{n_i}\} \subset [0, \infty)$ such that $\sum_{n=1}^\infty r_{n_i} < \infty$ and for some $0 \leq k_i < 1$. Then there exists a constant $k \in [0, 1)$ and sequences $\{r_n\}, \{s_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty} s_n = 0$ such that for any $x, y \in C$ and for each $i = 1, 2, \ldots, N$ and each $n \geq 1$, the following holds:*

$$\begin{aligned} \|T_i^n x - T_i^n y\| &\leq (1 + r_n)\|x - y\|^2 \\ &\quad + k\|(I - T_i^n)x - (I - T_i^n)y\|^2 + s_n. \end{aligned} \tag{1.17}$$

Proof. Since for each $i = 1, 2, \ldots, N$, T_i is asymptotically k_i-strictly pseudocontractive in the intermediate sense mapping, where $k_i \in [0, 1)$ and $\{r_{n_i}\}, \{s_{n_i}\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_{n_i} = 0$ and $\lim_{n\to\infty} s_{n_i} = 0$. Taking $r_n = \max\{r_{n_i}, i = 1, 2, \ldots, N\}$, $s_n = \max\{s_{n_i}, i = 1, 2, \ldots, N\}$ and $k = \max\{k_i, i = 1, 2, \ldots, N\}$, hence, for each $i = 1, 2, \ldots, N$, we have from (1.13)

$$
\begin{aligned}
\|T_i^n x - T_i^n y\| &\le (1 + r_{n_i})\|x - y\|^2 \\
&\quad + k_i\|(x - T_i^n x) - (y - T_i^n y)\|^2 + s_{n_i}, \\
&\le (1 + r_n)\|x - y\|^2 \\
&\quad + k\|(x - T_i^n x) - (y - T_i^n y)\|^2 + s_n.
\end{aligned}
\tag{1.18}
$$

The conclusion (1.17) is proved. This completes the proof of Lemma 1.4. □

It is the purpose of this paper to modify iteration process (1.16) by hybrid method as follows: chosen arbitrary $x_0 \in C$ and

$$
\begin{cases}
y_n = \alpha_n x_n + (1 - \alpha_n) T_i^k x_n, \\
C_n = \Big\{ z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\
\qquad + (1 - \alpha_n)(k - \alpha_n)\|x_n - T_i^k x_n\|^2 + \theta_n \Big\}, \\
Q_n = \big\{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \big\}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{cases}
\tag{1.19}
$$

where $n = (k - 1)N + i$, $i \in I = \{1, 2, \ldots, N\}$, $\theta_n = r_n \Delta_n^2 + (1 - \alpha_n) s_n \to 0$ $(n \to \infty)$ and

$$
\Delta_n = \sup \left\{ \|x_n - z\| : z \in F = \bigcap_{i=1}^{N} F(T_i) \right\}.
$$

The purpose of this paper is to establish strong convergence theorem of newly proposed (CQ) algorithm (1.19) for a finite family of asymptotically k-strictly pseudo-contractive mappings in the intermediate sense in Hilbert spaces. Our result extends the corresponding result of Thakur [21] and many others.

In the sequel, we will need the following lemmas.

Lemma 1.5. *(see [21]) Let H be a real Hilbert space. The following identities hold:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \quad \forall\, x, y \in H.$

(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2,$
$\forall\, t \in [0, 1], \forall\, x, y \in H.$

(iii) If $\{x_n\}$ is a sequence in H weakly converges to z, then

$$
\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2 \quad \forall\, y \in H.
$$

Lemma 1.6. *(see [12]) Let H be a real Hilbert space. Let $C \subset H$ be a closed convex subset, $x, y, z \in H$ points and $a \in \mathbb{R}$ a real number. The set*

$$\Big\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\Big\}$$

is convex (and closed).

Lemma 1.7. *(see [12]) Let K be a closed convex subset of a real Hilbert space H. For given $x \in H$ and $y \in K$, we have that $z = P_K x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \leq 0 \quad \forall y \in K,$$

where P_K is the nearest point projection from H onto K, that is, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| \leq \|x - y\| \quad \forall x \in K.$$

We will use the following notations:
1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x : \exists\ x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Lemma 1.8. *(see [11]) Let K be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n. \tag{1.20}$$

Then $x_n \to q$.

Lemma 1.9. *(see [16]) Let $\{a_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + r_n)a_n + \beta_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

Theorem 2.1. *Let C be a closed convex subset of a Hilbert space H. Let $N \geq 1$ be an integer. Let for each $1 \leq i \leq N$, $T_i \colon C \to C$ be N uniformly L_i-Lipschitzian and asymptotically k_i-strictly pseudo-contraction in the intermediate sense mappings for some $0 \leq k_i < 1$ and $I - T_n$ is demiclosed at zero. Let $k = \max\{k_i : 1 \leq i \leq N\}$ and $L = \max\{L_i : 1 \leq i \leq N\}$. Assume that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Given $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by an explicit iteration scheme (1.16). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k + \epsilon < \alpha_n < 1 - \epsilon$ for all n and for some $\epsilon \in (0, 1)$, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max\{r_{n_i} : 1 \leq i \leq N\}$ and $s_n = \max\{s_{n_i} : 1 \leq i \leq N\}$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$.*

Proof. Let $p \in F = \bigcap_{i=1}^{N} F(T_i)$. It follows from (1.16) and Lemma 1.5(ii) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)T_i^k x_n - p\|^2 \\
&= \|\alpha_n(x_n - p) + (1-\alpha_n)(T_i^k x_n - p)\|^2 \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T_i^k x_n - p\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_i^k x_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big[(1+r_n)\|x_n - p\|^2 \\
&\quad +k\|x_n - T_i^k x_n\|^2 + s_n\Big] - \alpha_n(1-\alpha_n)\|x_n - T_i^k x_n\|^2 \\
&\leq \Big[\alpha_n(1+r_n) + (1-\alpha_n)(1+r_n)\Big]\|x_n - p\|^2 + (1-\alpha_n)s_n \\
&\quad -(\alpha_n - k)(1-\alpha_n)\|x_n - T_i^k x_n\|^2 \\
&= (1+r_n)\|x_n - p\|^2 - (\alpha_n - k)(1-\alpha_n)\|x_n - T_i^k x_n\|^2 \\
&\quad +(1-\alpha_n)s_n \qquad (2.1)
\end{aligned}$$

Since $k + \epsilon < \alpha_n < 1 - \epsilon$ for all n and for some $\epsilon \in (0,1)$, from (2.1) we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1+r_n)\|x_n - p\|^2 - \epsilon^2\|x_n - T_i^k x_n\|^2 \\
&\quad +(1-k-\varepsilon)s_n. \qquad (2.2)
\end{aligned}$$

Now (2.2) implies that

$$\|x_{n+1} - p\|^2 \leq (1+r_n)\|x_n - p\|^2 + (1-k-\varepsilon)s_n. \qquad (2.3)$$

Since $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, it follows from Lemma 1.9, that $\lim_{n\to\infty}\|x_n - p\|$ exists and so $\{x_n\}$ is bounded. Consider (2.2) again yields that

$$\begin{aligned}
\|x_n - T_i^k x_n\|^2 &\leq \frac{1}{\epsilon^2}\left[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right] \\
&\quad +\frac{r_n}{\epsilon^2}\|x_n - p\|^2 + \Big(\frac{1-k-\varepsilon}{\varepsilon^2}\Big)s_n. \qquad (2.4)
\end{aligned}$$

Since $\{x_n\}$ is bounded, $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$. So, we get

$$\|x_n - T_i^k x_n\| \to 0 \text{ as } n \to \infty. \qquad (2.5)$$

From the definition of $\{x_n\}$, we have

$$\|x_{n+1} - x_n\| = (1-\alpha_n)\|x_n - T_i^k x_n\| \to 0, \quad \text{as} \quad n \to \infty. \qquad (2.6)$$

Thus,

$$\|x_n - x_{n+l}\| \to 0 \text{ as } n \to \infty \qquad (2.7)$$

and for all $l < N$. Now for $n \geq N$, and since T is uniformly Lipschitzian with

Lipschitz constant $L > 0$, so we have

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - T_n^k x_n\| + \|T_n^k x_n - T_n x_n\| \\ &\leq \|x_n - T_n^k x_n\| + L\|T_n^{k-1} x_n - x_n\| \\ &\leq \|x_n - T_n^k x_n\| + L\Big[\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\| \\ &\quad +\|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)}\| \\ &\quad +\|x_{(n-N)} - x_n\|\Big]. \end{aligned} \tag{2.8}$$

Since for each $n \geq N$, $n \equiv (n-N)\ (mod\ N)$. Thus $T_n = T_{n-N}$, therefore from (2.8), we have

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - T_n^k x_n\| + L^2\|x_n - x_{n-N}\| \\ &\quad +L\|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)}\| \\ &\quad +L\|x_{(n-N)} - x_n\|. \end{aligned} \tag{2.9}$$

From(2.5), (2.7) and (2.9), we obtain

$$\|x_n - T_n x_n\| \to 0 \text{ as } n \to \infty. \tag{2.10}$$

Consequently, for any $l \in I = \{1, 2, \dots, N\}$,

$$\begin{aligned} \|x_n - T_{n+l} x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| \\ &\quad +\|T_{n+l} x_{n+l} - T_{n+l} x_n\| \\ &\leq (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| \\ &\quad \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.11}$$

This implies that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall\ l \in I = \{1, 2, \dots, N\}. \tag{2.12}$$

Since $I - T_n$ is demiclosed at zero, (2.10) imply that $x_n \rightharpoonup x$ where x is a weak limit of $\{x_n\}$ and hence $\omega_w(x_n) \subset F = \bigcap_{i=1}^N F(T_i)$. Now we show that $\{x_n\}$ is weakly convergent. Let $p_1, p_2 \in \omega_w(x_n)$ and $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ which converge weakly to some p_1 and p_2 respectively.

Since $\lim_{n\to\infty} \|x_n - z\|$ exists for every $z \in F$ and since $p_1, p_2 \in F$, we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p_1\|^2 &= \lim_{j\to\infty} \|x_{m_j} - p_1\|^2 \\ &= \lim_{j\to\infty} \|x_{m_j} - p_2\|^2 + \|p_2 - p_1\|^2 \\ &= \lim_{i\to\infty} \|x_{n_i} - p_1\|^2 + 2\|p_2 - p_1\|^2 \\ &= \lim_{n\to\infty} \|x_n - p_1\|^2 + 2\|p_2 - p_1\|^2. \end{aligned}$$

Hence $p_1 = p_2$. Thus $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^N$. This completes the proof. □

Theorem 2.2. *Let C be a closed convex compact subset of a Hilbert space H. Let $N \geq 1$ be an integer. Let for each $1 \leq i \leq N$, $T_i : C \to C$ be N uniformly L_i-Lipschitzian and asymptotically k_i-strictly pseudo-contraction in the intermediate sense mappings for some $0 \leq k_i < 1$. Let $k = \max\{k_i : 1 \leq i \leq N\}$ and $L = \max\{L_i : 1 \leq i \leq N\}$. Assume that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Given $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by an explicit iteration scheme (1.16). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k+\epsilon < \alpha_n < 1-\epsilon$ for all n and for some $\epsilon \in (0,1)$, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max\{r_{n_i} : 1 \leq i \leq N\}$ and $s_n = \max\{s_{n_i} : 1 \leq i \leq N\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$.*

Proof. We only prove the difference between this theorem and Theorem 2.1. By compactness of C this immediately implies that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a common fixed point of $\{T_i\}_{i=1}^{N}$, say, p. Combining (2.3) with Lemma 1.9, we have $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof. □

For our next result, we shall need the following definition:

Definition 2.3. A mapping $T : C \to C$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = x \in C$.

Theorem 2.4. *Let C be a closed convex subset of a Hilbert space H. Let $N \geq 1$ be an integer. Let for each $1 \leq i \leq N$, $T_i : C \to C$ be N uniformly L_i-Lipschitzian and asymptotically k_i-strictly pseudo-contraction in the intermediate sense mappings for some $0 \leq k_i < 1$. Let $k = \max\{k_i : 1 \leq i \leq N\}$ and $L = \max\{L_i : 1 \leq i \leq N\}$. Suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Given $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by an explicit iteration scheme (1.16). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k+\epsilon < \alpha_n < 1-\epsilon$ for all n and for some $\epsilon \in (0,1)$, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max\{r_{n_i} : 1 \leq i \leq N\}$ and $s_n = \max\{s_{n_i} : 1 \leq i \leq N\}$. Assume that one member of the family $\{T_i\}_{i=1}^{N}$ is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$.*

Proof. Without loss of generality, we can assume that T_1 is semi-compact. It follows from (2.12) that

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{2.13}$$

By the semi-compactness of T_1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to u \in C$ strongly. Since C is closed, $u \in C$, and furthermore,

$$\lim_{n_k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = \|u - T_l u\| = 0, \tag{2.14}$$

for all $l \in I = \{1, 2, \ldots, N\}$. Thus $u \in F$. Since $\{x_{n_k}\}$ converges strongly to u and $\lim_{n\to\infty} \|x_n - u\|$ exists, it follows from Lemma 1.9 that $\{x_n\}$ converges strongly to u. This completes the proof. □

We now prove strong convergence of k-strictly asymptotically pseudo-contractive mappings in the intermediate sense using iteration scheme (1.19).

Theorem 2.5. *Let C be a closed convex subset of a Hilbert space H. Let $N \geq 1$ be an integer. Let for each $1 \leq i \leq N$, $T_i : C \to C$ be N uniformly L_i-Lipschitzian and asymptotically k_i-strictly pseudo-contraction in the intermediate sense mappings for some $0 \leq k_i < 1$ and $I - T_n$ is demiclosed at zero. Let $k = \max\{k_i : 1 \leq i \leq N\}$ and $L = \max\{L_i : 1 \leq i \leq N\}$. Assume that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Given $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by an explicit iterative process (1.19). Assume that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Assume that the sequence $\{\alpha_n\}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1$, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max\{r_{n_i} : 1 \leq i \leq N\}$ and $s_n = \max\{s_{n_i} : 1 \leq i \leq N\}$. Then $\{x_n\}$ converges strongly to $P_F(x_0)$.*

Proof. By Lemma 1.6, we observe that C_n is convex.

Now, for all $p \in F$, using **Lemma 1.5**(ii), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)T_i^k x_n - p\|^2 \\
&= \|\alpha_n(x_n - p) + (1-\alpha_n)(T_i^k x_n - p)\|^2 \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T_i^k x_n - p\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_i^k x_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big[(1+r_n)\|x_n - p\|^2 \\
&\quad +k\|x_n - T_i^k x_n\|^2 + s_n\Big] - \alpha_n(1-\alpha_n)\|x_n - T_i^k x_n\|^2 \\
&\leq \Big[\alpha_n(1+r_n) + (1-\alpha_n)(1+r_n)\Big]\|x_n - p\|^2 + (1-\alpha_n)s_n \\
&\quad -(\alpha_n - k)(1-\alpha_n)\|x_n - T_i^k x_n\|^2 \\
&= (1+r_n)\|x_n - p|^2 - (\alpha_n - k)(1-\alpha_n)\|x_n - T_i^k x_n\|^2 \\
&\quad +(1-\alpha_n)s_n \\
&= (1+r_n)\|x_n - p\|^2 + (k-\alpha_n)(1-\alpha_n)\|x_n - T_i^k x_n\|^2 \\
&\quad +(1-\alpha_n)s_n \\
&\leq \|x_n - p\|^2 + (k-\alpha_n)(1-\alpha_n)\|x_n - T_i^k x_n\|^2 + \theta_n \qquad (2.15)
\end{aligned}$$

so $p \in C_n$ for all n. Thus $F \subset C_n$ for all n.

Next we show that $F \subset Q_n$ for all $n \geq 0$, for this we use induction.

For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_n$.

Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 1.6, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0 \quad \forall z \in C_n \cap Q_n.$$

As $F \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F$. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Hence $F \subset Q_n$ for all $n \geq 0$.

Now, since $x_n = P_{Q_n}(x_0)$ (by the definition of Q_n), and since $F \subset Q_n$, we have

$$\|x_n - x_0\| \leq \|p - x_0\| \quad \forall p \in F.$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|q - x_0\|, \quad \text{where } q = P_F(x_0). \tag{2.16}$$

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$. This together with Lemma 1.5(i), implies that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{2.17}$$

This implies that the sequence $\{\|x_n - x_0\|\}$ is increasing. Since it is also bounded, we get that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. It turns out from (2.17) that

$$\|x_{n+1} - x_n\| \to 0. \tag{2.18}$$

By the fact $x_{n+1} \in C_n$, we get

$$\begin{aligned} \|x_{n+1} - y_n\|^2 \leq\ & \|x_{n+1} - x_n\|^2 \\ & +(k - \alpha_n)(1 - \alpha_n)\|x_n - T_i^k x_n\|^2 + \theta_n. \end{aligned} \tag{2.19}$$

Moreover, since $y_n = \alpha_n x_n + (1 - \alpha_n)T_i^k x_n$, we deduce that

$$\begin{aligned} \|x_{n+1} - y_n\|^2 =\ & \alpha_n\|x_{n+1} - x_n\|^2 \\ & +(1 - \alpha_n)\|x_{n+1} - T_i^k x_n\|^2 \\ & -\alpha_n(1 - \alpha_n)\|x_n - T_i^k x_n\|^2. \end{aligned} \tag{2.20}$$

Substituting (2.20) into (2.19) to get

$$\begin{aligned} (1 - \alpha_n)\|x_{n+1} - T_i^k x_n\|^2 \leq\ & (1 - \alpha_n)\|x_{n+1} - x_n\|^2 \\ & +k(1 - \alpha_n)\|x_n - T_i^k x_n\|^2 + \theta_n. \end{aligned}$$

Since $\limsup_{n\to\infty} \alpha_n < 1$, the last inequality becomes,

$$\begin{aligned} \|x_{n+1} - T_i^k x_n\|^2 \leq\ & \|x_{n+1} - x_n\|^2 + k\|x_n - T_i^k x_n\|^2 \\ & +\frac{\theta_n}{1 - \tau}, \end{aligned} \tag{2.21}$$

for some positive number $\tau > 0$, such that $\alpha_n \leq \tau < 1$.

But on the other hand, we compute

$$\begin{aligned} \|x_{n+1} - T_i^k x_n\|^2 =\ & \|x_{n+1} - x_n\|^2 + 2\langle x_{n+1} - x_n, x_n - T_i^k x_n\rangle \\ & +\|x_n - T_i^k x_n\|^2. \end{aligned} \tag{2.22}$$

By (2.21) and (2.22), we get

$$(1 - k)\|x_n - T_i^k x_n\|^2 \leq \frac{\theta_n}{1 - \tau} - 2\langle x_{n+1} - x_n, x_n - T_i^k x_n\rangle. \tag{2.23}$$

Therefore

$$\|x_n - T_i^k x_n\|^2 \leq \frac{\theta_n}{(1-\tau)(1-k)} - \frac{2}{1-k}\langle x_{n+1} - x_n, x_n - T_i^k x_n\rangle$$
$$\to 0 \text{ as } n \to \infty. \tag{2.24}$$

Now,

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - T_n^k x_n\| + \|T_n^k x_n - T_n x_n\| \\ &\leq \|x_n - T_n^k x_n\| + [(1+r_1)\|T_n^{k-1}x_n - x_n\| + s_1] \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.25}$$

Now, since $I - T_n$ is demiclosed at zero, (2.25) imply that $x_n \rightharpoonup x$, where x is a weak limit of $\{x_n\}$ and hence $\omega_w(x_n) \subset F(T_i)$ for any $i = 1, 2, \ldots, N$. So, $\omega_w(x_n) \subset F = \bigcap_{i=1}^N F(T_i)$. This fact, the inequality (2.16) and Lemma 1.8 imply that $x_n \to q = P_F(x_0)$, that is, $\{x_n\}$ converges strongly to $P_F(x_0)$. This completes the proof. □

Since asymptotically nonexpansive mappings are asymptotically 0-strict pseudo-contractions in the intermediate sense (by remark 1.3), we have the following consequence.

Corollary 2.6. *Let C be a closed bounded convex subset of a Hilbert space H. Let $N \geq 1$ be an integer. Let for each $1 \leq i \leq N$, $T_i\colon C \to C$ be N uniformly L_i-Lipschitzian and asymptotically nonexpansive mappings and $I - T_n$ is demiclosed at zero. Let $L = \max\{L_i : 1 \leq i \leq N\}$. Assume that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Given $x_0 \in C$, let $\{x_n\}_{n=1}^\infty$ be the sequence generated by the following (CQ) algorithm:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T_i^k x_n, \\ C_n = \Big\{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \qquad - \alpha_n(1-\alpha_n)\|x_n - T_i^k x_n\|^2 + \theta_n\Big\}, \\ Q_n = \big\{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\big\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases}$$

where $n = (k-1)N + i$, $i \in I = \{1, 2, \ldots, N\}$, $\theta_n = r_n\Delta^2 \to 0$ $(n \to \infty)$ and $\Delta = diam\, C$. Assume that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Assume that the sequence $\{\alpha_n\}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1$ and $\sum_{n=1}^\infty r_n < \infty$ where $r_n = \max\{r_{n_i} : 1 \leq i \leq N\}$. Then $\{x_n\}$ converges strongly to $P_F(x_0)$.

Remark 2.7. If the closed convex subset C in Theorem 2.5 is bounded, we can replace the Δ_n in the definition of θ_n in the algorithm (1.19) with the diameter of C, i.e., $\Delta_n = diam\ C$ for all n and thus $\theta_n = r_n(diam\ C)^2 + (1-\alpha_n)s_n$.

Remark 2.8. Theorem 2.1 extends and improves the corresponding result of Reich [18] and Marino and Xu [12] from nonexpansive and strict pseudo-contraction mapping to the more general class of finite family of asymptotically k-strictly pseudo-contractive in the intermediate sense mappings and explicit iteration process considered in this paper.

Remark 2.9. Theorem 2.1 also extends and improves the corresponding result of Acedo and Xu [2] from k-strictly pseudo-contraction mapping to the more general class of asymptotically k-strictly pseudo-contractive in the intermediate sense mappings and explicit iteration process considered in this paper.

Remark 2.10. Theorem 2.1 also extends and improves the corresponding result of Xu and Ori [22] from nonexpansive mapping to the more general class of asymptotically k-strictly pseudo-contractive in the intermediate sense mappings and explicit iteration process considered in this paper.

Remark 2.11. Theorem 2.2 extends and improves the corresponding result of Liu [9] in the following ways:

(i)A k-strictly asymptotically pseudo-contractive mapping is replaced by finite family of asymptotically k-strictly pseudo-contractive in the intermediate sense mappings.

(ii) The modified Mann iteration process is replaced by explicit iteration process for a finite family of mappings.

Remark 2.12. Theorem 2.4 extends and improves the corresponding result of Kim and Xu [7].

Remark 2.13. Theorem 2.4 also extends and improves Theorem 1.6 of Osilike and Akuchu [15] to the case of the more general class of asymptotically pseudocontractive mappings and explicit iteration process considered in this paper.

Remark 2.14. Theorem 2.5 extends Theorem 3.1 of Thakur [21] to the case of finite family of asymptotically k-strictly pseudo-contractive in the intermediate sense mappings and explicit iteration process considered in this paper.

Remark 2.15. Our results also extend the corresponding results of Sahu et al. [19] to the case of explicit iteration process considered in this paper.

Example 2.16. ([19]) Let $X = \mathbb{R}$ be a normed linear space and $C = [0, 1]$. For each $x \in C$, we define

$$T(x) = \begin{cases} kx, & \text{if } x \in [0, 1/2], \\ 0, & \text{if } x \in (1/2, 1], \end{cases}$$

where $0 < k < 1$. Then $T\colon C \to C$ is discontinuous at $x = 1/2$ and hence T is not Lipschitzian. Set $C_1 := [1, 1/2]$ and $C_2 := (1/2, 1]$. Hence

$$|T^n x - T^n y| = k^n |x - y| \leq |x - y|$$

for all $x, y \in C_1$ and $n \in \mathbb{N}$ and

$$|T^n x - T^n y| = 0 \leq |x - y|$$

for all $x, y \in C_2$ and $n \in \mathbb{N}$.

For $x \in C_1$ and $y \in C_2$, we have

$$\begin{aligned} |T^n x - T^n y| &= |k^n x - 0| = |k^n(x-y) + k^n y| \\ &\leq k^n|x-y| + k^n|y| \\ &\leq |x-y| + k^n, \quad \text{for all} \quad n \in \mathbb{N}. \end{aligned}$$

Thus,

$$\begin{aligned} |T^n x - T^n y|^2 &\leq (|x-y| + k^n)^2 \\ &\leq |x-y|^2 + k|x - T^n x - (y - T^n y)|^2 + k^n K, \end{aligned}$$

for all $x, y \in C$, $n \in \mathbb{N}$ and for some $K > 0$. Therefore, T is an asymptotically k-strictly pseudocontractive mapping in the intermediate sense.

Example 2.17. Let $X = \ell_2 = \{\bar{x} = \{x_i\}_{i=1}^{\infty} : x_i \in C, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, and let $\bar{B} = \{\bar{x} \in \ell_2 : \|x\| \leq 1\}$. Define $T\colon \bar{B} \to \ell_2$ by

$$T\bar{x} = (0, x_1^2, a_2 x_2, a_3 x_3, \dots)$$

where $\{a_j\}_{j=1}^{\infty}$ is a real sequence satisfying: $a_2 > 0$, $0 < a_j < 1$, $j \neq 2$, and $\prod_{j=2}^{\infty} a_j = 1/2$. Then

$$\begin{aligned} &\|T^n\bar{x} - T^n\bar{y}\|^2 \\ &\leq 2\Big(\prod_{j=2}^{n} a_j\Big)\|\bar{x} - \bar{y}\|^2 \\ &\leq 2\Big(\prod_{j=2}^{n} a_j\Big)\|\bar{x} - \bar{y}\|^2 + k\|(I - T^n)\bar{x} - (I - T^n)\bar{y}\|^2 + k^n Q \end{aligned}$$

for all $k \in (0, 1)$, $n \geq 2$, $\bar{x}, \bar{y} \in X$ and for some $Q > 0$. Since

$$\lim_{n\to\infty} 2\Big(\prod_{j=2}^{n} a_j\Big) = 1,$$

it follows that T is an asymptotically k-strictly pseudo-contractive mapping in the intermediate sense.

Acknowledgement

The author would like to thanks the referee for careful reading and valuable suggestions on the manuscript.

References

[1] Abbas, M., Jovanović, M., Radenović, S., Sretenović, A., Simić, S., Abstract metric spaces and approximating fixed points of a pair of contractive type mappings. J. Comput. Anal. Appl. 13(2) (2011), 243-253.

[2] Acedo, G.L., Xu, H.K., Iterative methods for strict pseudo-contractions in Hilbert spaces. Nonlinear Anal. 67 (2007), 2258-2271.

[3] Bruck, R., Kuczumow, T., Reich, S., Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Collo. Math. 65(2) (1993), 169-179.

[4] Círić, Lj., Rafiq, A., Radenović Stojan, Rajović Miloje, Ume Jeong Sheok, On Mann implicit iterations for strongly accretive and strongly pseudo-contractive mappings. Appl. Math. Comput. 198 (2008), 128-137.

[5] Dukić, D., Paunović, Lj., Radenović, S., Convergence of iterates with errors of unifomly quasi-Lipschitzian mappings in cone metric spaces. Kragujevac J. Math. 35(3) (2011), 399-410.

[6] Goebel, K., Kirk, W.A., A fixed point theorem for asymptotically nonexpansive mappings. Proc. Amer. Math. Soc. 35 (1972), 171-174.

[7] Kim, T.H., Xu, H.K., Convergence of the modified Mann's iteration method for asymptotically strictly pseudocontractive mapping. Nonlinear Anal. 68 (2008), 2828-2836.

[8] Kirk, W.A., Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type. Israel J. Math. 17 (1974), 339-346.

[9] Liu, Q., Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings. Nonlinear Anal. 26 (1996), 1835-1842.

[10] Mann, W.R., Mean value methods in iteration. Proc. Amer. Math. Soc. 4 (1953), 506-510.

[11] Martinez-Yanex, C., Xu, H.K., Strong convergence of the CQ method for fixed point processes. Nonlinear Anal. 64 (2006), 2400-2411.

[12] Marino, G., Xu, H.K., Weak and strong convergence theorems for strict pseudocontractions in Hilbert spaces. J. Math. Anal. Appl. 329 (2007), 336-346.

[13] Osilike, M.O., Iterative approximation of fixed points of asymptotically demicontractive mappings, Indian J. pure appl. Math. 29(12), December 1998, 1291-1300.

[14] Osilike, M.O., Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps. J. Math. Anal. Appl. 294(1) (2004), 73-81.

[15] Osilike, M.O., Akuchu, B.G., Common fixed points of a finite family of asymptotically pseudocontractive maps. Fixed Point Theory and Applications 2 (2004), 81-88.

[16] Osilike, M.O., Aniagbosor, S.C., Akuchu, B.G., Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces. PanAm. Math. J. 12 (2002), 77-88.

[17] Osilike, M.O., Udomene, A., Igbokwe, D.I., Akuchu, B.G., Demiclosedness principle and convergence theorems for k-strictly asymptotically pseudocontractive maps. J. Math. Anal. Appl. 326(2) (2007), 1334-1345.

[18] Reich, S., Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. 67 (1979), 274-276.

[19] Sahu, D.R., Xu, H.K., Yao, J.C., Asymptotically strict pseudocontractive mappings in the intermediate sense, Nonlinear Anal. TMA 70(10) (2009), 3502-3511.

[20] Sun, Z.H., Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings. J. Math. Anal. Appl. 286 (2003), 351-358.

[21] Thakur, B.S., Convergence of strictly asymptotically pseudo-contractions. Thai J. Math. 5(1) (2007), 41-52.

[22] Xu, H.K., Ori, R.G., An implicit iteration process for nonexpansive mappings. Numer. Funct. Anal. Optim. 22 (2001), 767-773.

Received by the editors December 30, 2012