Novi Sad J. Math.
Vol. 44, No. 1, 2014, 91-104

A GENERAL FIXED POINT THEOREM FOR PAIRS OF NON WEAKLY COMPATIBLE MAPPINGS IN G - METRIC SPACES

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Abstract. In this paper a general fixed point theorem for pairs of non weakly compatible pairs of mappings in G - metric spaces is proved. In the case of a single mapping some results from [2], [3], [11], [13], [17], [30] are obtained.

AMS Mathematics Subject Classification (2010): 54H25

Key words and phrases: G - metric space, common fixed point, implicit relation

1. Introduction

In [4] and [5] Dhage introduced a new class of generalized metric spaces, named D - metric spaces. Mustafa and Sims [14], [15] proved that most of the claims concerning the fundamental structures on D - metric spaces are incorrect and introduced an appropriate notion of D - metric space, named G - metric spaces. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G - metric spaces under certain conditions [1], [2], [16], [17], [29] and other papers.

In [19], [20] and other papers, the first author initiated the study of fixed points for mappings satisfying implicit relations. Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, Tychonoff spaces, probabilistic metric spaces, convex metric spaces, in two and three metric spaces, for single valued mappings, hybrid pair of mappings and set valued mappings. Quite recently, the method is used in the study of fixed points for mappings satisfying contractive conditions of integral type, in fuzzy metric spaces and intuitionistic metric spaces. There exists a vast literature in this topic which cannot be completely cited here. The method unified different types of contractive and extensive conditions, some proofs of fixed points theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

Quite recently, the present authors initiated the study of fixed points in G - metric spaces using implicit relations in [21], [23], [24], [25]. In [24] a general fixed point theorem for pairs of weakly compatible mappings in G - metric spaces is proved.

In this paper, a general fixed point theorem for pairs of non weakly compatible mappings in G - metric space is proved. In the case of a single mapping some results from [2], [3], [11], [13], [17] and [30] are obtained.

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2. Preliminaries

Definition 2.1 ([15]). Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

$(G_1) : G(x, y, z) = 0$ if $x = y = z$,

$(G_2) : 0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

$(G_3) : G(x, x, y) \leq G(x, y, z)$ for all $x, y \in X$ with $z \neq y$,

$(G_4) : G(x, y, z) = G(y, z, x) = G(z, x, y) = ...$ (symmetry in all three variables),

$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

The function G is called a G - metric and the pair (X, G) is called a G - metric space.

Note that if $G(x, y, z) = 0$ then $x = y = z$ [15].

Definition 2.2 ([15]). Let (X, G) be a G - metric space. A sequence (x_n) in X is said to be:

a) G - convergent if for $\varepsilon > 0$, there exist an $x \in X$ and $k \in \mathbb{N}$ such that for $m, n \geq k$, $m, n \in \mathbb{N}$, $G(x, x_n, x_m) < \varepsilon$.

b) G - Cauchy if for each $\varepsilon > 0$, there exist $k \in \mathbb{N}$ such that for all $m, n, p \in \mathbb{N}$ with $m, n, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$, that is $G(x_n, x_m, x_p) \to 0$ as $n, m, p \to \infty$.

A G - metric space is said to be G - complete if every G - Cauchy sequence is G - convergent.

Lemma 2.3 ([15]). *Let (X, G) be a G - metric space. Then the following properties are equivalent:*

1) (x_n) is G - convergent to x,

2) $G(x, x_n, x_n) \to 0$ as $n \to \infty$,

3) $G(x, x, x_n) \to 0$ as $n \to \infty$,

4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Lemma 2.4 ([15]). *Let (X, G) be a G - metric space. Then the following properties are equivalent:*

1) The sequence (x_n) is G - Cauchy;

2) For every $\varepsilon > 0$, there exist $k \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $m, n \geq k$, $G(x_n, x_m, x_m) < \varepsilon$.

Lemma 2.5 ([15]). *Let (X, G) be a G - metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Lemma 2.6 ([15]). *Let (X, G) be a G - metric space. Then $G(x, x, y) \leq 2G(x, y, y)$.*

3. Implicit relations

Definition 3.1. Let $\mathfrak{F}_G$ be the set of all real continuous functions $F(t_1, ..., t_6) : \mathbb{R}^6_+ \to \mathbb{R}$ such that

(F_1) : F is non - increasing in variable t_6,

(F_2) : There exists $h \in [0, 1)$ such that for all $u, v \geq 0$, $F(u, v, u, v, 0, u+v) \leq 0$ implies $u \leq hv$.

(F_3) : There exists $g \in [0, 1)$ such that for all $t, t' > 0$, $F(t, t, 0, 0, t', t) \leq 0$ implies $t \leq gt'$.

Example 3.2. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, where $k \in \left[0, \frac{1}{2}\right)$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, 0, u + v) = u - k \max u, v, u + v = h - k(u + v) \leq 0$. Hence $u \leq hv$, where $0 \leq h = \frac{k}{1-k} < 1$.

(F_3) : Let $t, t' > 0$ and $F(t, t, 0, 0, t', t) = t - k \max\{t, t'\} \leq 0$. If $t > t'$, then $t(1 - k) \leq 0$, a contradiction. Hence, $t \leq t'$ which implies $t \leq gt'$, where $0 \leq g = k < 1$.

Example 3.3. $F(t_1, ..., t_6) = t_1 - at_2 - bt_4 - c \max\{2t_3, t_5 + t_6\}$, where $a, b, c \geq 0$ and $0 < a + b + 2c < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, 0, u+v) = u - av - bv - c \max\{2u, u+v\} \leq 0$. If $u > v$, then $u(1 - (a + b + 2c)) \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a + b + 2c < 1$.

(F_3) : Let $t, t' > 0$ and $F(t, t, 0, 0, t', t) = t - at - c(t + t') \leq 0$ which implies $t \leq gt'$, where $0 \leq g = \frac{c}{a+c} < 1$.

Example 3.4. $F(t_1, ..., t_6) = t_1 - k \max\{t_3 + t_6, t_4 + t_5\} - at_2$, where $0 \leq a + 3k < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, 0, u + v) = u - av - k(2u + v) \leq 0$ which implies $u \leq hv$, where $0 \leq h = \frac{a+k}{1-2k} < 1$.

(F_3) : Let $t, t' > 0$ and $F(t, t, 0, 0, t', t) = t - at - k \max\{t, t'\} \leq 0$. If $t > t'$, then $t(1 - (a + k)) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq gt'$ where $0 \leq g = \frac{k}{1-a} < 1$.

Example 3.5. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \dfrac{t_3 + t_5}{2}, \dfrac{t_5 + t_6}{2}\right\}$, where $k \in [0, 1)$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, 0, u + v) = u - k \max\left\{u, v, \frac{u}{2}, \frac{u+v}{2}\right\} \leq 0$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k < 1$.

(F_3) : Let $t, t' > 0$ and $F(t, t, 0, 0, t', t) = t - k \max\{t, \frac{t'}{2}, \frac{t+t'}{2}\} \leq 0$. If $t > t'$, then $t(1 - k) \leq 0$, a contradiction. Hence $t \leq t'$, which implies $t \leq gt'$, where $0 \leq g = k < 1$.

Example 3.6. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \dfrac{t_5 + 2t_6}{3}, \dfrac{2t_3 + t_4}{3}\right\}$, where $k \in \left[0, \frac{3}{4}\right)$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ and $F(u,v,u,v,0,u+v) = u - k\max\{u, v, \frac{2(u+v)}{3}, \frac{2u+v}{3}\} \leq 0$. If $u > v$, then $u\left(1 - \frac{4k}{3}\right) \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \frac{4k}{3} < 1$.

(F_3) : Let $t,t' > 0$ and $F(t,t,0,0,t',t) = t - k\max\{t, , \frac{2t+t'}{3}\} \leq 0$. If $t > t'$, then $t(1-k) \leq 0$, a contradiction. Hence $t \leq t'$, which implies $t \leq gt'$, where $0 \leq g = k < 1$.

Example 3.7. $F(t_1, ..., t_6) = t_1 - at_2 - (b+d)t_3 - ct_4 - e\max\{t_3, t_5, t_6\}$, where $a,b,c,d,e \geq 0$ and $a+b+c+d+2e < 1$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ and $F(u,v,u,v,0,u+v) = u - av - (b+d)u - cv - e(u+v) \leq 0$ which implies $u \leq hv$, where $0 \leq h = \dfrac{a+c+e}{1-b-d-e} < 1$.

(F_3) : Let $t,t' > 0$ and $F(t,t,0,0,t',t) = t - at - e\max\{t,t'\} \leq 0$. If $t > t'$ then $t(1-(a+e)) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq gt'$, where $0 \leq g = \dfrac{e}{1-a} < 1$.

Example 3.8. $F(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3, t_4, t_5, t_6, t_1+t_3, t_3+t_6, 2t_5\}$, where $k \in \left[0, \frac{1}{3}\right)$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ and $F(u,v,u,v,0,u+v) = u - k\max\{u, v, u+v, 2u, 2u+v\} \leq 0$. If $u > v$ then $u(1-3k) > 0$, a contradiction. Hence $u \leq v$ which implies $u \leq hv$, where $0 \leq h = 3k < 1$.

(F_3) : Let $t,t' > 0$ and $F(t,t,0,0,t',t) = t - k\max\{t, t', 2t'\} \leq 0$. If $t > t'$, then $t(1-2k) \leq 0$, a contradiction. Hence, $t \leq t'$ which implies $t \leq gt'$, where $0 \leq g = 2k < 1$.

Example 3.9. $F(t_1, ..., t_6) = t_1 - at_2 - k\max\{t_3+t_5+t_6, 2t_3+t_4\}$, where $a,k \geq 0$ and $a+3k < 1$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ and $F(u,v,u,v,0,u+v) = u - av - k(2u+v) \leq 0$ which implies $u \leq hv$, where $0 \leq h = \dfrac{a+k}{1-2k} < 1$.

(F_3) : Let $t,t' > 0$ and $F(t,t,0,0,t',t) = t - at - k(t+t') \leq 0$ which implies $t \leq gt'$, where $0 \leq g = \frac{k}{1-a-k} < 1$.

Example 3.10. $F(t_1, ..., t_6) = t_1 - at_2 - bt_4 - k\max\{t_1+t_3+t_5+t_6, 2t_3+t_6\}$, where $a,b,k \geq 0$ and $a+b+4c < 1$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ and $F(u,v,u,v,0,u+v) = u - av - bv - k(3u+v) \leq 0$ which implies $u \leq hv$, where $0 \leq h = \frac{a+b+k}{1-3k} < 1$.

(F_3) : Let $t,t' > 0$ and $F(t,t,0,0,t',t) = t - at - k(2t+t') \leq 0$ which implies $t \leq gt'$, where $0 \leq g = \frac{k}{1-a-2k} < 1$.

Example 3.11. $F(t_1, ..., t_6) = t_1^2 - at_2^2 - \frac{bt_5t_6}{1+t_3^2+t_4^2}$, where $a,b > 0$ and $a+b < 1$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ and $F(u,v,u,v,0,u+v) = u^2 - av^2 \leq 0$, which implies $u \leq hv$, where $0 \leq h = \sqrt{a} < 1$.

(F_3) : Let $t, t' > 0$ and $F(t, t, 0, 0, t', t) = t^2 - at^2 - bt't \leq 0$, which implies $t \leq gt'$, where $0 \leq g = \dfrac{b}{1-a} < 1$.

Example 3.12. $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_3, t_5\} - d\max\{t_2, t_3, t_4, \frac{t_3+t_5}{2}\} - et_6$, where $a, b, c, d, e \geq 0$ and $0 \leq a + b + c + d + 2e < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, 0, u + v) = u - av - b\max\{u, v\} - cu - d\max\{v, u, \frac{u}{2}\} - e(u + v) \leq 0$. If $u > v$, then $u(1 - a - b - c - d - 2e) \leq 0$, a contradiction. Hence $u \leq v$ which implies $u \leq hv$, where $0 \leq h = \frac{a+b+d+e}{1-c-e} < 1$.

(F_3) : Let $t, t' > 0$ and $F(t, t, 0, 0, t', t) = t - at - ct' - d\max\{t, \frac{t'}{2}\} - et \leq 0$. If $t > t'$ then $t[1 - (a + c + d + e)] \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq gt'$ where $0 \leq g = a + c + d + e < 1$.

Example 3.13. $F(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3 + t_4, t_5 + t_6\}$, where $k \in \left[0, \frac{1}{2}\right)$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, 0, u + v) = u - k(u + v) \leq 0$, which implies $u \leq hv$, where $0 \leq h = k < 1$.

(F_3) : Let $t, t' > 0$ and $F(t, t, 0, 0, t', t) = t - k(t + t') \leq 0$ which implies $t \leq gt'$ where $0 \leq g = \frac{k}{1-k} < 1$.

Example 3.14. $F(t_1, ..., t_6) = t_1 - at_2 - k\max\{t_3, t_4, t_5, t_6\}$, where $a, k \geq 0$ and $a + 2k < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, 0, u + v) = u - k(u + v) \leq 0$, which implies $u \leq hv$, where $0 \leq h = \frac{a+k}{1-k} < 1$.

(F_3) : Let $t, t' > 0$ and $F(t, t, 0, 0, t', t) = t - at - k\max\{t, t'\} \leq 0$. If $t > t'$, then $t(1 - (a + k)) \leq 0$, a contradiction. Hence, $t \leq t'$ which implies $t \leq gt'$, where $0 \leq g = \frac{k}{1-a} < 1$.

Example 3.15. $F(t_1, ..., t_6) = t_1^4 - k\frac{t_3^2t_4^2 + t_5^2t_6^2}{1+t_2^2}$, where $k \in (0, 1)$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, 0, u + v) = u^4 - k\frac{u^2v^2}{1+v} \leq 0$. If $u > 0$, then $u^2 \leq k\frac{v^2}{1+v} \leq kv^2$. Hence $u \leq hv$, where $0 \leq h = \sqrt{k} < 1$. If $u = 0$, then $u \leq hv$.

(F_3) : Let $t, t' > 0$ and $F(t, t, 0, 0, t', t) = t^4 - k\frac{t'^2t^2}{1+t} \leq 0$, which implies $t^2 \leq k\frac{t'^2}{1+t} \leq kt'^2$. Hence, $t \leq gt'$, where $0 \leq g = \sqrt{k} < 1$.

4. Main results

Theorem 4.1. *Let (X, G) be a G - complete metric space and let $S, T : X \to X$ be two functions satisfying the following inequalities for all $x, y \in X$:*

$$\left\{ \begin{array}{c} \phi_1(G(Tx, Tx, Sy), G(x, x, y), G(x, Tx, Tx), \\ G(y, Sy, Sy), G(x, Sy, Sy), G(y, Tx, Tx)) \leq 0 \\ \phi_2(G(Sx, Sx, Ty), G(x, x, y), G(x, Sx, Sx), \\ G(y, Ty, Ty), G(x, Ty, Ty), G(y, Sx, Sx)) \leq 0 \end{array} \right. \tag{4.1}$$

where $\phi_1, \phi_2 \in \mathfrak{F}_G$. Then S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$ for $n = 0, 1, 2, \dots$. By (ϕ_1) we have successively:

$$\phi_1(G(Tx_{2n+1}, Tx_{2n+1}, Sx_{2n}), G(x_{2n+1}, x_{2n+1}, x_{2n}), G(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \\ G(x_{2n}, Sx_{2n}, Sx_{2n}), G(x_{2n+1}, Sx_{2n}, Sx_{2n}), G(x_{2n}, Tx_{2n+1}, Tx_{2n+1})) \leq 0,$$

$$\phi_1(G(x_{2n+2}, x_{2n+2}, x_{2n+1}), G(x_{2n+1}, x_{2n+1}, x_{2n}), G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ G(x_{2n}, x_{2n+1}, x_{2n+1}), 0, G(x_{2n}, x_{2n+2}, x_{2n+2})) \leq 0.$$

By (G_5) we have

$$G(x_{2n}, x_{2n+2}, x_{2n+2}) \leq G(x_{2n}, x_{2n+1}, x_{2n+1}) + G(x_{2n+1}, x_{2n+2}, x_{2n+2}).$$

By (F_1) and (G_4) we obtain

$$\phi_1(G(x_{2n+2}, x_{2n+2}, x_{2n+1}), G(x_{2n+1}, x_{2n+1}, x_{2n}), G(x_{2n+2}, x_{2n+2}, x_{2n+1}), \\ G(x_{2n+1}, x_{2n+1}, x_{2n}), 0, G(x_{2n+1}, x_{2n+1}, x_{2n}) + G(x_{2n+2}, x_{2n+2}, x_{2n+1})) \leq 0$$

which implies by (F_2) that

$$G(x_{2n+2}, x_{2n+2}, x_{2n+1}) \leq hG(x_{2n+1}, x_{2n+1}, x_{2n}),$$

where $h = \max\{h_1, h_2\}$. Similarly, by (ϕ_2) we have successively

$$\phi_2(G(Sx_{2n+2}, Sx_{2n+2}, Tx_{2n+1}), G(x_{2n+2}, x_{2n+2}, x_{2n+1}), \\ G(x_{2n+2}, Sx_{2n+2}, Sx_{2n+2}), G(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \\ G(x_{2n+2}, Tx_{2n+1}, Tx_{2n+1}), G(x_{2n+1}, Sx_{2n+2}, Sx_{2n+2})) \leq 0,$$

$$\phi_2(G(x_{2n+3}, x_{2n+3}, x_{2n+2}), G(x_{2n+2}, x_{2n+2}, x_{2n+1}), G(x_{2n+2}, x_{2n+3}, x_{2n+3}), \\ G(x_{2n+1}, x_{2n+2}, x_{2n+2}), 0, G(x_{2n+1}, x_{2n+3}, x_{2n+3})) \leq 0.$$

By (G_5) we have

$$G(x_{2n+1}, x_{2n+3}, x_{2n+3}) \leq G(x_{2n+1}, x_{2n+2}, x_{2n+2}) + G(x_{2n+2}, x_{2n+3}, x_{2n+3}).$$

By (F_1) and (G_4) we obtain

$$\phi_2(G(x_{2n+3}, x_{2n+3}, x_{2n+2}), G(x_{2n+2}, x_{2n+2}, x_{2n+1}), \\ G(x_{2n+3}, x_{2n+3}, x_{2n+2}), G(x_{2n+2}, x_{2n+2}, x_{2n+1}), \\ 0, G(x_{2n+2}, x_{2n+2}, x_{2n+1}) + G(x_{2n+3}, x_{2n+3}, x_{2n+2})) \leq 0$$

which implies by (F_2) that

$$G(x_{2n+3}, x_{2n+3}, x_{2n+2}) \leq hG(x_{2n+2}, x_{2n+2}, x_{2n+1}),$$

where $h = \max\{h_1, h_2\}$.

Hence

$$G(x_{n+1}, x_{n+1}, x_n) \leq hG(x_n, x_n, x_{n-1}) \text{ for } n = 1, 2, \dots,$$

which implies

$$G(x_{n+1}, x_{n+1}, x_n) \leq h^n G(x_1, x_1, x_0).$$

Then, for $m, n \in \mathbb{N}$, $m > n$, by repeating use of (G_5) we have

$$\begin{aligned} G(x_m, x_m, x_n) &\leq G(x_{n+1}, x_{n+1}, x_n) + G(x_{n+2}, x_{n+2}, x_{n+1}) + \\ &\quad +... + G(x_m, x_m, x_{m-1}) \\ &\leq (h^n + h^{n+1} + ... + h^{m-1})G(x_1, x_1, x_0) \\ &\leq \frac{h^n}{1-h} G(x_1, x_1, x_0). \end{aligned}$$

This implies that (x_n) is a G - Cauchy sequence. Since (X, G) is G - complete, there exists $u \in X$ such that (x_n) is G - convergent to u. We prove that u is a common fixed point for S and T.

By (ϕ_1) we have successively

$$\begin{gathered} \phi_1(G(Tu, Tu, Sx_{2n}), G(u, u, x_{2n}), G(u, Tu, Tu), \\ G(x_{2n}, Sx_{2n}, Sx_{2n}), G(u, Sx_{2n}, Sx_{2n}), G(x_{2n}, Tu, Tu)) \leq 0, \end{gathered}$$

$$\begin{gathered} \phi_1(G(Tu, Tu, x_{2n+1}), G(u, u, x_{2n}), G(u, Tu, Tu), \\ G(x_{2n}, x_{2n+1}, x_{2n+1}), G(u, x_{2n+1}, x_{2n+1}), G(x_{2n}, Tu, Tu)) \leq 0. \end{gathered}$$

Letting n tend to infinity and by (G_4) we obtain

$$\phi_1(G(Tu, Tu, u), 0, G(Tu, Tu, u), 0, 0, G(Tu, Tu, u)) \leq 0.$$

By (F_2) we obtain $G(Tu, Tu, u) = 0$ which implies $u = Tu$.

Similarly, by (ϕ_2) we have successively

$$\begin{gathered} \phi_2(G(Su, Su, Tx_{2n+1}), G(u, u, x_{2n+1}), \\ G(u, Su, Su), G(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \\ G(u, Tx_{2n+1}, Tx_{2n+1}), G(x_{2n+1}, Su, Su)) \leq 0, \end{gathered}$$

$$\begin{gathered} \phi_2(G(Su, Su, x_{2n+2}), G(u, u, x_{2n+1}), \\ G(u, Su, Su), G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ G(u, x_{2n+2}, x_{2n+2}), G(x_{2n+1}, Su, Su)) \leq 0. \end{gathered}$$

Letting n tend to infinity and (G_4) we obtain

$$\phi_2(G(Su, Su, u), 0, G(Su, Su, u), 0, 0, G(Su, Su, u)) \leq 0.$$

By (F_2) we obtain $G(Su, Su, u) = 0$ which implies $u = Su$.

Hence $u = Su = Tu$ and u is a common fixed point of S and T.

We prove that u is the unique common fixed point of S and T.

Let $v = Sv = Tv$ be another common fixed point of S and T.

Then, by (ϕ_1) and (G_4) we have successively

$$\begin{gathered} \phi_1(G(Tu, Tu, Sv), G(u, u, v), G(u, Tu, Tu), \\ G(v, Sv, Sv), G(u, Sv, Sv), G(v, Tu, Tu)) \leq 0 \end{gathered}$$

$$\phi_1(G(u, u, v), G(u, u, v), 0, 0, G(u, v, v), G(u, u, v)) \leq 0$$

which implies by (F_2) that

$$G(u, u, v) \leq h_1 G(v, v, u).$$

Similarly,

$$G(v,v,u) \leq h_1 G(u,u,v).$$

Hence

$$G(u,u,v)(1-h_1^2) \leq 0,$$

which implies $G(u,u,v)=0$, i.e. $u=v$. □

Remark 4.2. By Theorem 4.1, using examples 3.2 - 3.13 we obtain new particular results.

Theorem 4.3. *Let (X,G) be a G - complete metric space and let $T : X \to X$ be a function satisfying the following inequality for all $x,y \in X$:*

$$\begin{array}{c} \phi(G(Tx,Tx,Ty),G(x,x,y),G(x,Tx,Tx), \\ G(y,Ty,Ty),G(x,Ty,Ty),G(y,Tx,Tx)) \leq 0 \end{array} \tag{4.2}$$

where $\phi \in \mathfrak{F}_G$. Then T has a unique fixed point.

Corollary 4.4 (Theorem 2.1 [17])**.** *Let (X,G) be a G - complete metric space and let $T : X \to X$ be a function satisfying the following inequality for all $x,y,z \in X$:*

$$\begin{array}{c} G(Tx,Ty,Tz) \leq k\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty), \\ G(y,Tz,Tz),G(x,Ty,Ty),G(y,Tz,Tz),G(z,Tx,Tx)) \end{array} \tag{4.3}$$

where $k \in \left[0,\frac{1}{2}\right)$. Then T has a unique fixed point.

Proof. For $z=x$ we obtain by (4.3) that

$$\begin{array}{c} G(Tx,Tx,Ty) \leq k\max\{G(x,x,y),G(x,Tx,Tx),G(y,Ty,Ty), \\ G(x,Ty,Ty),G(y,Tx,Tx)) \leq 0. \end{array}$$

By Theorem 4.3 and Example 3.2, T has a unique fixed point. □

Corollary 4.5 (Theorem 3.1 [17])**.** *Let (X,G) be a G - complete metric space and let $T : X \to X$ be a function satisfying the following inequality for all $x,y,z \in X$:*

$$\begin{array}{c} G(Tx,Ty,Tz) \leq k\max\{G(x,Ty,Ty)+G(y,Tx,Tx), \\ G(y,Tz,Tz)+G(z,Ty,Ty),G(x,Tz,Tz)+G(z,Tx,Tx)) \end{array} \tag{4.4}$$

where $k \in \left[0,\frac{1}{2}\right)$. Then T has a unique fixed point.

Proof. For $z=x$ we obtain by (4.4) that

$$G(Tx,Tx,Ty) \leq k\max\{G(x,Ty,Ty)+G(y,Tx,Tx),2G(x,Tx,Tx)) .$$

By Theorem 4.3 and Example 3.3, T has a unique fixed point. □

Corollary 4.6 (Theorem 2.8 [17]). *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a function satisfying the following inequality for all $x, y, z \in X$:*

$$\begin{aligned} G(Tx, Ty, Tz) \leq k \max\{&G(z, Tx, Tx) + G(y, Tx, Tx), \\ &G(y, Tz, Tz) + G(x, Tz, Tz), G(x, Ty, Ty) + G(y, Ty, Ty)) \end{aligned} \tag{4.5}$$

where $k \in \left[0, \frac{1}{3}\right)$. Then T has a unique fixed point.

Proof. For $z = x$ we obtain by (4.5) that

$$\begin{aligned} G(Tx, Tx, Ty) \leq k \max\{&G(x, Tx, Tx) + G(y, Tx, Tx), \\ &G(x, Ty, Ty) + G(y, Ty, Ty)) \end{aligned}.$$

By Theorem 4.3 and Example 3.4, T has a unique fixed point. □

Corollary 4.7 (Theorem 2.1 [3]). *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a function satisfying the following inequality for all $x, y, z \in X$:*

(4.6)

$$\begin{aligned} G(Tx, Ty, Tz) \leq k \max\{&G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\ &\frac{G(x, Ty, Ty) + G(z, Tx, Tx)}{2}, \frac{G(x, Ty, Ty) + G(y, Tx, Tx)}{2}, \\ &\frac{G(y, Tz, Tz) + G(z, Ty, Ty)}{2}, \frac{G(x, Tz, Tz) + G(z, Tx, Tx)}{2}\} \end{aligned}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Proof. For $z = x$ we obtain by (4.6) that

$$\begin{aligned} G(Tx, Tx, Ty) \leq k \max\{&G(x, x, y), G(x, Tx, Tx), G(y, Ty, Ty), \\ &\frac{G(x, Ty, Ty) + G(x, Tx, Tx)}{2}, \frac{G(x, Ty, Ty) + G(y, Tx, Tx)}{2}\} \end{aligned}.$$

By Theorem 4.3 and Example 3.5, T has a unique fixed point. □

Corollary 4.8. *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a function satisfying the following inequality for all $x, y, z \in X$:*

(4.7)

$$\begin{aligned} G(Tx, Ty, Tz) \leq k \max\{&G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\ &\frac{G(y, Tx, Tx) + G(z, Ty, Ty) + G(y, Tz, Tz)}{3}, \\ &\frac{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)}{3}\} \end{aligned}$$

where $k \in \left[0, \frac{3}{4}\right)$. Then T has a unique fixed point.

Proof. For $z = x$ we obtain by (4.7) that

$$\begin{aligned} G(Tx, Tx, Ty) \leq k \max\{&G(x, x, y), G(x, Tx, Tx), G(y, Ty, Ty), \\ &\frac{2G(y, Tx, Tx) + G(x, Ty, Ty)}{3}, \frac{2G(x, Tx, Tx) + G(y, Ty, Ty)}{3}\} \end{aligned}.$$

By Theorem 4.3 and Example 3.6, T has a unique fixed point. □

Remark 4.9. This Corollary is a generalization of Theorem 2.6 [2], where $h \in \left[0, \frac{1}{2}\right)$.

Corollary 4.10 (Theorem 3.1 [13]). *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a mapping such that:*

$$
\begin{array}{c}
G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(x, Tx, Tx) \\
+cG(y, Ty, Ty) + dG(z, Tz, Tz) \\
+e \max\{G(x, Ty, Ty), G(y, Tx, Tx), G(y, Tz, Tz), \\
G(z, Ty, Ty), G(z, Tx, Tx), G(x, Tz, Tz)\}
\end{array}
\tag{4.8}
$$

for all $x, y, z \in X$, where $a, b, c, d, e \geq 0$ and $a + b + c + d + 2e < 1$. Then T has a unique fixed point.

Proof. For $z = x$ by (4.8) we obtain

$$
\begin{array}{c}
G(Tx, Tx, Ty) \leq aG(x, x, y) + bG(x, Tx, Tx) \\
+cG(y, Ty, Ty) + dG(x, Tx, Tx) \\
+e \max\{G(x, Ty, Ty), G(y, Tx, Tx), G(x, Tx, Tx)\}.
\end{array} .
$$

By Theorem 4.3 and Example 3.7, T has a unique fixed point. ☐

Corollary 4.11 (Theorem 3.7 [13]). *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a mapping such that:*

$$
\begin{array}{c}
G(Tx, Ty, Tz) \leq k \max\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\
G(x, Ty, Ty), G(y, Tz, Tz), G(z, Tx, Tx), G(x, Tz, Tz), \\
G(y, Tx, Tx), G(z, Ty, Ty), G(x, Ty, Ty), G(y, Tz, Tx), \\
G(z, Tx, Ty), G(x, y, Tz), G(y, z, Tx), G(z, x, Ty), G(x, y, z)\},
\end{array}
\tag{4.9}
$$

for all $x, y, z \in X$, where $k \in \left[0, \frac{1}{3}\right)$. Then T has a unique fixed point.

Proof. For $z = x$ by (4.9) we obtain

$$
\begin{array}{c}
G(Tx, Tx, Ty) \leq k \max\{G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), \\
G(y, Tx, Tx), G(x, Ty, Tx), G(x, y, Tx), G(x, x, Ty), G(x, x, y)\}.
\end{array} .
$$

By (G_5) and Lemma 2.6 we obtain

$$
\begin{array}{c}
G(Tx, Tx, Ty) \leq k \max\{G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), \\
G(y, Tx, Tx), G(x, Tx, Tx) + G(Tx, Tx, Ty), \\
G(x, Tx, Tx) + G(y, Tx, Tx), 2G(x, Ty, Ty), G(x, x, y)\}.
\end{array}
$$

By Theorem 4.3 and Example 3.8, T has a unique fixed point. ☐

Corollary 4.12. *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a mapping such that for all $x, y, z \in X$:*

(4.10)

$$
\begin{array}{c}
G(Tx, Ty, Tz) \leq aG(x, y, z) + k \max\{G(x, Ty, Ty) + G(y, Tx, Tx) + \\
+G(z, Tz, Tz), G(y, Tz, Tz) + G(z, Ty, Ty) + G(x, Tx, Tx), \\
G(z, Tx, Tx) + G(x, Tz, Tz) + G(y, Ty, Ty)\},
\end{array}
$$

where $a, k \geq 0$ and $a + 3k < 1$. Then T has a unique fixed point.

Proof. For $z = x$ by (4.10) we obtain

$$\begin{gathered} G(Tx, Tx, Ty) \leq aG(x, x, y) \\ +k \max\{G(x, Ty, Ty) + G(y, Tx, Tx) + G(x, Tx, Tx), \\ 2G(x, Tx, Tx) + G(y, Ty, Ty)\}, \end{gathered}$$

By Theorem 4.3 and Example 3.9, T has a unique fixed point. □

Remark 4.13. If $a = 0$, by Corollary 4.12 we obtain Theorem [13].

Corollary 4.14. *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a mapping such that for all $x, y, z \in X$:*

$$\begin{gathered} G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(y, Ty, Ty) \\ +k \max\{G(x, Ty, Ty) + G(y, Tx, Tx) + G(z, Tx, Ty), \\ G(y, Tz, Tz) + G(z, Ty, Ty) + G(x, Ty, Tz), \\ G(z, Tx, Tx) + G(x, Tz, Tz) + G(y, Tz, Tx)\}, \end{gathered} \tag{4.11}$$

where $a, b, k \geq 0$ and $a + b + 4k < 1$. Then T has a unique fixed point.

Proof. For $z = x$ and (G_5) we obtain

$$\begin{gathered} G(Tx, Tx, Ty) \leq aG(x, x, y) + bG(y, Ty, Ty) \\ +k \max\{G(x, Ty, Ty) + G(y, Tx, Tx) + G(x, Tx, Ty), \\ 2G(x, Tx, Tx) + G(y, Tx, Tx)\} \\ \leq aG(x, x, y) + bG(y, Ty, Ty) \\ +k \max\{G(x, Ty, Ty) + G(y, Tx, Tx) + \\ G(x, Tx, Tx) + G(Tx, Tx, Ty), \\ 2G(x, Tx, Tx) + G(y, Tx, Tx)\}. \end{gathered}$$

By Theorem 4.3 and Example 3.10, T has a unique fixed point. □

Remark 4.15. If $a = b = 0$, by Corollary 4.14 we obtain Theorem 3.11 [13].

Corollary 4.16. *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a mapping such that for all $x, y, z \in X$:*

$$\begin{gathered} G(Tx, Ty, Tz) \leq aG(x, y, z) + b \max\{G(x, Tx, Tx), G(y, Ty, Ty)\} \\ +c \max\{G(x, Tz, Tz), G(x, Ty, Ty)\} \\ +d \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), \\ \tfrac{G(x,Ty,Ty)+G(x,Tz,Tz)}{2}\} + eG(y, Tx, Tx), \end{gathered} \tag{4.12}$$

where $a, b, c, d, e \geq 0$ and $a + b + c + d + 2e < 1$. Then T has a unique fixed point.

Proof. For $z = x$ by (4.12) we obtain

$$\begin{gathered} G(Tx, Tx, Ty) \leq aG(x, x, y) + b \max\{G(x, Tx, Tx), G(y, Ty, Ty)\} \\ +c \max\{G(x, Tx, Tx), G(x, Ty, Ty)\} \\ +d \max\{G(x, x, y), G(x, Tx, Tx), G(y, Ty, Ty), \\ \tfrac{G(x,Ty,Ty)+G(x,Tx,Tx)}{2}\} + eG(y, Ty, Ty). \end{gathered}$$

By Theorem 4.3 and Example 3.12, T has a unique fixed point. □

Remark 4.17. Corollary 4.16 is a generalization of Theorem 3.2 [11] because in this theorem $a+b+2c+2d<1$ and $e=0$.

Corollary 4.18 (Corollary 5 [11]). *Let (X,G) be a G - complete metric space and let $T:X\to X$ be a mapping such that for all $x,y,z\in X$:*

$$G(Tx,Ty,Tz)\leq k\max\{G(x,y,z),G(x,Tx,Tx)+G(y,Ty,Ty),\\ G(x,Ty,Ty)+G(y,Tx,Tx)\}, \tag{4.13}$$

where $k\in\left[0,\frac{1}{2}\right)$. Then T has a unique fixed point.

Proof. For $z=x$ by (4.13) we obtain

$$G(Tx,Tx,Ty)\leq k\max\{G(x,x,y),G(x,Tx,Tx)+G(y,Ty,Ty),\\ G(x,Ty,Ty)+G(y,Tx,Tx)\}.$$

By Theorem 4.3 and Example 3.13, T has a unique fixed point. □

Corollary 4.19. *Let (X,G) be a G - complete metric space and let $T:X\to X$ be a mapping such that for all $x,y,z\in X$:*

$$G(Tx,Ty,Tz)\leq aG(x,y,z)+k\max\{G(x,Tx,Tx),G(x,Ty,Ty),\\ G(x,Tz,Tz),G(y,Ty,Ty),G(y,Tx,Tx),\\ G(y,Tz,Tz),G(z,Tz,Tz),G(z,Tx,Tx),G(z,Ty,Ty)\}, \tag{4.14}$$

where $x,y,z\in X$, where $a,k\geq 0$ and $0<a+2k<1$. Then T has an unique fixed point.

Proof. For $z=x$ by (4.14) we obtain

$$G(Tx,Tx,Ty)\leq aG(x,x,y)+k\max\{G(x,Tx,Tx),\\ G(y,Ty,Ty),G(x,Ty,Ty),G(y,Tx,Tx)\},$$

and the proof it follows by Theorem 4.1 and Example 3.14. □

Remark 4.20. If $a=0$ we obtain Theorem 1 [30].

Note. 1) The notions of quasi - metric spaces are introduced in [31]. Some fixed point theorems for mappings in quasi - metric spaces are proved in [6], [7], [8], [12], [26], [27] and other papers.

Let (X,Q) be a quasi - metric space, where $q(x,y)$ is a quasi - metric on X. In the proofs of Theorem 1 [26] and Theorem 1 [27] is used that $d(x,y)=\max\{q(x,y),q(y,x)\}$ is a metric on X.

Let (X,G) be a G - metric space. It is proved in Theorem 2.1 [9], Lemma 5.1 [22], Lemma 2.4 [25] that $G(x,x,y)$ is a quasi - metric $q(x,y)$ on X, hence $D(x,y)=\max\{G(x,x,y),G(y,y,x)\}$ is a metric on X.

Using this fact, the study of ([9], Theorem 3.1) and ([28], Theorems 17, 19, 21) is reduced to the study of fixed points in metric spaces.

2) Let $F(t_1,...,t_6)\leq 0$ be a contractive condition in G - metric spaces. To use a technique as in [9], [28] it is necessary that $F(t_1,...,t_6)$ to be non - increasing in variables $t_2,...,t_6$.

In the present paper, F is non - increasing only in variable t_6. In Example 3.11 F is non - decreasing in variables t_3 and t_4 and in Example 3.15, F is non - decreasing in variable t_2. Hence, Theorems 4.1 and 4.3 cannot be derived as the results from [9], [28].

3) In [10] there exists a function F non - increasing in variables $t_2, t_3, ...$ which is not derived as the results from [9], [28].

Acknowledgement

The authors thank the anonymous referee for his/her suggestion, that improve the initial version of the paper.

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Received by the editors February 13, 2013