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HARMONIC MAPS BETWEEN TWO HOLOMORPHIC JETS BUNDLE

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Abstract. In the geometry of the holomorphic jets bundle of order two $J^{(2,0)}M$, we have studied a special linear connection, named the Chern-Lagrange connection and, with respect to this, the geodesic curves are characterized in [20, 21, 22].

In the present paper, we consider a holomorphic function $f : M \to N$ between two complex manifolds, which carries the curves from $J^{(2,0)}M$ into curves on $J^{(2,0)}N$, and we find when this mapping is harmonic. We prove that if a curve on $J^{(2,0)}M$ is a geodesic and f defines a harmonic map between $J^{(2,0)}M$ and $J^{(2,0)}N$, then the f-image of this curve on $J^{(2,0)}N$ is a geodesic too.

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1. Introduction

The geometry of harmonic maps is vast and it knows numerous and various applications. Many results are obtained for Riemannian manifolds and their submanifolds (for general references, see [6, 8, 14] etc.).

A harmonic map between Riemannian manifolds is a smooth map such that, the divergence of its differential vanishes. More exactly, if $f : M \to N$ is a smooth map, with (M, g) a compact Riemannian manifold and N with a Riemannian metric h, first of all we define the *energy* functional

$$E(f) = \frac{1}{2}\int_M |df|^2 d\mu = \frac{1}{2}\int_M g^{ij}(x) h_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} d\mu$$

and its extrema, via the Euler-Lagrange variation, which leads to the *tension* field of f,

$$\tau(f) = div(df) = g^{ij}(\nabla df)_{ij}.$$

Thus, f is a *harmonic map* iff the tension $\tau(f)$ vanishes and, if $\nabla df = 0$ then f is said to be *totally geodesic.* As we know, if f is totally geodesic then it linearly maps the parameterized geodesic curves of M into parameterized geodesic curves of N. Obviously, any totally geodesic mapping is harmonic.

The notion of harmonic maps has been extended in last decade to the real and complex Finsler manifolds, see [11, 13, 17]. Mainly, the geometry of harmonic maps of Finsler manifolds encounters a difficulty in the fact that the energy is defined on the non-compact domain TM. Thus, the integral of

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the energy is considered on the indicatrix bundle, or on the projective sphere bundle.

In this note, we shall extend the theory of the harmonic maps on the parameterized curves of two holomorphic $(2,0)$ - jets bundle, avoiding the difficulty related to the problem of compactness.

In the geometry of the real k - jets bundle, significant results have been obtained by R. Miron' s school and his collaborators, [9, 3, 5] etc. The geometry of holomorphic $(k,0)$ - jets bundle has many similarities with this geometry, but also differences.

In previous papers [20, 21], we studied some aspects concerning the geometry of holomorphic $(2,0)$ - jets bundles $J^{(2,0)}M$. In [22], we extended the approach made in [1], to the geodesic parameterized curves on $(2,0)$ - jet bundles.

Here, considering a mapping $f : J^{(2,0)}M \to J^{(2,0)}N$, we obtain a curve on $J^{(2,0)}N$, from a geodesic curve on $J^{(2,0)}M$. The necessary and sufficient conditions that f should be a harmonic map are established, (Theorem 2.1). Taking into account the Corollary 2.1, we prove that f is a harmonic map if we obtain a geodesic curve using f.

1.1. Geometry of $J^{(2,0)}M$

We further summarize some basic notions on the geometry of $J^{(2,0)}M$ bundle.

Let M be a complex manifold, $\dim_{\mathbf{C}} M = n$, (z^i) be complex coordinates in a local chart. The complexified tangent bundle $T_{\mathbf{C}}M$ admits the classical decomposition $T_{\mathbf{C}}M = T'M \oplus T''M$, where $T'M$ is the holomorphic vector bundle over M (also denoted by $T^{(1,0)}M$) and its conjugate $T''M$ is the anti-holomorphic tangent bundle.

The holomorphic bundle of k-th *order jets differential* was introduced by Green and Griffiths in [7, 16], as the classes of sheaves of germs of holomorphic curves $\{f : \Delta_r \to M,\ f \in \mathcal{H}_{z_0},\ f(0) = z_0\}$, depending on a complex parameter θ, whose partial derivatives up to order k coincide. A detailed construction of $\pi^{(k,0)} : J^{(k,0)}M \to M$ fibre bundle structure, was discussed in [20]. $J^{(k,0)}M$ has a structure of complex differentiable manifold, whose geometry will be briefly recalled bellow for $k = 2$.

On the complex manifold $J^{(2,0)}M$, in a local chart, the coordinates are denoted by $Z = (z^k, \eta^k, \zeta^k)$, $k = \overline{1,n}$, and their changes are according to the following rules

$$
\begin{aligned}
z'^i &= z'^i(z); \\
\eta'^i &= \frac{\partial z'^i}{\partial z^j}\eta^j; \\
2\zeta'^i &= \frac{\partial \eta'^i}{\partial z^j}\eta^j + 2\frac{\partial \eta'^i}{\partial \eta^j}\zeta^j.
\end{aligned} \tag{1.1}
$$

A local base in the holomorphic bundle $T'(J^{(2,0)}M)$ is $\left\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^i}\right\}$. The changes of the local base are made according to the following rules

$$(1.2)\qquad \begin{aligned} \frac{\partial}{\partial z^j} &= \frac{\partial z'^i}{\partial z^j}\frac{\partial}{\partial z'^i} + \frac{\partial \eta'^i}{\partial z^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial z^j}\frac{\partial}{\partial \zeta'^i};\\ \frac{\partial}{\partial \eta^j} &= \frac{\partial \eta'^i}{\partial \eta^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial \eta^j}\frac{\partial}{\partial \zeta'^i};\\ \frac{\partial}{\partial \zeta^j} &= \frac{\partial \zeta'^i}{\partial \zeta^j}\frac{\partial}{\partial \zeta'^i}. \end{aligned}$$

By conjugation everywhere in (1.2), we obtain the corresponding conjugate base from $T''_z(J^{(2,0)}M)$.

A complex nonlinear connection, *(c.n.c.)* in brief, is given by a distribution $H(J^{(2,0)}M)$ at every point $Z \in J^{(2,0)}M$, which is supplementary to $V_1(J^{(2,0)}M)$ in $T'(J^{(2,0)}M)$, where $V_{1\,Z}(J^{(2,0)}M)$ is spanned by $\left\{\frac{\partial}{\partial \eta^j}, \frac{\partial}{\partial \zeta^j}\right\}$ in a local chart. By $V_2(J^{(2,0)}M)$ we denote the vertical bundle which is spanned in Z by $\left\{\frac{\partial}{\partial \zeta^j}\right\}$. By conjugation, we obtain the decomposition for $T_C(J^{(2,0)}M)$. A local base in $H_Z(J^{(2,0)}M)$ is $\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - \overset{(1)}{N}{}^i_j\frac{\partial}{\partial \eta^i} - \overset{(2)}{N}{}^i_j\frac{\partial}{\partial \zeta^i}$ and it is called *the adapted base* of the (c.n.c.) iff $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$. If F is the natural almost tangent structure on $J^{(2,0)}M$, (given by $F(\frac{\partial}{\partial z^j}) = \frac{\partial}{\partial \eta^j}$; $F(\frac{\partial}{\partial \eta^j}) = \frac{\partial}{\partial \zeta^j}$; $F(\frac{\partial}{\partial \zeta^j}) = 0$, which sends $H(J^{(2,0)}M)$ into $V_1(J^{(2,0)}M)$ and this into $V_2(J^{(2,0)}M) = \ker F$), then $F(\frac{\delta}{\delta z^j}) =: \frac{\delta}{\delta \eta^j} = \frac{\partial}{\partial \eta^j} - \overset{(1)}{N}{}^i_j\frac{\partial}{\partial \zeta^i}$ spans a local adapted base in $V_{1\,Z}(J^{(2,0)}M)$. For other details see [20]. Further on, we use for the adapted base $\{\frac{\delta}{\delta z^i}, \frac{\delta}{\delta \eta^i}, \frac{\partial}{\partial \zeta^i}\}$ the abbreviations $\{\delta_{0i} := \frac{\delta}{\delta z^i}, \delta_{1i} := \frac{\delta}{\delta \eta^i}, \delta_{2i} := \frac{\partial}{\partial \zeta^i}\}$. The adapted base on $T''(J^{(2,0)}M)$ is obtained by conjugation. Let $\{dz^i, \delta\eta^i, \delta\zeta^i\}_{i=\overline{1,n}}$ be the dual of the obtained adapted base.

A metric structure on $J^{(2,0)}M$ is given in general form by

$$(1.3)\qquad G = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j + g_{i\bar{j}}\delta\zeta^i \otimes \delta\bar{\zeta}^j$$

where $g_{i\bar{j}}$ is the metric tensor of a complex Lagrangian function $L : J^{(2,0)}M \to \mathbf{R}$ by $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \zeta^i \partial \bar{\zeta}^j}$, see [21].

Finally, we recall that in [20] a special derivative law on $J^{(2,0)}M$ was introduced, namely the *normal complex linear connection*, N-(c.l.c.), which preserves the distributions and has some special properties. A N - (c.l.c.) is well given in an adapted frame by a set of coefficients $D\Gamma = (L^i_{jk}, L^{\bar{i}}_{\bar{j}k}, F^i_{jk}, F^{\bar{i}}_{\bar{j}k}, C^i_{jk}, C^{\bar{i}}_{\bar{j}k})$, Locally, for $a = 1, 2, 3$ its coefficients are given by: $D_{\delta_{0k}}\delta_{aj} = L^i_{jk}\delta_{ai}$; $D_{\delta_{0k}}\delta_{a\bar{j}} = L^{\bar{i}}_{\bar{j}k}\delta_{a\bar{i}}$; $D_{\delta_{1k}}\delta_{aj} = F^i_{jk}\delta_{ai}$; $D_{\delta_{1k}}\delta_{a\bar{j}} = F^{\bar{i}}_{\bar{j}k}\delta_{a\bar{i}}$; $D_{\delta_{2k}}\delta_{aj} = C^i_{jk}\delta_{ai}$ and $D_{\delta_{2k}}\delta_{a\bar{j}} = C^{\bar{i}}_{\bar{j}k}\delta_{a\bar{i}}$.

In a second order complex Lagrange space there exists a special complex nonlinear connection named the Chern-Lagrange connection, [21] , where

$$(1.4)\qquad \overset{(1)CL}{M^i_j} = g^{\overline{m}i}\frac{\partial^2 L}{\partial \eta^j \partial \overline{\zeta}^m} \quad ; \quad \overset{(2)CL}{M^i_j} = g^{\overline{m}i}\frac{\partial^2 L}{\partial z^j \partial \overline{\zeta}^m}$$

and with respect to its adapted frame the following connection named the *Chern-Lagrange complex linear connection*, given by

$$(1.5)\qquad L^i_{jk}=g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k};\ \ F^i_{jk}=g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta \eta^k};\ \ C^i_{jk}=g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta \zeta^k}$$

and $L^{\bar{\imath}}_{\bar{j}k}=F^{\bar{\imath}}_{\bar{j}k}=C^{\bar{\imath}}_{\bar{j}k}=0$, i.e. D is of $(1,0)$ - type, is a normal complex linear connection. Moreover, the Chern-Lagrange N - (c.l.c.) is one metrical, i.e., $DG=0$.

1.2. Curves on $J^{(2,0)}M$

Let $\sigma:[a,b]\to M$, $\sigma(t)=(z^k(t))$, be a regular curve on M, which can be extended to a curve on $J^{(2,0)}M$ as a differentiable map $c:[a,b]\to J^{(2,0)}M$ given by $t\to(z^k(t),\eta^k(t),\zeta^k(t))$, where $\eta^k(t)=\frac{dz^k}{dt}$ and $\zeta^k(t)=\frac{1}{2}\frac{d^2z^k}{dt^2}$.

The tangent vector field of the curve c is given by $V:=\frac{dc}{dt}$, which, in the adapted frame of a (c.n.c.), is written as

$$(1.6)\ \ \frac{dc}{dt}:=\dot{c}_t+\overline{\dot{c}_t}=\frac{dz^k}{dt}\delta_{0k}+\frac{\delta\eta^k}{dt}\delta_{1k}+\frac{\delta\zeta^k}{dt}\delta_{2k}+\frac{d\bar{z}^k}{dt}\delta_{0\bar{k}}+\frac{\delta\bar{\eta}^k}{dt}\delta_{1\bar{k}}+\frac{\delta\bar{\zeta}^k}{dt}\delta_{2\bar{k}}.$$

It follows that $V=\overset{(p)}{V^k}\delta_{pk}+\overset{(p)}{\bar{V}^k}\delta_{p\bar{k}}=\dot{c}_t+\overline{\dot{c}_t}$, where $\overset{(0)}{V^k}=\frac{dz^k}{dt}$, $\overset{(1)}{V^k}=\frac{\delta\eta^k}{dt}$, $\overset{(2)}{V^k}=\frac{\delta\zeta^k}{dt}$ are the derivations with respect to the parameter $t\in\mathbf{R}$, along the curve c of the (c.n.c.) dual base and $\overset{(p)}{\bar{V}^k}$ are obtained by conjugation in $\overset{(p)}{V^k}$.

We note that $\pi:T'M\to M$ is a vector bundle structure, while $\pi_2:J^{(2,0)}M\to M$ and $\pi_1:J^{(2,0)}M\to T'M$ are only fibre bundles. Of course, $\pi_2(c)=\sigma$.

Let us consider $\Sigma:[-\varepsilon,\varepsilon]\times[a,b]\to M$ a variation $(s,t)\to\Sigma^k(s,t)\equiv z^k(s,t)$ of σ with $\Sigma(0,t)=\sigma(t)$. For a fixed s, the curves Σ define the curves $\Gamma(s,t)$ on $J^{(2,0)}M$ and we have the following commutative diagram

$$(1.7)\qquad \begin{array}{ccc}\gamma^*(J^{(2,0)}M) & \xrightarrow{\Gamma} & J^{(2,0)}M\\ \downarrow p_1 & & \downarrow \pi_1\\ \Sigma^*(T'M) & \xrightarrow{\gamma} & T'M\\ \downarrow p & & \downarrow\pi\\ [-\varepsilon,\varepsilon]\times[a,b] & \xrightarrow{\Sigma} & M\end{array}$$

where $\Sigma^*(T'M)$ and $\gamma^*(J^{(2,0)}M)$ are the pull-back bundles of $T'M$ and $J^{(2,0)}M$, using the maps Σ and γ, respectively. Thus we have the maps $\gamma:(s,t)\to\left(z^k(s,t),\eta^k(s,t)\right)$ and $\Gamma:(s,t)\to\left(z^k(s,t),\eta^k(s,t),\zeta^k(s,t)\right)$. Taking into account these curves, we have two kinds of vectors, which are well defined with respect to the adapted base of a (c.n.c),

$\frac{d}{dt}=\dot{c}_t(s)+\overline{\dot{c}_t}(s)=\overset{(p)}{V^k}(s)\ \delta_{pk}+\overset{(p)}{\bar{V}^k}(s)\ \delta_{p\bar{k}}$ and

$\frac{d}{ds}=\dot{c}_s(t)+\overline{\dot{c}_s}(t)=\overset{(p)}{U^k}(t)\ \delta_{pk}+\overset{(p)}{\bar{U}^k}(t)\ \delta_{p\bar{k}}.$

We consider the curves Σ with fixed endpoints, $\Gamma(s,a) = c(a)$, $\Gamma(s,b) = c(b)$ and $V(a) = V(b) = 0$.

Now, we define the *energy of the Lagrangian* structure, along the curve $c : t \to (z^k(t), \eta^k(t), \zeta^k(t))$, with fixed endpoints,

$$\mathcal{G}(c) = \int_a^b G(\dot{c}_t, \overline{\dot{c}_t})dt, \tag{1.8}$$

where G is given by (1.3), with respect to adapted frames of Chern-Lagrange (c.n.c), (1.4).

The condition which describes the extreme values of the energy function is

$$\frac{1}{2}\int_a^b \frac{dG(\dot{c}_t, \overline{\dot{c}_t})}{ds}dt = 0$$

and its solutions are called *the geodesic curves* of the metric structure G on $J^{(2,0)}M$. This computation lead us to the relation

$$\frac{dG(\dot{c}_t, \overline{\dot{c}_t})}{ds}|_{s=0} = \sum_{p=0,1,2} (\overset{(p)}{U} + \overset{(p)}{\overline{U}})G(\overset{(p)}{V}, \overset{(p)}{\overline{V}}).$$

Using the Chern-Lagrange linear connection D, (1.5), with the torsion $\mathbf{T}(X,Y)$, which is one metrical connection, we have proved:

Theorem 1.1. *[21]. The geodesic curves of the metric G satisfy the equations*

$$Re\ G(\dot{c}_s, D_{\frac{dc}{dt}}\overline{\dot{c}_t}) = Re\ G(\mathbf{T}(\frac{dc}{ds}, \dot{c}_t), \overline{\dot{c}_t}). \tag{1.9}$$

2. Harmonic maps

Let M and N be two complex manifolds of dimensions m and n, respectively. Consider $J^{(2,0)}M$ and $J^{(2,0)}N$ the holomorphic $(2,0)$ - jets bundles over M and N. We assume that $J^{(2,0)}M$ is endowed with a Lagrangian structure $L_M : J^{(2,0)}M \to \mathbf{R}$, which determines a metric structure G as in (1.3). Similarly, $J^{(2,0)}N$ is endowed with a Lagrangian structure $L_N : J^{(2,0)}N \to \mathbf{R}$ which determines a metric structure H.

As complex manifolds, we denote by (z^i, η^i, ζ^i) , $i = \overline{1,m}$ and $(u^\alpha, v^\alpha, w^\alpha)$, $\alpha = \overline{1,n}$ the complex coordinates on $J^{(2,0)}M$ and $J^{(2,0)}N$, respectively.

Now, let $f : M \to N$ be a homomorphic function, given locally by $u^\alpha = f^\alpha(z)$, $\alpha = \overline{1,n}$, and $\sigma : t \to (z^i(t))$ a curve on M. By the holomorphy, we have $\frac{\partial f^\alpha}{\partial \bar{z}^i} = 0$.

The map f can be lifted, (see [15]), to a map $f'^* : T'M \to T'N$, defined by $\eta^i \frac{\partial}{\partial z^i} \xrightarrow{f'^*} \eta^i \frac{\partial f^\alpha}{\partial z^i}\frac{\partial}{\partial v^\alpha}$. Thus, on $T'N$ we can consider the following complex coordinates $(u^\alpha = f^\alpha(z),\ v^\alpha = \eta^i \frac{\partial f^\alpha}{\partial z^i})$. By conjugation everywhere, this reasoning can be considered on $T''N$.

This idea can be extended to the whole $J^{(2,0)}N$.

Proposition 2.1. *The map* $f'^{**} : J^{(2,0)}M \to J^{(2,0)}N$ *given locally by*

$$(z^i, \eta^i, \zeta^i) \xrightarrow{f'^{**}} (u^\alpha = f^\alpha(z),\ v^\alpha = \eta^i \frac{\partial u^\alpha}{\partial z^i},\ w^\alpha = \zeta^i \frac{\partial v^\alpha}{\partial \eta^i}) \tag{2.1}$$

is well defined.

Proof. Using (1.1) and (1.2), we have,

$$v'^\alpha = \eta'^i \frac{\partial u'^\alpha}{\partial z'^i} = \eta^j \frac{\partial z'^i}{\partial z^j} \frac{\partial f'^\alpha}{\partial z'^i} = \eta^j \frac{\partial f'^\alpha}{\partial z^j} = \eta^j \frac{\partial u'^\alpha}{\partial z^j} = \eta^j \frac{\partial u^\beta}{\partial z^j} \frac{\partial u'^\alpha}{\partial u^\beta} = \frac{\partial u'^\alpha}{\partial u^\beta} v^\beta.$$

Next, since f^α depends only on z, we have $w^\alpha = \zeta^i \frac{\partial f^\alpha}{\partial z^i}$ and hence,

$$\begin{aligned}
2w'^\alpha &= 2\zeta'^i \frac{\partial f'^\alpha}{\partial z'^i} = (\frac{\partial \eta'^i}{\partial z^k}\eta^k + 2\frac{\partial \eta'^i}{\partial \eta^k}\zeta^k) \frac{\partial z^j}{\partial z'^i} \frac{\partial f'^\alpha}{\partial z'^j} \\
&= 2\zeta^k \frac{\partial f'^\alpha}{\partial z^k} + \frac{\partial \eta'^i}{\partial z^k}\eta^k \frac{\partial z^j}{\partial z'^i} \frac{\partial f'^\alpha}{\partial z'^j} \\
&= 2\zeta^k \frac{\partial u^\beta}{\partial z^j} \frac{\partial u'^\alpha}{\partial u^\beta} + \eta^k \frac{\partial \eta'^i}{\partial z^k} \frac{\partial u'^\alpha}{\partial z^k} + \eta'^i \eta'^j \frac{\partial^2 f'^\alpha}{\partial z'^i \partial z'^j} = 2w^\beta \frac{\partial u'^\alpha}{\partial u^\beta} + v^\beta \frac{\partial v'^\alpha}{\partial u^\beta}.
\end{aligned}$$

By these, we conclude that for the coordinates $(u^\alpha, v^\alpha, w^\alpha)$ we have the changes of coordinates analogous to (1.2). □

With this, we have the following diagram

$$\begin{array}{ccc}
J^{(2,0)}M & \xrightarrow{f'^{**}} & J^{(2,0)}N \\
\downarrow \pi_1 & & \downarrow \Pi_1 \\
T'M & \xrightarrow{f'^{*}} & T'N \\
\downarrow \pi & & \downarrow \Pi \\
M & \xrightarrow{\Sigma} & N
\end{array} . \tag{2.2}$$

Let $\sigma : [a, b] \to M$ be a curve on M and let $c : [a, b] \to J^{(2,0)}M$ be its extension. Combining the diagrams (1.7) and (2.2), we obtain a curve $l = f \circ c : [a, b] \to J^{(2,0)}N$.

The curve $f \circ \Sigma : [-\varepsilon, \varepsilon] \times [a, b] \to N$ is a variation $(s, t) \to u^\alpha(s, t) = (f^\alpha(z^k(s, t)))$, with $u^\alpha(0, t) = f^\alpha(\sigma(t))$.

The mappings $f'^* \circ \gamma$ and $f'^{**} \circ \Gamma$ pull-back the bundles $\Sigma^*(T'M)$ and $\gamma^*(J^{(2,0)}M)$ into $T'N$ and $J^{(2,0)}N$, respectively.

Further on, because $J^{(2,0)}M$ and $J^{(2,0)}N$ are endowed with Lagrangian structures, let us consider adapted base of the Chern-Lagrange (c.n.c), (1.4) on $T'(J^{(2,0)}M)$ and on $T'(J^{(2,0)}N)$, denoted by $\{\overset{M}{\delta}_{0i}, \overset{M}{\delta}_{1i}, \overset{M}{\delta}_{2i}\}$ and $\{\overset{N}{\delta}_{0i}, \overset{N}{\delta}_{1i}, \overset{N}{\delta}_{2i}\}$. By conjugation, we obtain the adapted base on $T''(J^{(2,0)}M)$ and on $T''(J^{(2,0)}N)$.

As we say, along the curves $\Gamma : (s, t) \to \big(z^k(s, t), \eta^k(s, t), \zeta^k(s, t)\big)$, which is a variation of c, we have the variation vector fields on $T_C(J^{(2,0)}M)$

$$\begin{aligned}
\frac{d}{dt} &: = \dot{c}_t(s) + \overline{\dot{c}_t}(s) = \overset{(p)}{V^k}(s) \overset{M}{\delta}_{pk} + \overset{(p)}{\bar{V}^k}(s) \overset{M}{\delta}_{p\bar{k}}; \\
\frac{d}{ds} &: = \dot{c}_s(t) + \overline{\dot{c}_s}(t) = \overset{(p)}{U^k}(t) \overset{M}{\delta}_{pk} + \overset{(p)}{\bar{U}^k}(t) \overset{M}{\delta}_{p\bar{k}},
\end{aligned}$$

defined as in the previous section.

Along the curves $f'^{**}\circ\Gamma:(s,t)\to(u^\alpha(s,t),v^\alpha(s,t),w^\alpha(s,t))$, which are variation of l, we have the variation vectors on $T_C(J^{(2,0)}N)$, still denoted by $\frac{d}{dt}$ and $\frac{d}{ds}$:

$$\begin{aligned}\frac{d}{dt} &: = \dot{l}_t(s)+\overline{\dot{l}_t}(s)=\overset{(p)}{X^\alpha}(s)\overset{N}{\delta}_{p\alpha}+\overset{(p)}{\bar{X}^\alpha}(s)\overset{N}{\delta}_{p\bar{\alpha}};\\ \frac{d}{ds} &: = \dot{l}_s(t)+\overline{\dot{l}_s}(t)=\overset{(p)}{Y^\alpha}(t)\overset{N}{\delta}_{p\alpha}+\overset{(p)}{\bar{Y}^\alpha}(t)\overset{N}{\delta}_{p\bar{\alpha}},\end{aligned}$$

where $\overset{(0)}{X^\alpha}=\frac{du^\alpha}{dt}$, $\overset{(1)}{X^\alpha}=\frac{\overset{N}{\delta}v^\alpha}{dt}$, $\overset{(2)}{X^\alpha}=\frac{\overset{N}{\delta}w^\alpha}{dt}$, $\overset{(0)}{Y^\alpha}=\frac{du^\alpha}{ds}$, $\overset{(1)}{Y^\alpha}=\frac{\overset{N}{\delta}v^\alpha}{ds}$ and $\overset{(2)}{Y^\alpha}=\frac{\overset{N}{\delta}w^\alpha}{ds}$.

We note that,

$\overset{(0)}{X^\alpha}=\frac{df^\alpha}{dt}=\frac{\partial f^\alpha}{\partial z^i}\frac{dz^i}{dt}=\frac{\partial f^\alpha}{\partial z^i}\overset{(0)}{V^i}=:\overset{(0)}{X^\alpha_{(b)i}}\overset{(0)}{V^i}$,

$\overset{(1)}{X^\alpha}=\frac{\delta v^\alpha}{\delta z^i}\frac{\delta z^i}{dt}+\frac{\delta v^\alpha}{\delta\eta^i}\frac{\delta\eta^i}{dt}+\frac{\delta v^\alpha}{\delta\zeta^i}\frac{\delta\zeta^i}{dt}=:\overset{(1)}{X^\alpha_{(q)i}}\overset{(q)}{V^i}$, and similarly

$\overset{(2)}{X^\alpha}=\frac{\delta w^\alpha}{\delta z^i}\frac{\delta z^i}{dt}+\frac{\delta w^\alpha}{\delta\eta^i}\frac{\delta\eta^i}{dt}+\frac{\delta w^\alpha}{\delta\zeta^i}\frac{\delta\zeta^i}{dt}=:\overset{(2)}{X^\alpha_{(q)i}}\overset{(q)}{V^i}$. From this, when $c(s,t)$ is with endpoints, i.e. $\overset{(p)}{V}=0$, for any $p=0,1,2$, then $\overset{(p)}{X}=0$ and thus, $l(s,t)$ is with endpoints.

A geodesic curve on $J^{(2,0)}N$ is defined as above for $J^{(2,0)}M$, and it is given by a variation of the curve l, such that the energy $\frac{1}{2}\frac{d\mathcal{H}(\tilde{c})(s,t)}{ds}|_{s=0}=\frac{1}{2}\int_a^b\frac{dH(\dot{c}_t,\overline{\dot{c}_t})}{ds}dt=0$ and, due to Theorem 1.1, it means

$$Re\ H(\dot{l}_s,\overset{N}{D}_{\frac{dl}{dt}}\overline{\dot{l}_t})=Re\ H(\mathbf{T}^N(\frac{dl}{ds},\dot{l}_t),\overline{\dot{l}_t}). \tag{2.3}$$

Now, our aim is to obtain a characterization of the harmonic map f. First, for the metric structure H, defined by the Lagrangian function on $J^{(2,0)}N$, let us compute

$$\begin{aligned}H(\dot{l}_t(s),\overline{\dot{l}_t}(s)) &= \sum_{p=0,1,2}H(\overset{(p)}{X^\alpha}\overset{N}{\delta}_{p\alpha},\overline{\overset{(p)}{X^\beta}\overset{N}{\delta}_{p\beta}})=\sum_{p=0,1,2}H(\overset{(p)}{X^\alpha}\overset{N}{\delta}_{p\alpha},\overset{(p)}{\bar{X}^\beta}\overset{N}{\delta}_{p\bar{\beta}})\\ &= \sum_{p=0,1,2}\overset{(p)}{X^\alpha}\overset{(p)}{\bar{X}^\beta}h_{\alpha\bar{\beta}}=\sum_{p,q=0,1,2}\overset{(p)}{X^\alpha_{(q)i}}\overset{(p)}{\bar{X}^\alpha_{(q)j}}\overset{(q)}{V^i}\overset{(q)}{\bar{V}^j}h_{\alpha\bar{\beta}}\\ &= \sum_{p,q=0,1,2}g^{\bar{j}i}\overset{(p)}{X^\alpha_{(q)i}}\overset{(p)}{\bar{X}^\alpha_{(q)j}}h_{\alpha\bar{\beta}}G(\dot{c}_t(s),\overline{\dot{c}_t}(s)).\end{aligned}\tag{2.4}$$

Definition 2.1. *A harmonic map of $J^{(2,0)}M$ into $J^{(2,0)}N$ is a holomorphic map $f:M\to N$ which gives the extreme of the integral*

$$\mathcal{E}(f)=\frac{1}{2}\int_a^b\sum_{p,q=0,1,2}g^{\bar{j}i}\overset{(p)}{X^\alpha_{(q)i}}\overset{(p)}{\bar{X}^\alpha_{(q)j}}h_{\alpha\bar{\beta}}dt:=\frac{1}{2}\int_a^b E(l(s,t))|_{s=0}dt \tag{2.5}$$

along the curve $l : t \to (u^\alpha(t), v^\alpha(t), w^\alpha(t))$, constructed as above.

Remark. Note that the above expression generalizes the classical integral of energy $\frac{1}{2}\int_a^b g^{\bar{j}i}\frac{\partial f^\alpha}{\partial z^i}\overline{\frac{\partial f^\beta}{\partial z^j}}h_{\alpha\bar{\beta}}dt$, which is contained as a term in (2.5). Moreover, if we reduce the computation to the horizontal curves on $J^{(2,0)}N$, which are characterized by $\overset{(1)}{X}{}^\alpha = \overset{(2)}{X}{}^\alpha = 0$, then $\mathcal{E}(f) = \frac{1}{2}\int_a^b g^{\bar{j}i}\frac{\partial f^\alpha}{\partial z^i}\overline{\frac{\partial f^\beta}{\partial z^j}}h_{\alpha\bar{\beta}}dt.$

Differentiating in (2.4), we have

$$\frac{dH(\dot{l}_t, \overline{\dot{l}_t})}{ds}|_{s=0} = G(\dot{c}_t, \overline{\dot{c}_t})\frac{dE(l(s,t))}{ds}|_{s=0} + E(l(t))\frac{dG(\dot{c}_t, \overline{\dot{c}_t})}{ds}|_{s=0}. \tag{2.6}$$

Further on, we recover somewhat the computation which yields the result from Theorem 1.1.

Taking into account that $H(\dot{l}_t, \overline{\dot{l}_t}) = \sum H(\overset{p}{X}, \overset{p}{\bar{X}})$ and $\frac{d}{ds} := \dot{l}_s(t) + \overline{\dot{l}_s}(t) = \overset{(q)}{Y}{}^\alpha \overset{N}{\delta}_{q\alpha} + \overset{(q)}{\bar{Y}}{}^{\alpha} \overset{N}{\delta}_{q\bar{\alpha}} =: Y + \bar{Y}$ and similarly, for $\frac{d}{dt} := X + \bar{X}$, we have

$$\frac{1}{2}\frac{dH(\dot{l}_t, \overline{\dot{l}_t})}{ds}|_{s=0} = \frac{1}{2}(Y + \bar{Y})H(X + \bar{X}).$$

Now, let us consider $\overset{N}{D}$ the Chern-Lagrange (1.5) metrical connection, i.e. $AH(B,C) = H(\overset{N}{D}_A B, C) + H(B, \overset{N}{D}_A C)$, with torsion $\mathbf{T}(A,B) = \overset{N}{D}_A B - \overset{N}{D}_B A - [A,B]$, for any vector fields of $T_C(J^{(2,0)}N)$. Taking into account that H is Hermitian, $H(A,\bar{B}) = \overline{H(B,\bar{A})}$, and expanding the calculation for the variation of H, we have

$$\begin{aligned}\frac{1}{2}\frac{dH(\dot{l}_t, \overline{\dot{l}_t})}{ds}|_{s=0} &= Re\,\{H(\mathbf{T}^N(\dot{l}_s, \dot{l}_t) + \overset{N}{D}_{\dot{l}_s}\dot{l}_t, \overline{\dot{l}_t}) + H(\mathbf{T}^N(\overline{\dot{l}_s}, \dot{l}_t) + \overset{N}{D}_{\dot{l}_t}\dot{l}_s, \overline{\dot{l}_t})\} \\ &= Re\,\{H(\mathbf{T}^N(\dot{l}_s + \overline{\dot{l}_s}, \dot{l}_t), \overline{\dot{l}_t}) + H(\overset{N}{D}_{\dot{l}_s}\dot{l}_t, \overline{\dot{l}_t}) + H(\overset{N}{D}_{\dot{l}_t}\dot{l}_s, \overline{\dot{l}_t}).\end{aligned}$$

Integrating by parts this variation, we obtain

$$\begin{aligned}\frac{1}{2}\frac{dH(\dot{l}_t, \overline{\dot{l}_t})}{ds}|_{s=0} &= \int_a^b Re\,\{H(\mathbf{T}^N(\dot{l}_s + \overline{\dot{l}_s}, \dot{l}_t), \overline{\dot{l}_t}) + \frac{d}{dt}H(\dot{l}_s, \overline{\dot{l}_t}) - H(\dot{l}_s, \overset{N}{D}_{\dot{l}_t}\overline{\dot{l}_t}) \\ &\quad + \frac{d}{dt}H(\dot{l}_s, \overline{\dot{l}_t}) - H(\dot{l}_s, \overset{N}{D}_{\overline{\dot{l}_t}}\overline{\dot{l}_t})\}dt,\end{aligned}$$

computed for $s = 0$.

Considering the curves $c(s,t)$ with fixed endpoints, i.e. $V(a) = V(b) = 0$, it yields $X(a) = X(b) = 0$. From (2.6), using the Chern-Lagrange connection $\overset{N}{D}$ with torsion $\mathbf{T}^N$, it follows that

$$\int_a^b Re\,\{H(\mathbf{T}^N(\dot{l}_s + \overline{\dot{l}_s}, \dot{l}_t), \overline{\dot{l}_t}) - H(\dot{l}_s, \overset{N}{D}_{\dot{l}_t + \overline{\dot{l}_t}}\overline{\dot{l}_t})\}_{|s=0}dt \tag{2.7}$$

$$= \frac{1}{2}\int_a^b G(\dot{c}_t,\overline{\dot{c}_t})\frac{dE(l(s,t))}{ds}|_{s=0}dt$$
$$+ \int_a^b E(l(t))Re\,\{G\left(\mathbf{T}^M(\dot{c}_s+\overline{\dot{c}_s},\dot{c}_t),\overline{\dot{c}_t}\right) - G(\dot{c}_s,\overset{M}{D}_{\dot{c}_t+\overline{\dot{c}_t}}\overline{\dot{c}_t})\}_{|s=0}dt.$$

Since G is nondegenerate on each curve $c(t)$, we conclude:

Theorem 2.1. *Let $c: t \to (z^k(t),\eta^k(t),\zeta^k(t))$ be a given curve on $J^{(2,0)}M$ and $l: t \to (u^\alpha(t),v^\alpha(t),w^\alpha(t))$ be the induced curve on $J^{(2,0)}N$ by the holomorphic map $f: M\to N$. Then,*

i) $f: M \to N$ is a harmonic map if and only if

$$Re\,\{H(\mathbf{T}^N(\dot{l}_s+\overline{\dot{l}_s},\dot{l}_t),\overline{\dot{l}_t}) - E(l(t))G\left(\mathbf{T}^M(\dot{c}_s+\overline{\dot{c}_s},\dot{c}_t),\overline{\dot{c}_t}\right)\}_{|s=0}$$
$$= Re\,\{H(\dot{l}_s,\overset{N}{D}_{\dot{l}_t+\overline{\dot{l}_t}}\overline{\dot{l}_t}) - E(l(t))G(\dot{c}_s,\overset{M}{D}_{\dot{c}_t+\overline{\dot{c}_t}}\overline{\dot{c}_t})\}_{|s=0}$$

and

$$Im\,E(l(t))\;Re\,\{G\left(\mathbf{T}^M(\dot{c}_s+\overline{\dot{c}_s},\dot{c}_t),\overline{\dot{c}_t}\right) - G(\dot{c}_s,\overset{M}{D}_{\dot{c}_t+\overline{\dot{c}_t}}\overline{\dot{c}_t})\}_{|s=0} = 0.$$

ii) If c is a geodesic curve on $J^{(2,0)}M$ and f is a harmonic map, then l is a geodesic curve on $J^{(2,0)}N$.

Corollary 2.1. *If $f: M \to N$ induces a totally geodesic mapping between $J^{(2,0)}M$ and $J^{(2,0)}N$, (i.e., it maps geodesics of $J^{(2,0)}M$ into geodesics of $J^{(2,0)}N$), then f is a harmonic map of $J^{(2,0)}M$ into $J^{(2,0)}N$.*

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